GAUSSIAN INTEGRALS AND THE ROGERS-RAMANUJAN IDENTITIES

D. Stanton *

School of Mathematics, University of Minnesota, Minneapolis, MN 55455

stanton@math.umn.edu

Abstract It is well known that the Fourier transform of a Gaussian is a Gaussian. In this paper it is shown that a q-analogue of this integral gives the Rogers-Ramanujan identities.

Keywords: Rogers-Ramanujan, Hermite polynomials

1. Introduction

The purpose of this paper is to show that a natural q-analogue of the elementary integral

$$I(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-xt+t^2/2} e^{-x^2/2} dx = e^{t^2} \tag{1.1}$$

immediately leads to the Rogers-Ramanujan identities [1]

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \frac{1}{\prod_{i=0}^{\infty}(1-q^{5i+1})(1-q^{5i+4})}, \tag{1.2a}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \frac{1}{\prod_{i=0}^{\infty}(1-q^{5i+2})(1-q^{5i+3})}. \tag{1.2b}$$

An abbreviated version of this idea appears in [2].

The paper is organized as follows. A Hermite polynomial evaluation of (1.1) is given in §2, and its q-analogue is in §3. This integral is evaluated for a special t in §4, giving the Rogers-Ramanujan identities. Two other q-analogues of Gaussian integrals are given in §5, the resulting identities are

*Partially supported by NSF grant DMS 99-70627.

Theorems 5.1 and 5.3. A mixed linearization result for q-Hermite polynomials is given in Theorem 7.1 of the Appendix.

2. A Hermite polynomial interpretation

It is clear that $I(t)$ may be evaluated by completing the square in the exponential. In this section we reinterpret the integral in terms of Hermite polynomials and give an alternate evaluation that $I(t) = e^{t^2}$.

We need the orthogonality relation and generating function for a rescaled version of the Hermite polynomials $\hat{H}_n(x)$. These two facts are

$$\int_{-\infty}^{\infty} \hat{H}_m(x)\hat{H}_n(x)w(x)dx = n!\delta_{mn} \qquad w(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \tag{Hermite orthogonality}$$

$$G(x,t) = \sum_{n=0}^{\infty} \hat{H}_n(x)\frac{t^n}{n!} = e^{xt-t^2/2}, \tag{Hermite GF}$$

We next restate $I(t)$ in terms of Hermite polynomials. Since

$$e^{-xt+t^2/2} = G(x,t)^{-1} = G(ix,it),$$

$$I(t) = \int_{-\infty}^{\infty} G(x,t)^{-1}w(x)dx \tag{2.1a}$$

$$= \int_{-\infty}^{\infty} G(ix,it)w(x)dx \tag{2.1b}$$

$$= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \int_{-\infty}^{\infty} \hat{H}_n(ix)w(x)dx. \tag{2.1c}$$

We need to know the constant term in the expansion of $\hat{H}_n(ix)$ in terms of $\hat{H}_n(x)$ to evaluate the integral in (2.1c). However,

$$\hat{H}_n(ix) = \sum_{k=0}^{n/2} \frac{n!i^n}{k!(n-2k)!}\hat{H}_{n-2k}(x). \tag{2.2}$$

So the constant term is 0 if n is odd, and is $i^n \frac{n!}{(n/2)!}$ if n is even. Thus

$$I(t) = \sum_{N=0}^{\infty} \frac{t^{2N}}{N!} = e^{t^2}. \tag{2.3}$$

3. The q-version of $I(t)$

In this section we set up a q-analogue of $I(t)$, based upon the interpretation of §2. We use the standard notation [3] from q-series,

$$(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i), \quad (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i).$$

The q-Hermite polynomials satisfy [3]

$$\int_0^\pi H_m(\cos\theta|q)H_n(\cos\theta|q)w_q(\cos\theta)d\theta = (q;q)_n\delta_{mn},$$
$$w_q(\cos\theta) = \frac{(q;q)_\infty}{2\pi}(e^{2i\theta}, e^{-2i\theta};q)_\infty$$
(q-Hermite orthogonality)

$$G_q(x,t) = \sum_{n=0}^{\infty} H_n(\cos\theta|q)\frac{t^n}{(q;q)_n} = \frac{1}{(te^{i\theta}, te^{-i\theta};q)_\infty}. \qquad (q\text{-Hermite GF})$$

These are q-analogues because the appropriate limiting cases as $q \to 1$ are

$$\lim_{q\to 1^-} \frac{H_n(x\sqrt{1-q}/2|q)}{(1-q)^{n/2}} = \hat{H}_n(x), \tag{3.1a}$$

$$\lim_{q\to 1^-} w_q(x\sqrt{1-q}/2) = w(x), \tag{3.1b}$$

$$\lim_{q\to 1^-} G_q(x\sqrt{1-q}/2, t\sqrt{1-q}) = G(x,t). \tag{3.1c}$$

Thus our q-analogue of $I(t)$ is

$$I_q(t) = \frac{(q;q)_\infty}{2\pi}\int_0^\pi (te^{i\theta}, te^{-i\theta}, e^{2i\theta}, e^{-2i\theta};q)_\infty d\theta$$
$$= \int_0^\pi G_q(\cos\theta, t)^{-1} w_q(\cos\theta)d\theta. \qquad (q\text{-2.1a})$$

We expect $I_q(t)$ to be a q-analogue of e^{t^2}.

To carry out the q-analogue of the proof in §2, we need a q-analogue of the functional equation $G(x,t)^{-1} = G(ix, it)$. This is

$$G_q(x,t)^{-1} = G_{1/q}(x, t/q) = (te^{i\theta}, te^{-i\theta};q)_\infty$$
$$= \sum_{n=0}^{\infty} H_n(x|q^{-1})\frac{q^{\binom{n}{2}}(-t)^n}{(q;q)_n}. \tag{3.2}$$

Since we have

$$\lim_{q\to 1}\frac{H_n(x\sqrt{1-q}/2|q^{-1})}{(1-q)^{n/2}}\frac{(1-q)^n(-t)^n}{(q;q)_n}=\hat{H}_n(ix)\frac{(it)^n}{n!},$$

(3.2) is a q-analogue of $G(x,t)^{-1}=G(ix,it)$. This allows the next two steps of (2.1) to be accomplished

$$I_q(t)=\int_0^\pi G_{1/q}(\cos\theta,t/q)w_q(\cos\theta)d\theta \qquad (q\text{-2.1b})$$

$$=\sum_{n=0}^\infty\frac{q^{\binom{n}{2}}(-t)^n}{(q;q)_n}\int_0^\pi H_n(\cos\theta|q^{-1})w_q(\cos\theta)d\theta. \qquad (q\text{-2.1c})$$

Thus we must find the constant term in the q-Hermite expansion of $H_n(x|q^{-1})$. However, all of the terms in this expansion were known to Rogers (see [3, (7.6.14)], [5, p. 335 (1)])

$$H_n(x|q^{-1})=\sum_{k=0}^{n/2}\frac{q^{k(k-n)}(q;q)_n}{(q;q)_k(q;q)_{n-2k}}H_{n-2k}(x|q). \qquad (3.3)$$

The constant term is 0 for n odd and is $q^{-n^2/4}\frac{(q;q)_n}{(q;q)_{n/2}}$ if n is even. Thus

$$I_q(t)=\sum_{N=0}^\infty\frac{q^{N^2-N}t^{2N}}{(q;q)_N} \qquad (q\text{-2.3})$$

which is clearly a q-version of $I(t)=e^{t^2}$ in view of the scaling $t\to t\sqrt{1-q}$.

4. The Rogers-Ramanujan identities

In this section we prove the Rogers-Ramanujan identities. Note that $I_q(\sqrt{q})$ and $I_q(q)$ are the sum sides of the two Rogers-Ramanujan identities. We now evaluate $I_q(\sqrt{q})$ and $I_q(q)$ in another way, which will give the product sides of the Rogers-Ramanujan identities. Instead of using q-Hermite orthogonality as in §3 we will use the classical exponential orthogonality

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{in\theta}e^{-im\theta}d\theta=\delta_{mn}.$$

First note that in $I_q(t)$ we can integrate on $[-\pi,\pi]$, at the expense of dividing by 2, since the integrand is an even function of θ.

Take $t = \sqrt{q}$, and expand both $G(x, \sqrt{q})^{-1}$ and $(q, e^{2i\theta}, e^{-2i\theta}; q)_\infty$ in powers of $e^{i\theta}$ using the Jacobi triple product identity [1]

$$(q, z, q/z; q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{(k^2-k)/2} z^k,$$

$$(q, \sqrt{q}e^{i\theta}, \sqrt{q}e^{-i\theta}; q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2/2} e^{-ik\theta}, \tag{4.1a}$$

$$(q, e^{2i\theta}, e^{-2i\theta}; q)_\infty = (1 - e^{2i\theta}) \sum_{j=-\infty}^{\infty} (-1)^j q^{(j^2+j)/2} e^{2ij\theta}. \tag{4.1b}$$

Thus

$$I_q(\sqrt{q}) = \frac{1}{4\pi(q;q)_\infty} \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} q^{k^2/2}(-1)^k q^{(j^2+j)/2}(-1)^j \times \int_{-\pi}^{\pi} e^{-ik\theta} e^{2ij\theta}(1 - e^{2i\theta}) d\theta.$$

The exponential orthogonality implies

$$\begin{aligned} I_q(\sqrt{q}) =& \frac{1}{2(q;q)_\infty} \sum_{j=-\infty}^{\infty} (q^{2j^2} q^{(j^2+j)/2} - q^{2(j+1)^2} q^{(j^2+j)/2})(-1)^j \\ =& \frac{1}{(q;q)_\infty} \sum_{j=-\infty}^{\infty} q^{2j^2} q^{(j^2+j)/2} (-1)^j. \end{aligned}$$

Finally applying the Jacobi triple product identity we arrive at

$$I_q(\sqrt{q}) = \frac{1}{(q^1, q^4; q^5)_\infty},$$

which is the product side of the first Rogers-Ramanujan identity (1.2a).

Note that the modulus 5 appears because the q-Hermite weight is on $e^{2i\theta}$ and $2^2 + 1 = 5$.

The choice $t = q$ similarly yields the second Rogers-Ramanujan identity (1.2b).

We may also prove (1.2b) by evaluating the integral

$$\hat{I}_q(t) = \int_0^{\pi} H_1(\cos\theta|q)(te^{i\theta}, te^{-i\theta}, e^{2i\theta}, e^{-2i\theta}; q)_\infty d\theta$$

at $t=\sqrt{q}$, which is a q-analogue of

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-xt+t^2/2}e^{-x^2/2}dx = -te^{t^2}.$$

Using (3.3) we see that

$$\hat{I}_q(t) = -t\sum_{N=0}^{\infty}\frac{q^{N^2}t^{2N}}{(q;q)_N},$$

and the evaluation

$$\hat{I}_q(\sqrt{q}) = -\sqrt{q}\frac{1}{(q^2,q^3;q^5)_\infty}$$

follows from Fourier orthogonality.

5. Two more Gaussian integrals

In this section we give two more Rogers-Ramanujan type identities, Theorems 5.1 and 5.3, which can be established using Gaussian integrals.

The evaluation of $I_q(t)$ in §3 was based upon the explicit form of the constant term when $H_n(x|q^{-1})$ is expanded in the q-Hermite basis. Given any set of polynomials $p_n(x)$ such that

$$p_n(x) = \sum_{k=0}^{n} c_{nk}H_k(x|q), \tag{5.1a}$$

if

$$H(x,t) = \sum_{n=0}^{\infty} p_n(x)t^n \tag{5.1b}$$

then

$$\int_0^\pi H(\cos\theta,t)w_q(\cos\theta)d\theta = \sum_{n=0}^{\infty} c_{n0}t^n. \tag{5.2}$$

For the first example, we choose

$$H(x,t) = \sum_{n=0}^{\infty}\frac{q^{\frac{1}{2}\binom{n+1}{2}}(-t)^n}{(q^{1/2};q^{1/2})_n}H_n(x|q^{-1}) = \frac{(tq^{1/2}e^{i\theta},tq^{1/2}e^{-i\theta};q^{1/2})_\infty}{(t^2q^{1/2};q)_\infty}$$

so that

$$\lim_{q\to 1} H(x\sqrt{1-q}/2, t\sqrt{1-q}) = G(ix,2it) = e^{-2xt+2t^2},$$

then (3.3) implies

$$\int_0^\pi H(\cos\theta, t) w_q(\cos\theta) d\theta = \sum_{N=0}^{\infty} \frac{(q;q)_{2N}}{(q^{1/2};q^{1/2})_{2N}} \frac{(q^{1/2}t^2)^N}{(q;q)_N}. \tag{5.3}$$

Equation (5.3) is a q-analogue of

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2xt+2t^2} e^{-x^2/2} dx = e^{4t^2}.$$

The following theorem results if we evaluate the integral in (5.3) at $t=1$ using Fourier orthogonality.

Theorem 5.1. *We have*

$$\sum_{N=0}^{\infty} \frac{(-q;q)_{2N}}{(q^2;q^2)_N} q^N = \frac{1}{(q,q^2,q^3,q^3,q^4,q^5;q^6)_\infty}.$$

For an integer partition interpretation of Theorem 5.1, we use colored partitions. Note that

$$\frac{(-q;q)_{2N}}{(q^2;q^2)_N} q^N = \frac{(-q^{N+1};q)_N}{(q;q)_N} q^N,$$

which may be interpreted as enumerating partitions with arbitrary parts from the set $1, 2, \cdots, N$, N is a part, and possibly distinct parts from the set $N+1, \cdots, 2N$.

Corollary 5.2. *Let $A(n)$ be the number of integer partitions of n into parts not congruent to 0 mod 6, whose parts mod 3 are colored red or blue. Let $B(n)$ be the number of integer partitions of n with red or blue parts, such that the if largest red part is N, then the blue parts must be distinct and lie between $N+1$ and $2N$. Then $A(n) = B(n)$.*

Theorem 5.1 is closely related to a result of Slater [6, (24)]

$$\sum_{N=0}^{\infty} \frac{(-1;q)_{2N}}{(q^2;q^2)_N} q^N = \frac{1}{(q,q,q^2,q^4,q^5,q^5;q^6)_\infty}. \tag{5.4}$$

If $H_1(\cos\theta|q)$ is inserted into the integrand in (5.3), the coefficient of $H_1(x|q)$ rather than the constant term of (3.3) is used to evaluate the integral. The resulting companion identity is

$$-q \sum_{N=0}^{\infty} \frac{(-q;q)_{2N+1}}{(q^2;q^2)_N} q^N = \frac{(q^6,q,q^5;q^6)_\infty - (q^6,q^3,q^3;q^6)_\infty}{(q;q)_\infty (q;q^2)_\infty}. \tag{5.5}$$

Slater's (5.4) is the difference of Theorem 5.1 and (5.5).

For the second example we choose the generating function

$$H(x,t)=\sum_{n=0}^{\infty}\frac{q^{2\binom{n+1}{2}}(-t)^n}{(q^2;q^2)_n}H_{2n}(x|q^{-1})=\frac{(tq^2e^{2i\theta},tq^2e^{-2i\theta};q^2)_\infty}{(-tq;q)_\infty}$$

which satisfies

$$\lim_{q\to 1}H(x\sqrt{1-q}/2,t)=\sum_{n=0}^{\infty}\hat{H}_{2n}(ix)\frac{(t/2)^n}{n!}=e^{-tx^2/2(1+t)}/\sqrt{1+t}.$$

This time (5.2) yields

$$\int_0^\pi H(\cos\theta,t)w_q(\cos\theta)d\theta=\sum_{N=0}^{\infty}\frac{(q;q^2)_N}{(q;q)_N}(-tq)^N, \tag{5.6}$$

which is a q-analogue of

$$\frac{1}{\sqrt{2\pi(1+t)}}\int_{-\infty}^{\infty}e^{-tx^2/2(1+t)}e^{-x^2/2}dx=\frac{1}{\sqrt{1+2t}}.$$

Again evaluating the integral at $t=1$ gives Theorem 5.3.

Theorem 5.3. *We have*

$$\sum_{N=0}^{\infty}\frac{(q;q^2)_N}{(q;q)_N}(-q)^N=\frac{(q,q^5;q^6)_\infty}{(q^2,q^4;q^6)_\infty}.$$

If we replace q by $-q$ in Theorem 5.3, an integer partition interpretation can be given. The right side interpretation is clear, while for the left side we note that

$$\frac{(-q;q^2)_N}{(-q;-q)_N}q^N=\begin{cases}\frac{(-q^{2m+1};q^2)_m}{(q^2;q^2)_m}q^{2m} & \text{if } N=2m,\\ \frac{(-q^{2m+3};q^2)_m}{(q^2;q^2)_m}q^{2m}q^1 & \text{if } N=2m+1.\end{cases}$$

Corollary 5.4. *Let $A(n)$ be the number of integer partitions of n into parts congruent to 2 or 4 mod 6, and distinct parts congruent to 1 or 5 mod 6. Let $B(n)$ be the number integer partitions of n*

(1) whose odd parts are distinct,

(2) whose odd parts (except possibly 1) are greater than the largest even part $2m$,

(3) whose largest odd part is at most $4m-1$ if 1 is not a part,

(4) *whose largest odd part is at most* $4m + 1$ *if 1 is a part, moreover in this case* $2m + 1$ *is not a part.*

Then $A(n) = B(n)$.

For example, if $n = 9$ the partitions enumerated by $A(9) = 7$ are

$$81,\ 72,\ 54,\ 522,\ 441,\ 4221,\ 22221$$

while those for $B(9) = 7$ are

$$81,\ 621,\ 54,\ 441,\ 4221, 3222,\ 22221.$$

6. Remarks

The proof of the Rogers-Ramanujan identities given here is fundamentally the same as Rogers' proof [5, p. 328]. He extracted constant terms without using integration. We have shown that his method is equivalent to evaluating a q-Gaussian integral.

In each of the three q-analogues of Gaussian integrals given in this paper, we used the constant term in the expansion of $H_n(x|q^{-1})$ in terms of $H_n(x|q)$. Several other polynomials besides $H_n(x|q^{-1})$ may be chosen whose constant terms c_{n0} in the q-Hermite basis explicitly factor, leading to integral evaluations. Many interesting choices involve changing the base in q-Hermite polynomials to other functions of q besides q^{-1}. For example, one choice involving base q^5 gives an integral which is a q-analogue of

$$\frac{e^{-t^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\sqrt{5}xt - t^2/2}\ e^{-x^2/2} dx = e^{t^2}. \tag{6.1}$$

This integral leads to a quintic transformation which generalizes the Rogers-Ramanujan identities [2, Theorem 7.1]. A more extensive study of these polynomials and their corresponding Rogers-Ramanujan type identities is given in [4].

We also note that (5.1), (5.2), and the q-Hermite orthogonality relation imply that

$$\int_0^{\pi} H(\cos\theta, t) H_k(\cos\theta|q) w_q(\cos\theta) d\theta = \sum_{n=0}^{\infty} c_{nk}(q;q)_k t^n.$$

If all of the coefficients c_{nk} are explicitly known, then generalized versions of the Rogers-Ramanujan identities may be given. This occurs in (3.3), thus generalizations may be given for (1.1), Theorem 5.1, and Theorem 5.3. The generalization for the classical Rogers-Ramanujan identities is explicitly stated for (1.1) in [2, (3.5)].

An integral which gives the multisum Rogers-Ramanujan identities is also given in [2, §4].

7. Appendix

Because of the interest in (3.3), we state and prove a mixed linearization result for q-Hermite polynomials which should be better known. The case $m = 0$ is (3.3).

Theorem 7.1. *We have*

$$H_m(x|q)H_n(x|q^{-1}) = \sum_{s=0}^{n} \frac{(q;q)_n(q;q)_{m+n-s}q^{s(s-n)}}{(q;q)_{n-s}(q;q)_s} \frac{H_{m+n-2s}(x|q)}{(q;q)_{m+n-2s}}.$$

Proof: Consider the product of the generating functions

$$F(t_1,t_2) = G_q(x,t_1)G_{1/q}(x,t_2) = \frac{(t_2e^{i\theta}, t_2e^{-i\theta}; q)_\infty}{(t_1e^{i\theta}, t_1e^{-i\theta}; q)_\infty}.$$

This is the generating function for the q-ultraspherical polynomials (see [3, (7.4.1)])

$$F(t_1,t_2) = \sum_{N=0}^{\infty} C_N(x; t_2/t_1|q)t_1^N.$$

However, the q-ultraspherical polynomials may be expanded in terms of q-Hermite polynomials [3, (7.6.14)]

$$C_N(x; t_2/t_1|q) = \sum_{s=0}^{N/2} \frac{(-t_2/t_1)^s q^{\binom{s}{2}} (t_2/t_1;q)_s}{(q;q)_s(q;q)_{N-2s}} H_{N-2s}(x|q).$$

We then have

$$F(t_1,t_2) = \sum_{N=0}^{\infty}\sum_{s=0}^{N/2} t_1^{N-s}(-t_2)^s(t_2/t_1;q)_{N-s}q^{\binom{s}{2}} \frac{H_{N-2s}(x|q)}{(q;q)_s(q;q)_{N-2s}}.$$

Expanding $(t_2/t_1;q)_{N-s}$ by the q-binomial theorem yields

$$F(t_1,t_2) = \sum_{N=0}^{\infty}\sum_{s=0}^{N/2}\sum_{k=0}^{N-s} \begin{bmatrix} N-s \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} t_1^{N-s-k} t_2^{s+k}$$
$$\times (-1)^s q^{\binom{s}{2}} \frac{H_{N-2s}(x|q)}{(q;q)_s(q;q)_{N-2s}}.$$

So the coefficient of $t_1^m t_2^n$ in $F(t_1,t_2)$ is

$$\sum_{s=0}^{n} \begin{bmatrix} m+n-s \\ n-s \end{bmatrix}_q q^{\binom{n-s}{2}+\binom{s}{2}}(-1)^n \frac{H_{m+n-2s}(x|q)}{(q;q)_s(q;q)_{m+n-2s}} = \frac{H_m(x|q)(-1)^n q^{\binom{n}{2}} H_n(x|q^{-1})}{(q;q)_m(q;q)_n}.$$

Several other connection results for the q-Hermite polynomials are given in [4].

Acknowledgment: Much of this paper is based on joint work with Tina Garrett and Mourad Ismail, and the author would like to thank each of them.

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