A UNIMODALITY IDENTITY FOR A SCHUR FUNCTION

Frederick M. Goodman*
Kathleen M. O'Hara**
and
Dennis Stanton***

Abstract. A polynomial identity in q for the principal specialization of the Schur function, $s_\lambda(1, q, \ldots, q^n)$ is given. The identity immediately proves that $s_\lambda(1, q, \ldots, q^n)$ is a unimodal polynomial in q.

It is well known that the principal specialization of the Schur function, $s_\lambda(1, q, \ldots, q^n)$, is a unimodal polynomial. A brief representation theoretic proof consists in identifying $s_\lambda(1, q, \ldots, q^n)$ as the character of a homogeneous polynomial representation of $GL(2, \mathbb{C})$ evaluated at $\mathrm{diag}(1, q)$, see [3, p. 67]. Recently O'Hara gave a combinatorial proof [4] of the unimodality of the Gaussian coefficient $s_{(k)}(1, q, \ldots, q^n) = \begin{bmatrix} n+k \\ k \end{bmatrix}_q$, and in [6] Zeilberger derived a polynomial identity from [4] which immediately implies the unimodality of $\begin{bmatrix} n+k \\ k \end{bmatrix}_q$. In this paper we give an analogous polynomial identity (1.4) which implies the unimodality of $s_\lambda(1, q, \ldots, q^n)$.

There are two ingredients to the proof of (1.4): first we give a bijection for column strict tableaux, and then we employ a remarkable bijection due to Kerov-Kirillov-Reshetikhin [1,2], which establishes a formula for the q-Kostka polynomial $K_{\lambda, 1^{|\lambda|}}(q)$. In fact, [1] and [2] provide the deep facts needed for the proof of (1.4). Finally (1.4) implies the unimodality of $s_\lambda(1, q, \ldots, q^n)$. The proof of (1.4) together with [4] constitutes a combinatorial proof of this fact.

We shall adopt notation for partitions etc. as found in [3]. By a descent of a standard tableau, we mean a cell i such that the cell labeled $i + 1$ lies in a row below i. We define the weight of a column strict tableau T, $w(T)$, as the sum of its entries, and the weight of a standard tableau S, $c(S)$, as the sum of its descents.

---

*Department of Mathematics, University of Iowa, Iowa City, Iowa 52422. Work partially supported by NSF grant DMS:8801329.

**Department of Mathematics, University of Iowa, Iowa City, Iowa 52422.

***School of Mathematics, University of Minnesota, Minneapolis, MN 55455. Work partially supported by NSF grant DMS:9001195.

Typeset by $\mathcal{AMS}$-TEX

Proposition 1. *There is a bijection between column strict tableaux T of shape λ with entries $\{0,1,2,\ldots,n\}$, and ordered pairs (μ, S), where S is a standard tableaux of shape λ and no more than n descents, and μ is a partition which lies inside a $|\lambda| \times (n-k)$ rectangle, where k is the number of descents of S. Moreover, in this bijection, $w(T) = |\mu| + |\lambda| k - c(S)$.*

Proof. Let w be the lattice permutation (or Yamanouchi word) corresponding to T, and let a be the entries of T listed in increasing order (which is the same as the order of the entries of w). The standard tableau S which corresponds to w has a descent at i if, and only if, w has an ascent at position i. The parts of μ (in increasing order) are defined by $\mu_i = a_i - (\#$ of ascents to the left of position i in $w)$. Thus μ has at most $|\lambda|$ parts, and its largest part is $\leq n-k$. An ascent in position i of w subtracts 1 from the next $|\lambda| - i$ entries of a_i, so $w(T) = |\mu| + k|\lambda| - c(S)$. $\square$

The generating function identity implied by Proposition 1 is

$$s_\lambda(1, q, \ldots, q^n) = \sum_k \begin{bmatrix} |\lambda| + n - k \\ n-k \end{bmatrix}_q q^{k|\lambda|} f_k^\lambda(q^{-1}) \tag{1.2}$$

where

$$f_k^\lambda(q) = \sum_S q^{c(S)}$$

is the generating function for standard tableaux of shape λ with k descents. Identity (1.2) is a special case of Stanley's theory of P-partitions [6, §8].

The polynomial $f_k^\lambda(q^{-1})$ can be rewritten in terms of the q-Kostka polynomial $K_{\lambda,1^{|\lambda|}}(q)$ (see [3, p.130]). If the descent set of S is D, the set of i such that $i+1$ is to the right of i is $D' = \{1, 2, \ldots, |\lambda|-1\} \setminus D$. Thus the sum of the elements of D' is $\binom{|\lambda|}{2} - c(S)$. The generating function for all standard tableaux of shape λ for descents to the right is $K_{\lambda,1^{|\lambda|}}(q)$, so

$$f_k^\lambda(q^{-1}) = q^{-\binom{|\lambda|}{2}} K^k_{\lambda,1^{|\lambda|}}(q), \tag{1.3}$$

where the k denotes that the tableaux have k descents.

We now appeal to Kirillov-Reshetikhin [2, Th. 4.2], who have an explicit formula for the q-Kostka polynomial,

$$K_{\lambda,1^{|\lambda|}}(q) = \sum_{(\alpha^{(0)},\alpha^{(1)},\ldots)} q^{c(\alpha)} \prod_{n,\ell\geq 1} \begin{bmatrix} P_n^\ell(\alpha) + m_n(\alpha^{(\ell)}) \\ m_n(\alpha^{(\ell)}) \end{bmatrix}_q,$$

where the summation is over certain allowable sequences of partitions (α), and $P_n^\ell(\alpha)$, $m_n(\alpha^{(\ell)})$, and $c(\alpha)$ are certain integers computed from (α). The exact definition of these functions will not be needed.

However in (1.3) we have the constraint that the tableaux have k descents. Fortunately, Kirillov-Reshetikhin prove [2, Cor. 4.7(iii)] that this constraint is that first column of $\alpha^{(1)}$ has length k. Thus, we have

$$\begin{aligned} s_\lambda(1, q, \ldots, q^n) = \sum_k & \begin{bmatrix} |\lambda| + n - k \\ n-k \end{bmatrix}_q q^{k|\lambda| - \binom{|\lambda|}{2}} \\ & \sum_{\substack{(\alpha^{(0)},\alpha^{(1)},\ldots) \\ \alpha^{(1)\prime}(1)=k}} q^{c(\alpha)} \prod_{n,\ell\geq 1} \begin{bmatrix} P_n^\ell(\alpha) + m_n(\alpha^{(\ell)}) \\ m_n(\alpha^{(\ell)}) \end{bmatrix}_q. \end{aligned} \tag{1.4}$$

We claim that (1.4) implies unimodality of $s_\lambda(1, q, \ldots, q^n)$. Since the individual terms in the sum are symmetric and unimodal, what remains is to show that they are all centered at $n|\lambda|/2$. Let x and y be the minimum and maximum degrees of the polynomial term in the sum corresponding to α. Then including $q^{k|\lambda|-\binom{|\lambda|}{2}}$ and $\begin{bmatrix} |\lambda|+n-k \\ n-k \end{bmatrix}_q$ the term has minimum degree $x + k|\lambda| - \binom{\lambda}{2}$, maximum degree $n|\lambda| - \binom{|\lambda|}{2} + y$. We must show that $x + y = 2\binom{|\lambda|}{2} - k|\lambda|$. Clearly $x = c(\alpha)$ and

$$y = c(\alpha) + \sum_{n,\ell} m_n(\alpha^{(\ell)}) P_n^\ell(\alpha).$$

However (4.2) of [2] is

$$2c(\alpha) + \sum_{n,\ell} m_n(\alpha^{(\ell)}) P_n^\ell(\alpha) = 2\binom{|\lambda|}{2} - |\lambda| \alpha^{(1)'}(1),$$

which is precisely what is required, since $\alpha^{(1)'}(1) = k$.

Other applications and properties of the constructions of [2] will appear in a forthcoming paper.

References

1. S. V. Kerov, A. N. Kirillov and N. Yu. Reshetikhin, *Combinatorics, Bethe Anstatz, and representations of the symmetric group*, J. Soviet Math. **41** (1988), 916-924.
2. A. N. Kirillov and N. Yu. Reshetikhin, *The Bethe ansatz and the combinatorics of Young tableaux*, J. Soviet Math. **41** (1988), 925-955.
3. I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford, London, 1979.
4. Kathleen M. O'Hara, *Unimodality of Gaussian coefficients: a constructive approach*, JCT A **53** (1990), 29-52.
5. R. Stanley, *Ordered structures and partitions*, Memoirs A.M.S. **119** (1972).
6. D. Zeilberger, *A one-line high school algebra proof of the unimodality of the Gaussian polynomials* $\begin{bmatrix} n \\ k \end{bmatrix}_q$ *for* $k < 20$, in q-Series and Partitions, D. Stanton, ed., IMA Volumes in Mathematics and Its Applications **18** (1989) 67-75..