OPEN POSITIVITY CONJECTURES FOR INTEGER PARTITIONS

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1. Introduction. The purpose of this report is to collect some open positivity conjectures for integer partitions. The conjectures in §2-§3 are simple to state and initially appear easy to do, but remain open. The conjectures in §4-§5 should be related to representation theory, since they generalize known facts in this area. The conjectures in §6 are speculative, and if true, would require new combinatorial insight.

We will use standard notation found in [1] throughout. The q-shifted factorials are defined by

$$(A;q)_n = \prod_{i=0}^{n-1}(1-Aq^i), \quad (A_1, A_2, \cdots, A_k; q)_n = \prod_{i=1}^{k}(A_i;q)_n$$

The q-binomial coefficient, which is the generating function for integer partitions which lie inside a $k\times(n-k)$ rectangle, is the polynomial in q given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n;q^{-1})_k}{(q;q)_k}.$$

2. A simple rational function. In [6] the following problem was discussed. Suppose that we consider all integer partitions whose parts lie in some fixed set P. The generating function for these partitions is

$$F_P(q) = \prod_{p\in P}\frac{1}{1-q^p} = \sum_{m=0}^{\infty} a_m q^m.$$

We wish to classify those sets P such that the coefficients a_m weakly increase for $m\geq 1$. This accomplished in [6, Th. 3.5], assuming the following conjecture.

Conjecture 1. *Suppose that $n\geq 3$ is an odd integer. If*

$$F_{\{n,n+1,\cdots,2n-1\}}(q) = \frac{1}{(1-q^n)(1-q^{n+1})\cdots(1-q^{2n-1})} = \sum_{m=0}^{\infty} a_m q^m,$$

then a_m is weakly increasing for $m\geq 1$.

Conjecture 1 has been verified by computer for $n\leq 37$. It is easy to see that a_m increases for m large, by considering the dominant pole of $(1-q)F_{\{n,n+1,\cdots,2n-1\}}(q)$.

There is a version for even values of n, which is somewhat more complicated.

A natural route of attack is to find an injection from the partitions of m with parts $\{n, n+1, \cdots, 2n-1\}$, to the partitions of $m+1$ with parts $\{n, n+1, \cdots, 2n-1\}$. This has been accomplished for $n \le 9$, but the details are numerous and not illuminating. Savage [16] has also partial results in this direction.

One may also hope to find a polynomial version which generalizes Conjecture 1 and is easier to prove.

3. The $+--$ conjecture. P. Borwein has conjectured [2] the sign behavior of the polynomial

$$p_n(q) = (q, q^2; q^3)_n = \prod_{i=1}^{n}(1-q^{3i-2})(1-q^{3i-1}) = \sum_{m=0}^{\infty} a_m q^m. \tag{3.1}$$

Conjecture 2. *(The $+--$ conjecture) If a_m is defined by (2.1), then for all m,*

$$a_{3m} \ge 0, \quad a_{3m+1} \le 0, \quad a_{3m+2} \le 0.$$

It is clear that Conjecture 2 may be reformulated as the existence of an injection. Let $D_n(m)$ be the set of integer partitions of m with distinct parts, and with no parts congruent to 0 modulo 3. Let $D_n^{even}(m)$ ($D_n^{odd}(m)$) denote those elements of $D_n(m)$ with an even (odd) number of parts. No one has been able to find injections

$$\begin{aligned} D_n^{odd}(3m) &\to D_n^{even}(3m), \\ D_n^{even}(3m+1) &\to D_n^{odd}(3m+1), \\ D_n^{even}(3m+2) &\to D_n^{odd}(3m+2). \end{aligned}$$

Another reasonable method is to use the q-binomial theorem to explicitly find the polynomials $A_n(q)$, $B_n(q)$, and $C_n(q)$ such that

$$p_n(q) = A_n(q^3) - qB_n(q^3) - q^2 C_n(q^3).$$

It is easy to see [2] that

$$\begin{aligned} A_n(q) &= \sum_k (-1)^k \begin{bmatrix} 2n \\ n+3k \end{bmatrix}_q q^{k(9k+1)/2}, \\ B_n(q) &= \sum_k (-1)^k \begin{bmatrix} 2n \\ n+3k-1 \end{bmatrix}_q q^{k(9k-5)/2}, \\ C_n(q) &= \sum_k (-1)^k \begin{bmatrix} 2n \\ n+3k+1 \end{bmatrix}_q q^{k(9k+7)/2}, \end{aligned}$$

and one could hope for a simple inductive proof if a recurrence for these polynomials with positive coefficients can be found. None one has yet.

However, very closely related alternating sums of q-binomial coefficients are known to be non-negative polynomials in q. It is known [3, Th.1], for example, that if α, β, K, M, and N are positive integers, with $\alpha+\beta < 2K$, $\beta - K \le N - M \le K - \alpha$, then

$$\sum_k (-1)^k \begin{bmatrix} M+N \\ M-kK \end{bmatrix}_q q^{Kk^2(\alpha+\beta)/2+Kk(\beta-\alpha)/2} \tag{3.2}$$

has non-negative coefficients. Moreover this polynomial is the generating function for partitions inside an $M \times N$ rectangle, whose hook differences satisfy certain inequalities involving α, β, and K, see [3].

If $M = N = n$, and $K = 3$, solving for α and β in (3.2) to obtain $A_n(q)$, we have $\alpha = 5/3$, $\beta = 4/3$. Thus if a combinatorial interpretation can be found for these rational values of α and β, Conjecture 2 is verified. A reasonable guess at such an interpretation is to use the bijection [11, p. 83],[7, p. 2] between partitions λ and $(\lambda_0, \mu_1, \mu_2\, \mu_3)$, where λ_0 is the 3-core of λ and (μ_1, μ_2, μ_3) is the 3-quotient. Perhaps the choices $\alpha = 5/3$, $\beta = 4/3$ put restrictions on the hook differences, say of μ_1 and μ_2. If this works it is clear that should be a conjecture for any modulus, not just 3, which is the next conjecture.

Conjecture 3. *Fix relatively prime positive integers a, K, with $a < K/2$. Let*

$$p_n(q) = (q^a, q^{K-a}; q^K)_n = \prod_{i=1}^{n} (1 - q^{a+(i-1)K})(1 - q^{Ki-a}) = \sum_{m=0}^{\infty} a_m q^m.$$

If K is odd, we have

$$\begin{aligned} &a_m \le 0 \text{ if } m \equiv \pm aj \mod K, \text{for some positive odd integer } j < K/2, \\ &a_m \ge 0 \text{ if } m \equiv \pm aj \mod K, \text{for some non-negative even integer } j < K/2. \end{aligned}$$

If K is even, we have $(-1)^m a_m \ge 0$.

The case $n = \infty$ of Conjecture 3 may be established by sieving the Jacobi triple product identity. For K odd we have

$$\begin{aligned} (q^a, q^{K-a}; q^K)_\infty = &\sum_{r=(1-K)/2}^{(K-1)/2} (-1)^r q^{K\binom{r+1}{2}-ar} \\ &\times \frac{(q^{K(\binom{K+1}{2}+Kr-a)}, q^{K(\binom{K}{2}-Kr+a)}, q^{K^3}; q^{K^3})_\infty}{(q^K; q^K)_\infty} \end{aligned} \tag{3.3}$$

Thus we see that (3.3) is

$$(q^a, q^{K-a}; q^K)_\infty = \sum_{r=(1-K)/2}^{(K-1)/2} (-1)^r q^{-ar} f_r(q^K), \tag{3.4}$$

where $f_r(q^K)$ is a power series in q^K with non-negative coefficients. So (3.4) verifies Conjecture 3 for K odd and $n = \infty$.

For K even we similarly have

$$(q^a, q^{K-a}; q^K)_\infty = \sum_{r=0}^{K-1} (-1)^r q^{K\binom{r}{2}+ar} \times \frac{(-q^{K(\binom{K+1}{2}-Kr-a)}, -q^{K(\binom{K}{2}+Kr+a)}, q^{K^3}; q^{K^3})_\infty}{(q^K; q^K)_\infty}, \tag{3.5}$$

so that we again obtain (3.4). Here K is even and a is odd, so ar will be odd modulo K if, and only if, r is odd modulo K.

One may try to prove Conjecture 2 by proving (3.3) for $K = 3$ by an involution, and then restricting the part sizes to be at most $3n - 1$.

Bressoud [4] has conjectured the positivity an appropriate fractional version of (3.2). It implies Conjecture 3. Some of the fractional cases for $K = 2$ have been verified by Greene [9].

These conjectures involving alternating sums of q-binomial coefficients are closely related to representations of the Virasoro algebra [12] and statistical mechanics [5].

Andrews found a beautiful generalization of Conjecture 2.

Conjecture 4. *(Andrews) If $f_{m,n}(z) = (q, q^2; q^3)_n (zq, zq^2; q^3)_m$, then for any t the coefficient of z^t is a polynomial in q whose sign behavior is $+--$ modulo 3.*

P. Borwein [2] had two other conjectures.

Conjecture 5. *(P. Borwein) The sign behavior of $(q, q^2; q^3)_n^2$ is $+--$ modulo 3.*

Conjecture 6. *(P. Borwein) The sign behavior of $(q, q^2, q^3, q^4; q^5)_n$ is $+----$ modulo 5.*

4. Unimodality conjectures. A polynomial $p(q)$ of degree N with real coefficients

$$p(q) = \sum_{m=0}^{N} a_m q^m$$

is called *unimodal* if there is some k such that

$$a_0 \le a_1 \le a_2 \le \cdots \le a_k \ge a_{k+1} \ge \cdots \ge a_N.$$

It is called *symmetric* if $a_m = a_{N-m}$ for all m. We could restate unimodality as a positivity condition by considering $(1-q)p(q)$.

The q-binomial coefficient is known [1, p. 48] to be a symmetric, unimodal polynomial in q. A general method [18] for proving that polynomials have this property is to realize them as formal characters of sl_2 representations. This method works for the q-binomial coefficient, and was generalized to certain differences of q-binomial coefficients in [15]. For example

$$q^{-k}\left(\begin{bmatrix} n \\ k \end{bmatrix}_q - \begin{bmatrix} n \\ k-1 \end{bmatrix}_q\right) = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q - q^{n-2k+1}\begin{bmatrix} n-1 \\ k-2 \end{bmatrix}_q \tag{4.1}$$

is symmetric and unimodal if n is odd and $2k+1 \le n$. A more general theorem involves differences of Schur functions. Motivated by rigged configurations, we made the following conjecture which is (4.1) if $r = 1$.

Conjecture 7. *If n is odd, and r and k are non-negative integers with $n \geq 2rk - 4r + 3$, then*

$$\begin{bmatrix} n-1 \\ k \end{bmatrix}_q - q^{n-2rk+4r-3} \begin{bmatrix} n-1+4(r-1) \\ k-2 \end{bmatrix}_q$$

is a symmetric, unimodal polynomial in q with non-negative coefficients.

It even takes some effort to verify that for $q = 1$, this difference is a non-negative integer.

Another unimodality conjecture concerns the interpretation of $\begin{bmatrix} n \\ k \end{bmatrix}_q$ as the generating function of all partitions which lie inside a $k \times (n-k)$ rectangle. For a shape λ, let $Y_\lambda(q)$ be the generating function for all partitions whose Ferrers diagram fit inside λ. $Y_\lambda(q)$ is not always unimodal [19], but special choices besides rectangles should be. If $\lambda = (n, n-1, \cdots, 1)$, the staircase shape, it is well-known that $Y_\lambda(q)$ is a q-analog of a Catalan number related to the Rogers-Ramanujan identities.

Conjecture 8. *If $\lambda = (n, n-1, \cdots, 1)$, then $Y_\lambda(q)$ is a unimodal polynomial in q.*

Perhaps more speculatively, I have made the following conjecture [19].

Conjecture 9. *If λ is self-conjugate, then $Y_\lambda(q)$ is a unimodal polynomial in q.*

Conjecture 9 has been verified for all partitions λ of all integers less than 125.

5. t-cores. A partition λ is called a *t-core* if none of the hook numbers of λ are multiples of t. For example, the only partitions which are 2-cores are the staircase partitions. We let $a_t(n)$ denote the number of partitions of n which are t-cores. A celebrated result of Granville and Ono [8] is that t-cores exist for $t \geq 4$, that is, $a_t(n) > 0$ if $t \geq 4$.

It is possible to give explicit formulas for $a_3(n)$, $a_5(n)$, and $a_7(n)$ [7], from the generating function

$$\sum_{n=0}^{\infty} a_t(n) q^n = \frac{(q^t; q^t)_\infty^t}{(q; q)_\infty}.$$

However, numerical evidence indicates that the following (possibly rash) conjecture may hold.

Conjecture 10. *If $t \geq 4$, then*

$$a_t(n) \leq a_{t+1}(n), \quad \textit{for } n \geq t+1.$$

6. Gaussian posets. A finite partially ordered set P is called *Gaussian* if the generating function for all order reversing maps $\sigma : P \to \{0, 1, \cdots, m\}$ is

$$\sum_\sigma q^{|\sigma|} = \prod_{i=1}^{|P|} \frac{1 - q^{m+h_i}}{1 - q^{h_i}}, \tag{6.1}$$

for some set of positive integers h_i. If P is just a chain with k elements, then an order reversing map σ can be considered as an integer partition with at most k parts, largest part $\leq m$. So the generating function (6.1) is the q-binomial coefficient, or Gaussian polynomial, thus the term Gaussian.

It turns out that each connected Gaussian poset must be ranked, with level numbers $a_1, a_2, \cdots, a_k$, and that the generating function above simplifies to

$$\sum_{\sigma} q^{|\sigma|} = \prod_{i=1}^{k} \frac{(1-q^{m+i})^{a_i}}{(1-q^i)^{a_i}} = F(\vec{a}, m, q). \tag{6.2}$$

If each $a_i = 1$, (6.2) is a q-binomial coefficient.

It is an open problem [17, p. 271] to classify all connected Gaussian posets. All of the known examples [14] are related to miniscule representation of simple Lie algebras. Nonetheless, MacMahon [13, Chap. V] considered products of the form (6.2), and classified, for a given sequence, $a_1, a_2, \cdots, a_k$, when the product is a polynomial in q. For connected Gaussian posets we also have the symmetry relation $a_i = a_{k+1-i}$ for all i. Let us call a sequence of positive integers $\vec{a} = (a_1, \cdots, a_k)$ a *fake Gaussian* sequence if it is symmetric and the rational function (6.2) is a polynomial for all non-negative integers m.

The conjectures in this sequence are speculative, but supported by numerical evidence.

Conjecture 11. *If $(a_1, \cdots, a_k)$ is a fake Gaussian sequence, then the coefficients of the polynomial $F(\vec{a}, m, q)$ in q are non-negative for all non-negative integers m.*

Using the geometry computer package `Porta`, Conjecture 11 was verified [20] for all fake Gaussian sequences with at most 10 parts. A stronger conjecture was verified, that the coefficients in the polynomial $W_s(\vec{a}, q)$ in the expansion

$$F(\vec{a}, m, q) = \sum_{s=0}^{a_1+\cdots+a_k} \begin{bmatrix} m + a_1 + \cdots + a_k - s \\ a_1 + \cdots + a_k \end{bmatrix}_q W_s(\vec{a}, q)$$

are non-negative if $\vec{a}$ is a symmetric fake Gaussian sequence.

Haglund [10] has interpreted Conjecture 11 in terms of rook theory. He has a more general conjecture concerning q-hit numbers of Ferrers boards.

A more speculative positivity conjecture in [20] involves a possible generalization of the q-multinomial coefficient.

Conjecture 12. *Suppose that $(a_1, a_2, \cdots, a_k)$ is a sequence of positive integers. If*

$$g(\vec{a}, q) = \frac{\prod_{i=1}^{a_1+a_2+\cdots+a_n} (1-q^i)}{\prod_{i=1}^{n} (1-q^i)^{a_i}},$$

is a polynomial in q, then the coefficients of $g(\vec{a}, q)$ are non-negative.

If $\vec{a}$ is a decreasing sequence, then $g(\vec{a}, q)$ is a q-multinomial coefficient, and Conjecture 12 is verified. It has also been verified for all sequences $\vec{a}$ whose sum is at most 18, and all $\vec{a}$ with at most 10 parts satisfying the hypotheses of Conjecture 11.

Conjecture 12 would indicate a way to generalize multiset permutations, presumably to a set with an appropriate statistic, which generalizes *inv* or *maj*. However, I do not have a combinatorial interpretation of Conjecture 12 for $q = 1$.

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