

MARKING AND SHIFTING A PART IN PARTITION THEOREMS

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ABSTRACT. Refined versions, analytic and combinatorial, are given for classical integer partition theorems. The examples include the Rogers-Ramanujan identities, the Göllnitz-Gordon identities, Euler's odd=distinct theorem, and the Andrews-Gordon identities. Generalizations of each of these theorems are given where a single part is "marked" or weighted. This allows a single part to be replaced by a new larger part, "shifting" a part, and analogous combinatorial results are given in each case. Versions are also given for marking a sum of parts.

1. INTRODUCTION

Many integer partition theorems can be restated as an analytic identity, as a sum equal to a product. One such example is the first Rogers-Ramanujan identity

$$(1) \quad \frac{1}{\prod_{k=0}^{\infty} (1 - q^{5k+1})(1 - q^{5k+4})} = 1 + \sum_{k=1}^{\infty} q^{k^2} \frac{1}{(1 - q)(1 - q^2) \cdots (1 - q^k)}.$$

MacMahon's combinatorial version of (1) uses integer partitions. The left side is the generating function for all partitions whose parts are congruent to 1 or 4 mod 5. The factor $1/(1 - q^9)$ on the left side allows an arbitrary number of 9's in an integer partition. If we "mark" or weight the 9 by an x , the factor $1/(1 - q^9)$ is replaced by

$$\frac{1}{1 - xq^9}.$$

One may ask how the right side is modified upon marking a part, and whether a refined combinatorial interpretation exists.

The result is known [8, (2.2)], and there is a refined combinatorial version. The key to the combinatorial result is that the terms in the sum side are positive as power series in q and x .

Theorem 1.1. *Let $M \geq 1$ be any integer congruent to 1 or 4 mod 5. Then*

$$\frac{1 - q^M}{1 - xq^M} \frac{1}{\prod_{k=0}^{\infty} (1 - q^{5k+1})(1 - q^{5k+4})} = 1 + q \frac{1 + q + \cdots + q^{M-2} + xq^{M-1}}{1 - xq^M} \\ + \sum_{k=2}^{\infty} q^{k^2} \frac{1 + q + \cdots + q^{M-1}}{1 - xq^M} \frac{1}{(1 - q^2)(1 - q^3) \cdots (1 - q^k)}.$$

Here is a combinatorial version of Theorem 1.1.

Theorem 1.2. *Let M be positive integer which is congruent to 1 or 4 mod 5. Then the number of partitions of n into parts congruent to 1 or 4 mod 5 with exactly k M 's is equal to the number of partitions λ of n with difference at least 2 and*

- (1) *if λ has one part, then $\lfloor n/M \rfloor = k$,*
- (2) *if λ has at least two parts, then $\lfloor (\lambda_1 - \lambda_2 - 2)/M \rfloor = k$.*

The purpose of this paper is to give the analogous results for several other classical partition theorems: the Göllnitz-Gordon identities, Euler's odd=distinct theorem, and the Andrews-Gordon identities. The main engine, Proposition 3.1, may be applied to many other single sum identities. The results obtained here by marking a part are refinements of the corresponding classical results.

We shall also consider "shifting" a part, for example replacing all 9's by 22's in (1). This is replacing the factor

$$\frac{1}{1 - q^9} \quad \text{by} \quad \frac{1}{1 - q^{22}}.$$

We shall see that the set of partitions enumerated by the sum side is an explicit subset of the partitions in the original identity.

Finally in section 6 we consider marking a sum of parts. We can extend Theorem 1.2 to allow other values of M , for example $M = 7$, by marking the partition $6 + 1$. See Corollary 6.8.

We use the standard notation,

$$(A; q)_k = \prod_{j=0}^{k-1} (1 - Aq^j), \quad [M]_q = \frac{1 - q^M}{1 - q}.$$

If the base q is understood we may write $(A; q)_k$ as $(A)_k$.

2. THE ROGERS-RAMANUJAN IDENTITIES

In this section we give prototypical examples for the Rogers-Ramanujan identities.

First we state a marked version of the second Rogers-Ramanujan identity, which follows from Proposition 3.1.

Theorem 2.1. *Let $M \geq 2$ be any integer congruent to 2 or 3 mod 5. Then*

$$\frac{1 - q^M}{1 - xq^M} \frac{1}{(q^2; q^5)_\infty (q^3; q^5)_\infty} = 1 + q^2 \frac{[M-2]_q + xq^{M-2} + q^{M-1}}{1 - xq^M} + \sum_{k=2}^{\infty} q^{k^2+k} \frac{[M]_q}{1 - xq^M} \frac{1}{(q^2; q)_{k-1}}.$$

Here is a combinatorial version of Theorem 2.1.

Theorem 2.2. *Let M be positive integer which is congruent to 2 or 3 mod 5. Then the number of partitions of n into parts congruent to 2 or 3 mod 5 with exactly k M 's is equal to the number of partitions λ of n with difference at least 2, no 1's, and*

- (1) *if λ has one part, then $n = Mk + j$, $2 \leq j \leq M - 1$, or $j = 0$ or $j = M + 1$,*
- (2) *if λ has at least two parts, then $\lfloor (\lambda_1 - \lambda_2 - 2)/M \rfloor = k$.*

Proof. We simultaneously prove Theorems 1.2 and 2.2. We need to understand the combinatorics of the replacement in the k^{th} term on the sum side

$$(2) \quad \frac{1}{1 - q} \rightarrow \frac{[M]_q}{1 - xq^M} = \sum_{p=0}^{\infty} q^p x^{\lfloor p/M \rfloor}.$$

In the classical Rogers-Ramanujan identities, the factor $1/(1 - q)$ represents the difference in the first two parts after the double staircase has been removed. This is the second case of each theorem. \square

Example 2.3. *Let $k = 2$, $M = 7$, and $n = 22$. The equinumerous sets of partitions for Theorem 2.2 are*

$$\{(8, 7, 7), (7, 7, 3, 3, 2), (7, 7, 2, 2, 2, 2)\} \leftrightarrow \{(22), (20, 2), (19, 3)\}.$$

Equivalent combinatorial versions of Theorems 1.2 and 2.2 may be given (see [8, Theorem 2, Theorem 3]). This time the terms $k \geq M$ of the sum side are considered, and the replacement considered is

$$\frac{1}{1 - q^M} \rightarrow \frac{1}{1 - xq^M},$$

namely the part M is marked on the sum side. We need notation for when a double staircase is removed from a partition with difference at least two.

Definition 2.4. *For any partition λ with k parts whose difference of parts is at least 2, let λ^* denote the partition obtained upon removing the double staircase $(2k - 1, 2k - 3, \dots, 1)$ from λ , and reading the result by columns.*

*For any partition λ with k parts and no 1's whose difference of parts is at least 2, let λ^{**} denote the partition obtained upon removing the double staircase $(2k, 2k - 2, \dots, 2)$ from λ , and reading the result by columns.*

Theorem 2.5. *Let M be positive integer which is congruent to 1 or 4 mod 5. Then the number of partitions of n into parts congruent to 1 or 4 mod 5 with exactly k M 's is equal to the number of partitions λ of n with difference at least 2 and*

- (1) *if λ has one part, then $\lfloor n/M \rfloor = k$,*
- (2) *if λ has between two and $M - 1$ parts, then $\lfloor (\lambda_1 - \lambda_2 - 2)/M \rfloor = k$,*
- (3) *if λ has at least M parts, then λ^* has exactly k M 's.*

Example 2.6. *Let $k = 2$, $M = 4$, and $n = 24$. The equinumerous sets of partitions for Theorem 2.5 are*

$$\begin{aligned} & \{(16, 4^2), (14, 4^2, 1^2), (11, 4^2, 1^5), (9, 6, 4^2, 1), (9, 4^2, 1^7), (6, 6, 4^2, 1^4), \\ & (6, 4^2, 1^{10}), (4^2, 1^{16})\} \leftrightarrow \\ & \{(9, 7, 5, 3), (18, 5, 1), (17, 6, 1), (17, 5, 2), (16, 6, 2), (16, 5, 3), (17, 7), (18, 6)\}. \end{aligned}$$

Theorem 2.7. *Let M be positive integer which is congruent to 2 or 3 mod 5. Then the number of partitions of n into parts congruent to 2 or 3 mod 5 with exactly k M 's is equal to the number of partitions λ of n with difference at least 2, no 1's and*

- (1) *if λ has one part, then $n = Mk + j$, $2 \leq j \leq M - 1$, or $j = 0$ or $j = M + 1$,*
- (2) *if λ has between two and $M - 1$ parts, then $\lfloor (\lambda_1 - \lambda_2 - 2)/M \rfloor = k$,*
- (3) *if λ has at least M parts, then λ^{**} has exactly k M 's.*

3. A GENERAL EXPANSION

In this section we give a general expansion, Proposition 3.1, for marking a single part.

Many partition identities have a sum side of the form

$$\sum_{j=0}^{\infty} \frac{\alpha_j}{(q; q)_j},$$

where α_j has non-negative coefficients as a power series in q .

These include

- (1) the Rogers-Ramanujan identities, $\alpha_j = q^{j^2}$ or q^{j^2+j} ,
- (2) Euler's odd=distinct theorem, $\alpha_j = q^{\binom{j+1}{2}}$,
- (3) the Göllnitz-Gordon identities, q replaced by q^2 , $\alpha_j = q^{j^2}(-q; q^2)_j$,
- (4) all partitions by largest part, $\alpha_j = q^j$,
- (5) all partitions by Durfee square, $\alpha_j = q^{j^2}/(q; q)_j$.

A part of size M may be marked in general using the next proposition.

Proposition 3.1. *For any positive integer M , if $\alpha_0 = 1$,*

$$\frac{1 - q^M}{1 - wq^M} \sum_{j=0}^{\infty} \frac{\alpha_j}{(q; q)_j} = 1 + \frac{\alpha_1[M]_q - q^M + wq^M}{1 - wq^M} + \sum_{j=2}^{\infty} \frac{[M]_q}{1 - wq^M} \frac{\alpha_j}{(q^2; q)_{j-1}}.$$

As long as α_1 has the property that

$$\alpha_1[M]_q - q^M$$

is a positive power series in q , the right side has a combinatorial interpretation.

There are two possible elementary combinatorial interpretations. For any $j \geq 2$, the factor

$$\frac{[M]_q}{1 - wq^M} = \sum_{p=0}^{\infty} q^p w^{[p/M]}$$

replaces $1/(1 - q)$, which accounts for parts of size 1 in a partition. This is a weighted form of the number of 1's.

The second interpretation holds for terms with $j \geq M$. Here

$$\frac{[M]_q}{1 - wq^M} \frac{1}{(q^2; q)_{j-1}} = \frac{1}{(1 - q) \cdots (1 - q^{M-1})(1 - wq^M)(1 - q^{M+1}) \cdots (1 - q^j)}.$$

In this case the part of size M is marked by w .

For a particular combinatorial application of Proposition 3.1 one must realize what the denominator factors $(1 - q)$ and $(1 - q^M)$ represent on the sum side. For example, in the Rogers-Ramanujan identities these factors account for 1's and M 's in λ^* . Since

$$(\#1's \text{ in } \lambda^*) = \lambda_1 - \lambda_2 - 2,$$

the two interpretations are Theorem 1.2 and Theorem 2.5.

3.1. Distinct parts. Choosing $\alpha_j = q^{\binom{j+1}{2}}$ in Proposition 3.1 gives distinct partitions, which by Euler's theorem are equinumerous with partitions into odd parts. Here is the marked version.

Corollary 3.2. *For any odd positive integer M ,*

$$\begin{aligned} & \frac{1}{(1 - q)(1 - q^3) \cdots (1 - q^{M-2})(1 - wq^M)(1 - q^{M+2}) \cdots} \\ &= 1 + \frac{q + q^2 + \cdots + q^{M-1} + wq^M}{1 - wq^M} + \sum_{j=2}^{\infty} \frac{q^{\binom{j+1}{2}}}{(q^2; q)_{j-1}} \frac{[M]_q}{1 - wq^M}. \end{aligned}$$

Definition 3.3. *For any partition λ with j distinct parts let λ^{St} be the partition obtained upon removing a staircase $(j, j - 1, \dots, 1)$ from λ , and reading the result by columns.*

Example 3.4. *If $\lambda = (8, 7, 3, 1)$, then $\lambda^{St} = (3, 2, 2, 2)$.*

Here is the combinatorial version of Corollary 3.2, generalizing Euler's theorem.

Theorem 3.5. *For any odd positive integer M , the number of partitions of n into odd parts with exactly k parts of size M , is equal to the number of partitions λ of n into distinct parts such that*

- (1) if λ has one part, then $\lfloor n/M \rfloor = k$,
- (2) if λ has at least two parts, then $\lfloor (\lambda_1 - \lambda_2 - 1)/M \rfloor = k$.

Example 3.6. Let $k = 2$, $M = 5$, and $n = 18$. The equinumerous sets of partitions for Theorem 3.5 are

$$\begin{aligned} & \{(7, 5, 5, 1), (5, 5, 3, 3, 1, 1), (5, 5, 3, 1^5), (5, 5, 1^8)\} \\ & \leftrightarrow \{(16, 2), (15, 3), (15, 2, 1), (14, 3, 1)\}. \end{aligned}$$

Proposition 3.7. There is an M -version of the Sylvester “fishhook” bijection which proves Theorem 3.5.

Proof. Let FH be the fishhook bijection from partitions with distinct parts to partitions with odd parts. If $FH(\lambda) = \mu$, it is known that the number of 1's in μ is $\lambda_1 - \lambda_2 - 1$, except for $FH(n) = 1^n$. This proves Theorem 3.5 if $M = 1$, and FH is the bijection for $M = 1$.

For the M -version, $M > 1$, let λ have distinct parts. For $\lambda = n$ a single part, Define the M -version by $FH^M(n) = (M^k, 1^{n-kM})$ which has k parts of size M . Otherwise λ has at least two parts, and

$$kM \leq \lambda_1 - \lambda_2 - 1 \leq (k+1)M - 1.$$

Let θ be the partition with distinct parts where λ_1 has been reduced by kM ,

$$0 \leq \theta_1 - \theta_2 - 1 \leq M - 1.$$

Finally put $\gamma = FH(\theta)$, and note that γ has at most $M - 1$ 1's.

There are 2 cases. If γ has no parts of size M , define $FH^M(\lambda) = \gamma \cup M^k$, so that $FH^M(\lambda)$ is a partition with odd parts, exactly k parts of size M , and at most $M - 1$ 1's.

If γ has $r \geq 1$ parts of size M , change all of them to rM 1's to obtain γ' with at least M 1's. Then put $FH^M(\lambda) = \gamma' \cup M^k$, so that $FH^M(\lambda)$ is a partition with odd parts, exactly k parts of size M , and at least M 1's. \square

Theorem 3.8. For any odd positive integer M , the number of partitions of n into odd parts with exactly k parts of size M , is equal to the number of partitions λ of n into distinct parts such that

- (1) if λ has one part, then $\lfloor n/M \rfloor = k$,
- (2) if λ has between two and $M - 1$ parts, then $\lfloor (\lambda_1 - \lambda_2 - 1)/M \rfloor = k$,
- (3) if λ has at least M parts, then λ^{St} has exactly k M 's.

Example 3.9. Let $k = 2$, $M = 3$, and $n = 18$. The equinumerous sets of partitions for Theorem 3.8 are

$$\begin{aligned} & \{(11, 3, 3, 1), (9, 3, 3, 1^3), (7, 5, 3, 3), (7, 3, 3, 1^5), (5, 5, 3, 3, 1, 1), (5, 3, 3, 1^7), (3, 3, 1^{12})\} \\ & \leftrightarrow \{(7, 6, 4, 1), (8, 5, 4, 1), (11, 4, 3), (10, 5, 3), (9, 6, 3), (8, 7, 3), (13, 5)\}. \end{aligned}$$

3.2. Göllnitz-Gordon identities. The Göllnitz-Gordon identities are (see [1], [4], [5])

$$(3) \quad \sum_{n=0}^{\infty} q^{n^2} \frac{(-q; q^2)_n}{(q^2; q^2)_n} = \frac{1}{(q; q^8)_\infty (q^4; q^8)_\infty (q^7; q^8)_\infty},$$

$$(4) \quad \sum_{n=0}^{\infty} q^{n^2+2n} \frac{(-q; q^2)_n}{(q^2; q^2)_n} = \frac{1}{(q^3; q^8)_\infty (q^4; q^8)_\infty (q^5; q^8)_\infty}$$

We apply Proposition 3.1 with q replaced by q^2 , M replaced by $M/2$, and $\alpha_j = q^{j^2}(-q; q^2)_j$ to obtain the next result.

Corollary 3.10. *Let M be a positive integer. Then*

$$\begin{aligned} & \frac{1 - q^M}{1 - wq^M} \frac{1}{(q; q^8)_\infty (q^4; q^8)_\infty (q^7; q^8)_\infty} \\ &= 1 + \frac{q[M-1]_q + wq^M}{1 - wq^M} + \sum_{j=2}^{\infty} q^{j^2} \frac{[M]_q}{1 - wq^M} \frac{(-q^3; q^2)_{j-1}}{(q^4; q^2)_{j-1}}. \end{aligned}$$

We used

$$\frac{q(1+q)[M/2]_{q^2} - q^M + wq^M}{1 - wq^M} = \frac{q[M-1]_q + wq^M}{1 - wq^M}$$

to simplify the second term in the sum in Corollary 3.10. Note that the numerator has positive coefficients, and thus a simple combinatorial interpretation.

Here is the combinatorial restatement [5, Theorem 2] of the first Göllnitz-Gordon identity.

Theorem 3.11. *The number of partitions of n into parts congruent to 1, 4, or 7 mod 8 is equal to the number of partitions of n into parts whose difference is at least 2, and greater than 2 for consecutive even parts.*

For the combinatorial version of Corollary 3.10, we need to recall why the sum side of (3) is the generating function for the restricted partitions with difference at least 2. In particular we must identify what the denominator factor $1 - q$ represents in the sum side.

Suppose λ is such a partition with j parts. This is equivalent to showing that the generating function for λ^* is

$$(5) \quad \frac{(-q; q^2)_j}{(q^2; q^2)_j} = \frac{1 + q}{1 - q^2} \frac{(-q^3; q^2)_{j-1}}{(q^4; q^2)_{j-1}}.$$

The partition $\mu = \lambda - (2j - 1, 2j - 3, \dots, 1)$ has at most j parts, and the odd parts of μ are distinct. The column read version $\lambda^* = \mu^t$ can be built in the following way. Take arbitrary parts from sizes $j, j - 1, \dots, 1$ with even multiplicity, whose generating function is $1/(q^2; q^2)_j$. The rows now have even length. Then choose a subset of the odd integers $1+0, 2+1, \dots, j+(j-1)$.

For each such odd part $k + (k - 1)$ add columns of length k and $k - 1$. This keeps all rows even, except the k^{th} row which is odd and distinct.

We see that the factor $(1 + q)/(1 - q^2) = 1/(1 - q)$ in (5) accounts for 1's in λ^* . In Corollary 3.10 this quotient is replaced by

$$\frac{1 + q}{1 - q^2} \rightarrow \frac{[M]_q}{1 - wq^M} = \sum_{p=0}^{\infty} q^p w^{\lfloor p/M \rfloor}.$$

There is one final opportunity for a 1 to appear in λ^* : when $3 = 2 + 1$ is chosen as an odd part. This occurs only when the second part of λ is even.

Theorem 3.12. *Let M be a positive integer which is congruent to 1, 4 or 7 mod 8. The number of partitions of $n \geq 1$ into parts congruent to 1, 4 or 7 mod 8 with exactly k M 's, is equal to the number of partitions λ of n into parts whose difference is at least 2, and greater than 2 for consecutive even parts such that*

- (1) if λ has a single part, then $\lfloor n/M \rfloor = k$,
- (2) if λ has at least two parts and the second part of λ is even,

$$\lfloor (\lambda_1 - \lambda_2 - 3)/M \rfloor = k,$$

- (3) if λ has at least two parts and the second part of λ is odd,

$$\lfloor (\lambda_1 - \lambda_2 - 2)/M \rfloor = k.$$

Example 3.13. *Let $k = 3$, $M = 7$, and $n = 31$. The equinumerous sets of partitions for Theorem 3.12 are*

$$\{(9, 7, 7, 7, 1), (7, 7, 7, 4, 4, 1, 1), (7, 7, 7, 4, 1^6), (7, 7, 7, 1^{10})\} \leftrightarrow \{(30, 1), (29, 2), (28, 3), (27, 3, 1)\}.$$

Note that $\lambda = (27, 4)$ is not allowed because the second part of λ is even.

For the second Göllnitz-Gordon identity, the version of Corollary 3.10 is

$$(6) \quad \frac{1 - q^M}{1 - wq^M} \frac{1}{(q^3; q^8)_{\infty} (q^4; q^8)_{\infty} (q^5; q^8)_{\infty}} \\ = 1 + \frac{q^3 + \dots + q^{M-1} + wq^M + q^{M+1} + q^{M+2}}{1 - wq^M} \\ + \sum_{j=2}^{\infty} q^{j^2+2j} \frac{[M]_q}{1 - wq^M} \frac{(-q^3; q^2)_{j-1}}{(q^4; q^2)_{j-1}}.$$

Here is the combinatorial refinement of [5, Theorem 3].

Theorem 3.14. *Let M be a positive integer which is congruent to 3, 4 or 5 mod 8. The number of partitions of $n \geq 1$ into parts congruent to 3, 4 or 5 mod 8 with exactly k M 's, is equal to the number of partitions λ of n into parts whose difference is at least 2, greater than 2 for consecutive even parts, smallest part at least 3, such that*

(1) if λ has a single part, then $n = Mk$, or $n = Mk + j$, $3 \leq j \leq M + 2$,
 $j \neq M$,

(2) if λ has at least two parts and the second part of λ is even,

$$\lfloor (\lambda_1 - \lambda_2 - 3)/M \rfloor = k,$$

(3) if λ has at least two parts and the second part of λ is odd,

$$\lfloor (\lambda_1 - \lambda_2 - 2)/M \rfloor = k.$$

4. AN ANDREWS-GORDON VERSION

The Andrews-Gordon identities are

Theorem 4.1. *If $0 \leq a \leq k$, then*

$$\frac{(q^{k+1-a}, q^{k+2+a}, q^{2k+3}; q)_\infty}{(q; q)_\infty} = \sum_{n_1 \geq n_2 \geq \dots \geq n_k \geq 0} \frac{q^{n_1^2 + n_2^2 + \dots + n_k^2 + n_{k+1-a} + \dots + n_k}}{(q)_{n_1 - n_2} \cdots (q)_{n_{k-1} - n_k} (q)_{n_k}}.$$

The Rogers-Ramanujan identities are the cases $k = 1$, $a = 0, 1$.

Because Theorem 4.1 has a multisum instead of a single sum, we cannot apply Proposition 3.1. Nonetheless the same idea can be applied to obtain a marked version of Theorem 4.1.

Let F_k^a denote the right side multisum of Theorem 4.1 for $0 \leq a \leq k$, and let $F_k^a = F_k^0$ for $a < 0$. So we have

$$F_k^a = F_{k-1}^{a-1} + \sum_{n_1 \geq n_2 \geq \dots \geq n_k \geq 1} \frac{q^{n_1^2 + n_2^2 + \dots + n_k^2 + n_{k+1-a} + \dots + n_k}}{(q)_{n_1 - n_2} \cdots (q)_{n_{k-1} - n_k} (q)_{n_k}}.$$

Multiplying by $\frac{1-q^M}{1-xq^M}$ yields

$$\begin{aligned} \frac{1-q^M}{1-xq^M} F_k^a &= \frac{1-q^M}{1-xq^M} F_{k-1}^{a-1} \\ &+ \sum_{n_1 \geq n_2 \geq \dots \geq n_k \geq 1} \frac{q^{n_1^2 + n_2^2 + \dots + n_k^2 + n_{k+1-a} + \dots + n_k}}{(q)_{n_1 - n_2} \cdots (q)_{n_{k-1} - n_k} (q^2; q)_{n_k - 1}} \frac{[M]_q}{1-xq^M}, \end{aligned}$$

which, upon iterating, is the following weighted version of the Andrews-Gordon identities.

Theorem 4.2. *For $0 \leq a \leq k$, let M be any positive integer not congruent to 0, $\pm(k+1-a)$ modulo $2k+3$. Then*

$$\begin{aligned} \frac{1-q^M}{1-xq^M} \frac{(q^{k+1-a}, q^{k+2+a}, q^{2k+3}; q)_\infty}{(q; q)_\infty} &= 1 + A + \sum_{n_1=2}^{\infty} \frac{q^{n_1^2+B}}{(q^2; q)_{n_1-1}} \frac{[M]_q}{1-xq^M} \\ &+ \sum_{r=2}^k \sum_{n_1 \geq n_2 \geq \dots \geq n_r \geq 1} \frac{q^{n_1^2 + n_2^2 + \dots + n_r^2 + n_{k+1-a} + \dots + n_r}}{(q)_{n_1 - n_2} \cdots (q)_{n_{r-1} - n_r} (q^2; q)_{n_r - 1}} \frac{[M]_q}{1-xq^M} \end{aligned}$$

where

$$(1) \text{ for } 0 \leq a < k, \quad B = 0, \quad A = q([M-1]_q + xq^{M-1})/(1-xq^M)$$

$$(2) \text{ for } a = k, \quad B = n_1, \quad A = q^2([M-2]_q + xq^{M-2} + q^{M-1})/(1-xq^M).$$

For a combinatorial version of Theorem 4.2 we use Andrews' Durfee dissections, and $(k+1, k+1-a)$ -admissible partitions, see [2].

Definition 4.3. *Let k be a positive integer and $0 \leq a \leq k$. A partition λ is called $(k+1, k+1-a)$ -admissible if λ may be dissected by $r \leq k$ successive Durfee rectangles, moving down, of sizes*

$$n_1 \times n_1, \dots, n_{k-a} \times n_{k-a}, (n_{k-a+1} + 1) \times n_{k-a}, \dots, (n_r + 1) \times n_r.$$

such that the $(n_1 + n_2 + \dots + n_{k-a+i} + i)^{\text{th}}$ part of λ is n_{k-a+i} , for $1 \leq i \leq r - (k - a)$.

Note that $r \leq k - a$ is allowed, in which case all of the Durfee rectangles are squares. Also, the parts of λ to the right of the Durfee rectangles are not constrained, except at the last row of the non-square Durfee rectangle, where it is empty.

Example 4.4. *Suppose $k = 3$ and $a = 2$. Then $\lambda = 91$ is not $(4, 2)$ -admissible: the Durfee square has size $n_1 = 1$, but the next Durfee rectangle of size 2×1 does not exist, so the second part cannot be covered if $r \geq 2$.*

Theorem 2 in [2] interprets Theorem 4.1.

Proposition 4.5. *The generating function for all partitions which are $(k+1, k+1-a)$ -admissible is given by the sum in Theorem 4.1.*

We need to understand the replacement

$$\frac{1}{(q)_{n_r}} = \frac{1}{(1-q)(q^2; q)_{n_r-1}} \rightarrow \frac{1}{(q^2; q)_{n_r-1}} \frac{[M]_q}{1-xq^M}$$

in the factor $(q)_{n_r}$ to give a combinatorial version of Theorem 4.2.

First we recall [2] that if the sizes of the Durfee rectangles are fixed by n_1, n_2, \dots, n_r , then the generating function for the partitions which have this Durfee dissection is

$$\frac{1}{(q)_{n_1}} \prod_{j=1}^{r-1} \begin{bmatrix} n_j \\ n_{j+1} \end{bmatrix}_q = \frac{1}{(q)_{n_r}} \prod_{j=1}^{r-1} \frac{1}{(q)_{n_j - n_{j+1}}}.$$

(A simple bijection for this fact is given in [6].) Upon multiplying by

$$(1 - q^M)/(1 - xq^M)$$

we have

$$\frac{[M]_q}{1-xq^M} \frac{1}{(q^2; q)_{n_1-1}} \prod_{j=1}^{r-1} \begin{bmatrix} n_j \\ n_{j+1} \end{bmatrix}_q = \frac{[M]_q}{1-xq^M} \frac{1}{(q^2; q)_{n_r-1}} \prod_{j=1}^{r-1} \frac{1}{(q)_{n_j - n_{j+1}}}.$$

Consider the factor $1/(q)_{n_1}$, which accounts for the portion of the partition to the right of the first Durfee rectangle of λ . In this factor we are replacing

$$\frac{1}{1-q} \rightarrow \frac{[M]_q}{1-xq^M}.$$

As before, the M 1's in the columns to the right of the first Durfee rectangle are weighted by x . These 1's are again a difference in the first two parts of λ .

Putting these pieces together, the following result is a combinatorial restatement of Theorem 4.2.

Theorem 4.6. *Fix integers a, k, M satisfying $0 \leq a \leq k$ and $M \not\equiv 0, \pm(k+1-a) \pmod{2k+3}$. The number of partitions of n into parts not congruent to $0, \pm(k+1-a) \pmod{2k+3}$ with exactly j M 's, is equal to the number of partitions λ of n which are $(k+1, k+1-a)$ -admissible with $r \leq k$ Durfee rectangles of sizes*

$$n_1 \times n_1, \dots, n_{k-a} \times n_{k-a}, (n_{k-a+1} + 1) \times n_{k-a}, \dots, (n_r + 1) \times n_r$$

of the following form:

- (1) if $r = n_1 = 1$, and $0 \leq a < k$, λ is a single part of size Mj , $Mj+1, \dots, Mj+(M-1)$, or
- (2) if $r = n_1 = 1$, and $a = k$, $\lambda = (\lambda_1, 1)$ has size $Mj, Mj+2, \dots, Mj+(M-1)$, or $Mj+(M+1)$.
- (3) if $n_1 = 1$ and $r \geq 2$, then $\lfloor (\lambda_1 - n_1)/M \rfloor = j$,
- (4) if $n_1 \geq 2$, then $\lfloor (\lambda_1 - \lambda_2)/M \rfloor = j$.

5. SHIFTING A PART

The weighted versions allow one to shift a part. For example in first Rogers-Ramanujan identity, what happens if parts of size 11 are replaced by parts of size 28? All we need to do is to choose $M = 11$ and $x = q^{17}$ in Theorem 1.1.

Corollary 5.1. *Let M be a positive integer which is congruent to 1 or 4 modulo 5. Let $N > M$ be an integer not congruent to 1 or 4 modulo 5. The number of partitions of n into parts congruent to 1 or 4 modulo 5, except M , or parts of size N , is equal to the number of partitions λ of n with difference at least 2, such that*

- (1) λ has a single part, which is congruent to $0, 1, \dots$, or $M-1 \pmod{N}$,
- (2) λ has at least two parts, and $\lambda_1 - \lambda_2 - 2$ is congruent to $0, 1, \dots$, or $M-1 \pmod{N}$.

Example 5.2. *Let $N = 8$, $M = 4$, and $n = 9$. The equinumerous sets of partitions for Corollary 5.1 are*

$$\{(9), (6, 1, 1, 1), (8, 1), (1^9)\} \leftrightarrow \{(9), (6, 3), (7, 2), (5, 3, 1)\}.$$

TABLE 1. Theorem 4.6 when $a = 2, k = 3, M = 3$

partition of 10 without 2,7,9	# of 3's	(4,2)-admissible partition of 10	value of j
10	0	10	3
811	0	61111	1
64	0	421111	0
631	1	322111	0
61111	0	331111	0
55	0	811	2
541	0	6211	1
5311	1	5311	0
511111	0	5221	1
4411	0	22222	0
433	2	4411	0
43111	1	4321	0
4111111	0	82	2
3331	3	73	1
331111	2	64	0
31111111	1	55	0
1111111111	0	433	0

An related example occurs when two parts are shifted: 1 and 4 are replaced by 2 and 3. The appropriate identity is

$$(7) \quad \frac{1}{(1-q^2)(1-q^3)(q^6; q)_\infty (q^9; q)_\infty} \\ = 1 + \frac{q^2(1+q)}{1-q^3} + \sum_{k=2}^{\infty} \frac{q^{k^2}}{(q^2; q)_{k-1}} \frac{1+q^2}{1-q^3}.$$

Theorem 5.3. *The number of partitions of n into parts from*

$$\{2, 3, 5k+1, 5k+4 : k \geq 1\}$$

is equal to the number of partitions λ of n with difference at least 2 and

- (1) *if λ has a single part, then $n \not\equiv 1 \pmod{3}$,*
- (2) *if λ has at least two parts, then $(\lambda_1 - \lambda_2 - 2) \not\equiv 1 \pmod{3}$.*

Example 5.4. *Let $n = 13$. The two equinumerous sets of partitions in Theorem 5.3 are*

$$\{(11, 2), (9, 2, 2), (6, 3, 2, 2), (3, 2, 2, 2, 2, 2), (3, 3, 3, 2, 2)\} \\ \leftrightarrow \{(12, 1), (10, 3), (9, 4), (8, 4, 1), (7, 5, 1)\}.$$

The possible partitions with difference at least 2

$$\{(13), (11, 2), (8, 5), (9, 3, 1), (7, 4, 2)\}$$

are disallowed.

Corollary 5.5. *Let M be an odd positive integer. Let $N > M$ be an even integer. The number of partitions of n into odd parts except M , or parts of size N , is equal to the number of partitions λ of n into distinct parts, such that*

- (1) λ has a single part, which is congruent to $0, 1, \dots, \text{ or } M-1 \pmod{N}$,
- (2) λ has at least two parts, and $\lambda_1 - \lambda_2 - 1$ is congruent to $0, 1, \dots, \text{ or } M-1 \pmod{N}$.

Example 5.6. *If $N = 8$, $M = 3$, and $n = 9$ the equinumerous sets in Corollary 5.5 are*

$$\{(9), (8, 1), (7, 1, 1), (5, 1^4), (1^9)\} \leftrightarrow \{(9), (5, 4), (6, 3), (5, 3, 1), (4, 3, 2)\}.$$

6. MARKING A SUM OF PARTS

One may ask if Theorems 1.1 and 2.1 have combinatorial interpretations without the modular conditions on M . The sum sides retain the interpretations given by Theorems 1.2 and 2.2 and are positive as a power series in q and w . It remains to understand what the product side represents as a generating function of partitions. We give in Proposition 6.4 a general positive combinatorial expansion for the product side. We call this “marking a sum of parts”.

As an example suppose that $M = A + B$, is a sum of two parts, where A and B are distinct integers congruent to 1 or $4 \pmod{5}$. The quotient in the product side of Theorem 1.1

$$\frac{1 - q^{A+B}}{1 - wq^{A+B}} \frac{1}{(1 - q^A)(1 - q^B)} = \frac{1}{(1 - q^B)(1 - wq^{A+B})} + \frac{q^A}{(1 - q^A)(1 - wq^{A+B})}$$

is a generating function for partitions with parts A or B . The first term allows the number of B 's to be at least as many as the number of A 's. The second term allows the number of A 's to be greater than the number of B 's. The exponent of w is the number of times a pair AB appears in a partition. For example, if $A = 6$, $B = 4$, the partition $(6, 6, 4, 4, 4, 4)$ contains 64 twice, along with two 4 's. We have found a prototypical result.

Proposition 6.1. *Let $M = A + B$ for some $A, B \equiv 1, 4 \pmod{5}$, $A \neq B$. Then*

$$\frac{1 - q^M}{1 - wq^M} \frac{1}{(q; q^5)_\infty (q^A; q^5)_\infty}$$

is the generating function for all partitions μ with parts $\equiv 1, 4 \pmod{5}$ by the number of occurrences of the pair AB

A more general statement holds for partitions other than $M = A + B$. To state this result, we need to define an analogue of the multiplicity of a single part to a multiplicity of a partition. We again use the multiplicity notation for a partition, for example $(7^3, 4^1, 2^3)$ denotes the partition $(7, 7, 7, 4, 2, 2, 2)$.

Definition 6.2. Let $\lambda = (A_1^{m_1}, \dots, A_k^{m_k})$ be a partition. We say λ is inside μ k times, $k = E_\lambda(\mu)$, if

$$k = \max\{j : j \geq 0, \mu \text{ contains at least } jm_s \text{ parts of size } A_s \text{ for all } s\}.$$

Example 6.3. Let $\lambda = (6^1, 4^2, 1^1)$, $\mu = (9^1, 6^7, 4^5, 1^8)$. Then $E_\lambda(\mu) = 2$ but not 3 because μ contains only five 4's.

With this definition, Proposition 6.1 holds for any partition.

Proposition 6.4. Let $\lambda \vdash M$ be a fixed partition into parts congruent to 1 or 4 mod 5. Then

$$\frac{1 - q^M}{1 - wq^M} \frac{1}{(q; q^5)_\infty (q^4; q^5)_\infty}$$

is the generating function for all partitions μ into parts congruent to 1 or 4 mod 5,

$$\sum_{\mu} q^{|\mu|} w^{E_\lambda(\mu)},$$

where $E_\lambda(\mu)$ is the number of times λ appears in μ .

The modular condition on the parts in Proposition 6.4 is irrelevant.

Proposition 6.5. Let $\mathbb{A} = \{A_1, A_2, \dots\}$ be any set of positive integers. Suppose that $\lambda = (B_1^{m_1}, \dots, B_k^{m_k})$ is a partition whose parts come from \mathbb{A} and $M = \sum_{i=1}^k m_i B_i$. Then

$$\frac{1 - q^M}{1 - wq^M} \prod_{i=1}^{\infty} (1 - q^{A_i})^{-1}$$

is the generating function for all partitions μ with parts from \mathbb{A}

$$\sum_{\mu} q^{|\mu|} w^{E_\lambda(\mu)}.$$

Proof. We start with the telescoping sum

$$1 - q^M = 1 - q^{m_1 B_1} + q^{m_1 B_1} (1 - q^{m_2 B_2}) + \dots + q^{\sum_{i=1}^{k-1} m_i B_i} (1 - q^{m_k B_k}).$$

which implies

$$\begin{aligned} & (1 - q^M) \prod_{i=1}^k (1 - q^{B_i})^{-1} \\ (8) \quad & = \sum_{i=1}^k q^{m_1 B_1 + \dots + m_{i-1} B_{i-1}} \prod_{j=1}^{i-1} (1 - q^{B_j})^{-1} \frac{1 - q^{m_i B_i}}{1 - q^{B_i}} \prod_{j=i+1}^k (1 - q^{B_j})^{-1}. \end{aligned}$$

We see that (8) is the generating function for partitions μ with parts from $\{B_1, B_2, \dots, B_k\}$ such that $E_\lambda(\mu) = 0$. The i^{th} term of the sum represents partitions $\mu = (B_1^{n_1}, B_2^{n_2}, \dots, B_k^{n_k})$

$$n_1 \geq m_1, n_2 \geq m_2, \dots, n_{i-1} \geq m_{i-1}, n_i < m_i.$$

These disjoint sets cover all μ with $E_\lambda(\mu) = 0$.

Adding back the multiples of λ by multiplying by $(1 - wq^M)^{-1}$, and also the unused parts from \mathbb{A} , gives the result. \square

Definition 6.6. Let \mathbb{A} be a set of parts. If λ has parts from \mathbb{A} , let $E_\lambda^\mathbb{A}(n, k)$ be the number of partitions μ of n with parts from \mathbb{A} such that $E_\lambda(\mu) = k$.

Corollary 6.7. For any set of part sizes \mathbb{A} , let λ_1 and λ_2 be two partitions of M into parts from \mathbb{A} . Then for all $n, k \geq 0$

$$E_{\lambda_1}^\mathbb{A}(n, k) = E_{\lambda_2}^\mathbb{A}(n, k).$$

Here are the promised versions of Theorem 1.2 and Theorem 2.2 when M does not satisfy the mod 5 condition.

Corollary 6.8. Suppose that λ is a partition of M into parts congruent to 1 or 4 mod 5. Then Theorem 1.2 holds if the number of partitions having M of multiplicity k is replaced by $E_\lambda^\mathbb{A}(n, k)$, $\mathbb{A} = \{1, 4, 6, 9, \dots\}$. Also, if λ is a partition of M into parts congruent to 2 or 3 mod 5, then Theorem 2.2 holds if the number of partitions having M of multiplicity k is replaced by $E_\lambda^\mathbb{B}(n, k)$, $\mathbb{B} = \{2, 3, 7, 8, \dots\}$.

Example 6.9. Let $\lambda = (6, 1)$, $M = 7$, and $n = 17$. The equinumerous sets of partitions for Corollary 6.8 are

$$\begin{aligned} & \{(9, 6, 1, 1), (6, 6, 4, 1), (6, 4, 4, 1, 1, 1), (6, 4, 1^7), (6, 1^{11})\} \leftrightarrow \\ & \{(16, 1), (15, 2), (14, 3), (13, 4), (13, 3, 1)\}. \end{aligned}$$

One corollary of the Rogers-Ramanujan identities is that there are more partitions of n into parts congruent to 1 or 4 mod 5 than into parts congruent to 2 or 3 mod 5. Kadell [7] gave an injection which proves this, and Berkovich-Garvan [3, Theorem 5.1] gave a stronger injection for modulo 8. We can use Corollary 6.7, Theorem 1.1, and Theorem 2.1 to generalize this fact.

Theorem 6.10. Let

$$\mathbb{A} = \{5k + 1, 5k + 4 : k \geq 0\}, \quad \mathbb{B} = \{5k + 2, 5k + 3 : k \geq 0\}.$$

Fix partitions $\lambda \vdash M$ and $\theta \vdash M$, $M \geq 3$, with parts from \mathbb{A} and \mathbb{B} respectively. Then for all $n, k \geq 0$

$$E_\theta^\mathbb{B}(n, k) \leq E_\lambda^\mathbb{A}(n, k).$$

Proof. By Corollary 6.7, Theorem 1.1, and Theorem 2.1 we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} q^n w^k (E_\lambda^\mathbb{A}(n, k) - E_\theta^\mathbb{B}(n, k)) \\ &= \frac{q - q^{M+1}}{1 - wq^M} + \sum_{k=2}^{\infty} q^{k^2} \frac{[M]_q}{1 - wq^M} \frac{1}{(q^2; q)_{k-2}}. \end{aligned}$$

All terms are positive except for the first term. If we add the $k = 2$ term to the first term we have

$$\frac{q - q^{M+1} + q^4[M]_q}{1 - wq^M}$$

whose numerator is positive for $M \geq 3$. □

7. REMARKS

In [8] marked versions of the 2nd Rogers-Ramanujan identity are given for

- (1) a single part $\{M\}$,
- (2) two parts $\{2, M\}$,
- (3) four parts $\{2, 3, 7, 8\}$.

We do not have a general version of Proposition 3.1 which gives the last marked version.

A q -analogue of Euler's odd=distinct theorem [9, Theorem 1] is the following. Let q be a positive integer. The number of partitions of N into q -odd parts $[2k+1]_q$ is equal to the the number of partitions of N into parts $[m]_q$ whose multiplicity is $\leq q^m$. A generating function identity equivalent to this result is

$$\prod_{n=0}^{\infty} \frac{1}{1 - t^{[2n+1]_q}} = 1 + \sum_{m=1}^{\infty} t^{[m]_q} \frac{1 - t^{q^m[m]_q}}{1 - t^{[m]_q}} \prod_{k=1}^{m-1} \frac{1 - t^{(q^k+1)[k]_q}}{1 - t^{[k]_q}}.$$

We do not know how to perturb this identity to mark a part.

Given λ and μ , $E_\lambda(\mu)$ is an integer which counts the number of λ 's in μ . One could imagine defining instead a rational value for this "multiplicity".

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