Ramanujan Continued Fractions Via Orthogonal Polynomials *

Mourad E.H. Ismail
Department of Mathematics
University of Central Florida
Orlando, FL 32816

Dennis Stanton
School of Mathematics
University of Minnesota
Minneapolis, MN 55455

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Abstract

Some Ramanujan continued fractions are evaluated using asymptotics of polynomials orthogonal with respect to measures with absolutely continuous components.

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1 Introduction

A sequence of orthogonal polynomials satisfies a three-term recurrence relation

$$y_{n+1}(x) = (x - b_n)y_n(x) - \lambda_n y_{n-1}(x), \quad n \geq 0, \tag{1.1}$$

with $\lambda_{n+1} > 0$ and $b_n \in \mathbb{R}$ for $n = 0, 1, \ldots$. There are two special independent solutions to (1.1), denoted p_n and p_n^*, with the initial conditions $p_0 = 1$, $p_{-1} = 0$, $p_0^* = 0$, $p_1^* = 1$. The finite continued fraction associated with (1.1) is

$$C_n(x) = \cfrac{1}{x - b_0 - \cfrac{\lambda_1}{x - b_1 - \cfrac{\lambda_2}{x - b_2 - \cfrac{\lambda_3}{\ddots - \cfrac{\lambda_{n-1}}{x - b_{n-1}}}}}} = \frac{p_n^*(x)}{p_n(x)}.$$

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The above $C_n(x)$ is the nth convergent of the continued fraction

$$\cfrac{1}{x-b_0-\cfrac{\lambda_1}{x-b_1-\cfrac{\lambda_2}{\ddots}}},$$

$n=1,2,\ldots$. Continued fractions of the above type are called J-fractions [12] and are associated with orthogonal polynomials.

It is clear that knowing the large n asymptotics of $p_n(x)$ and $p_n^*(x)$ enables us to evaluate the infinite continued fraction $C_\infty(x)$ via

$$C_\infty(x)=\lim_{n\to\infty}C_n(x).$$

This has been done for a generalization of the Rogers-Ramanujan continued fraction in [1].

Andrews et al. [4] (see our Theorem 4.7) have recently evaluated another Ramanujan continued fraction $C_n(1/2)$, whose limit depends upon the congruence class of n modulo 3. In this paper we reprove the evaluation via asymptotics of orthogonal polynomials, and explain that the modulo 3 behavior is typical for continued J-fractions evaluated at $x=\cos(\pi/3)$ when the corresponding orthogonal polynomials are orthogonal with respect to a weight function supported on $[-1,1]$. As a byproduct we evaluate several new continued fractions, see Theorems 4.1-4.6. The modulo 3 behavior can be generalized to any modulus k, by choosing a suitable value for x which depends upon k. For the Ramanujan continued fraction, this result is given in Theorem 4.8. Moreover the modulo k behavior holds for a wide class of orthogonal polynomials which have the appropriate asymptotics. We discuss this in §5, and give two such general theorems for modulo k limits, Theorems 5.2 and 5.6. In §6 we demonstrate these ideas using the Al-Salam-Chihara and q-ultraspherical polynomials. After completing this manuscript the work of Bowman and McLaughlin [7] was brought to our attention. In §7 we will compare their results with our own.

2 A family of q-Chebyshev polynomials

In this section we collect the basic facts about a system of orthogonal polynomials introduced by Ismail and Mulla in [11]. We state the defining recurrence relation, give generating functions and asymptotics. The polynomial system considered is a q extension of the Chebyshev polynomials. These polynomials will be used to explain and extend the results by Andrews et al.

Consider the three-term recurrence relation

$$y_{n+1}(x)=(2x+aq^n)y_n(x)-y_{n-1}(x). \tag{2.1}$$

We define the polynomial system $\{p_n(x;a,q)\}$ to be the solution of (2.1) with the initial conditions

$$p_0(x;a,q)=1, \quad p_1(x;a,q)=2x+a.$$

The polynomials of the second kind $p_n^*(x;a,q)$ are defined as the solutions to (2.1) with the initial conditions

$$p_0^*(x;a,q)=0, \quad p_1^*(x;a,q)=2.$$

It readily follows that

$$p_n^*(x;a,q)=2p_{n-1}(x;aq,q).$$

The generating function of $\{p_n(x;a,q)\}$ is easily found from (2.1) to be

$$P(x,t)=\sum_{n=0}^{\infty} p_n(x;a,q)t^n=\sum_{k=0}^{\infty}\frac{a^k t^k q^{\binom{k}{2}}}{(te^{i\theta},te^{-i\theta};q)_{k+1}}, \quad x=\cos\theta, \tag{2.2}$$

see [11].

One can apply Darboux's method to the generating function (2.2) and find the leading term in the large n asymptotics of $p_n(x;a,q)$. Let

$$F(z,a,q):=\sum_{n=0}^{\infty}\frac{a^n z^{-n} q^{\binom{n}{2}}}{(q,q/z^2;q)_n}. \tag{2.3}$$

We parameterize x by $x=\cos\theta$, so that $e^{\pm i\theta}=x\pm\sqrt{x^2-1}$. The branch of the square root is chosen such that $\sqrt{x^2-1}/x\to 1$ as $x\to\infty$. We shall use the notation

$$\rho(x), 1/\rho(x)=x\pm\sqrt{x^2-1}, \ |\rho(x)|>1 \text{ for } x\notin[-1,1]. \tag{2.4}$$

Proposition 2.1. *If $x\notin[-1,1]$, then the asymptotic relation*

$$p_n(x;a,q)=\frac{[\rho(x)]^{n+1}}{\rho(x)-1/\rho(x)}F(z,a,q)(1+o(1)), \quad \text{as } n\to\infty, \tag{A}$$

holds. On the other hand if $x\in[-1,1]$, and $z=e^{i\theta}$ then

$$p_n(x;a,q)=\frac{z^n}{1-z^{-2}}F(z,a,q)+\frac{z^{-n}}{1-z^2}F(1/z,a,q)+o(1), \tag{B}$$

as $n\to\infty$.

Proposition 2.2. *The asymptotic function $F(z,a,q)$ has the alternative representation*

$$F(z,a,q)=(-a/z;q)_\infty\sum_{k=0}^{\infty}\frac{(aqz^{-3})^k q^{3\binom{k}{2}}}{(q,q/z^2,-a/z;q)_k}.$$

Proof. Apply the ${}_2\phi_1$ to ${}_2\phi_2$ transformation

$${}_2\phi_1(a,b;c;c;q,z)=\frac{(az;q)_\infty}{(z;q)_\infty}{}_2\phi_2(a,c/b;c,az;q,bz), \tag{2.5}$$

[9, (III.4)], with $a=q^{-n}$, $b=q^n$, $c=qz^{-2}$, $z=-aq^n/z$, and let $n\to\infty$. ☐

We choose $b_n=-aq^n$, $\lambda_n=1$, and replace x by $2x$ in $C_n(x)$. In view of Proposition 2.1, to evaluate $C_\infty(x)$, one must choose a and z so that $F(a,z,q)$ is evaluable. We collect these choices in the next section.

3 Evaluations of the asymptotic function

In this section we evaluate the asymptotic function $F(z,a,q)$ for various choices of z and a.

Proposition 3.1. *If $\omega = e^{2\pi i/3}$, we have*

$$\text{(A)} \qquad F(e^{i\pi/3}, q, q) = (\omega q; q)_\infty (q^2; q^3)_\infty,$$

$$\text{(B)} \qquad F(e^{i\pi/3}, 1, q) = (\omega q; q)_\infty \left[(q; q^3)_\infty - \omega (q^2; q^3)_\infty\right],$$

$$\text{(C)} \qquad F(e^{i\pi/3}, 1/q, q) = \frac{\omega}{q} (\omega q; q)_\infty \left[(1+q)(q; q^3)_\infty + \omega (q^2; q^3)_\infty\right]$$

$$\text{(D)} \qquad F(e^{i\pi/3}, q^2, q) = (\omega q; q)_\infty \left[(1+\omega^2)(q; q^3)_\infty - \omega^2 (q^2; q^3)_\infty\right].$$

Proof. For the first evaluation, put $z = e^{i\pi/3}$ in Proposition 2.2 and use the q^3-binomial theorem. For (B), note that by Proposition 2.2, if $\omega = e^{2\pi i/3}$

$$\begin{aligned} F(e^{i\pi/3}, 1, q) &= \frac{(\omega; q)_\infty}{1-\omega} \sum_{k=0}^{\infty} \frac{q^{k+3\binom{k}{2}} (-1)^k (1-\omega q^k)}{(q, q\omega^2, q\omega; q)_k} \\ &= (\omega q; q)_\infty \sum_{k=0}^{\infty} \frac{q^{k+3\binom{k}{2}} (-1)^k (1-\omega q^k)}{(q^3; q^3)_k}. \end{aligned}$$

Then use the q^3-binomial theorem twice to obtain the stated result. The proofs of (C) and (D) are analogous to that of (B) and will be omitted. ☐

The evaluation in (A) was first proved by Andrews in [2].

Proposition 3.2. *The following Rogers-Ramanujan identities*

$$\begin{aligned} F(p^{-1/2}, p^{3/2}, p^2) &= (1-p) \sum_{n=0}^{\infty} \frac{p^{n^2+n}}{(p;p)_{2n+1}} \\ &= (p^3, p^7, p^{10}; p^{10})_\infty (p^4, p^{16}; p^{20})_\infty / (p^2; p)_\infty, \end{aligned} \tag{3.1}$$

$$\begin{aligned} F(p^{-1/2}, p^{5/2}, p^2) &= (1-p) \sum_{n=0}^{\infty} \frac{p^{n^2+2n}}{(p;p)_{2n+1}} \\ &= (p^4, p^6, p^{10}; p^{10})_\infty (p^2, p^{18}; p^{20})_\infty / (p^2; p)_\infty, \end{aligned} \tag{3.2}$$

hold for $0 < p < 1$.

Proof. These are equations (94) and (96) on Slater's list of Rogers-Ramanujan type identities [16]. ☐

The results in Proposition 3.2 are valid for complex p with $|p| < 1$. The square roots present no problem because Proposition 2.2 indicates that the left-hand sides of (3.1) and (3.2) are analytic functions of p for $|p| < 1$. We shall need the evaluations that correspond to shifting a by q in Proposition 2.4.

Proposition 3.3. *The following evaluations of the function F hold*

$$
\begin{aligned}
F(p^{-1/2}, p^{7/2}, p^2) &= (1-p)\textstyle\sum_{n=0}^{\infty} \frac{p^{n^2+3n}}{(p;p)_{2n+1}} \\
&= \tfrac{1}{p(p^2;p)_\infty}\left[-(p^1, p^9, p^{10}; p^{10})_\infty (p^8, p^{12}; p^{20})_\infty \right. \\
&\quad \left. +(p^3, p^7, p^{10}; p^{10})_\infty (p^4, p^{16}; p^{20})_\infty\right],
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
F(p^{-1/2}, p^{9/2}, p^2) &= (1-p)\textstyle\sum_{n=0}^{\infty} \frac{p^{n^2+4n}}{(p;p)_{2n+1}} \\
&= \tfrac{1}{p(p^2;p)_\infty}\left[(p^2, p^8, p^{10}; p^{10})_\infty (p^6, p^{14}; p^{20})_\infty \right. \\
&\quad \left. +(p-1)(p^4, p^6, p^{10}; p^{10})_\infty (p^2, p^{18}; p^{20})_\infty\right].
\end{aligned} \tag{3.4}
$$

Proof. Observe that

$$
\begin{aligned}
F(p^{-1/2}, p^{5/2}a, p^2) &= (1-p)\sum_{k=0}^{\infty} \frac{p^{k^2+2k}a^k}{(p;p)_{2k+1}} \\
&= p^{-1}(1-p)\sum_{k=0}^{\infty} \frac{p^{k^2}a^k(p^{2k+1}-1+1)}{(p;p)_{2k+1}} \\
&= -p^{-1}(1-p)\sum_{k=0}^{\infty} \frac{p^{k^2}a^k}{(p;p)_{2k}} + p^{-1}(1-p)\sum_{k=0}^{\infty} \frac{p^{k^2}a^k}{(p;p)_{2k+1}}
\end{aligned}
$$

Thus if $a = p$, these two sums may be evaluated by Slater (99) and (94) respectively. This is the first result. For $a = p^2$, the second sum may be evaluated using formula (96) of [16], while the first sum is

$$
\sum_{k=0}^{\infty} \frac{p^{k^2+2k}}{(p;p)_{2k}} = \sum_{k=0}^{\infty} \frac{p^{k^2}(p^{2k}-1+1)}{(p;p)_{2k}} = \sum_{k=0}^{\infty} \frac{p^{(k+1)^2}}{(p;p)_{2k+1}} + \sum_{k=0}^{\infty} \frac{p^{k^2}}{(p;p)_{2k}},
$$

again evaluable from (98) and (96) of Slater [16]. This establishes the second part and the proof is complete. □

Analogous to Proposition 3.3, we state without proof three other evaluations which also follow from Slater, (94), (96), (98), and (99).

Proposition 3.4. *We have*

$$
\begin{aligned}
F(p^{-3/2}, p^{1/2}, p^2) &= (1-p)(1-p^3)\sum_{n=0}^{\infty} \frac{p^{n^2+n}}{(p;p)_{2n+1}(1-pq^{2n+3})} \\
&= \frac{(1-p^3)}{(p^2;p)_\infty}\left[(p^1, p^9, p^{10}; p^{10})_\infty (p^8, p^{12}; p^{20})_\infty \right. \\
&\quad \left. +\, p(p^3, p^7, p^{10}; p^{10})_\infty (p^4, p^{16}; p^{20})_\infty\right],
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
F(p^{-3/2},p^{5/2},p^2) &= (1-p)(1-p^3)\sum_{n=0}^{\infty}\frac{p^{n^2+3n}}{(p;p)_{2n+1}(1-p^{2n+3})} \\
&= \frac{(1-p^3)}{p^3(p^2;p)_\infty}\left[(p^1,p^9,p^{10};p^{10})_\infty(p^8,p^{12};p^{20})_\infty\right. \\
&\qquad \left.+(p-1)(p^3,p^7,p^{10};p^{10})_\infty(p^4,p^{16};p^{20})_\infty\right],
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
F(p^{-3/2},p^{9/2},p^2) &= (1-p)(1-p^3)\sum_{n=0}^{\infty}\frac{p^{n^2+5n}}{(p;p)_{2n+1}(1-p^{2n+3})} \\
&= \frac{(1-p^3)}{p^6(p^2;p)_\infty}\left[(p^2+1)(p^1,p^9,p^{10};p^{10})_\infty(p^8,p^{12};p^{20})_\infty\right. \\
&\qquad \left.+(-p^2+p-1))(p^3,p^7,p^{10};p^{10})_\infty(p^4,p^{16};p^{20})_\infty\right].
\end{aligned} \tag{3.7}
$$

Slater's (94), (96), (98) and (99) give the next set of evaluations.

Proposition 3.5. *We have*

$$
F(i,iq,q^2)+F(-i,iq,q^2) = 2\frac{(q^8,q^{32},q^{40};q^{40})_\infty(q^{24},q^{56};q^{80})_\infty}{(q^4;q^4)_\infty}, \tag{A}
$$

$$
F(i,iq^3,q^2)+F(-i,iq^3,q^2) = 2\frac{(q^4,q^{36},q^{40};q^{40})_\infty(q^{32},q^{48};q^{80})_\infty}{(q^4;q^4)_\infty}, \tag{B}
$$

$$
F(i,iq,q^2)-F(-i,iq,q^2) = 2q\frac{(q^{12},q^{28},q^{40};q^{40})_\infty(q^{16},q^{64};q^{80})_\infty}{(q^4;q^4)_\infty}, \tag{C}
$$

$$
F(i,iq^3,q^2)-F(-i,iq^3,q^2) = 2q^3\frac{(q^{16},q^{24},q^{40};q^{40})_\infty(q^8,q^{72};q^{80})_\infty}{(q^4;q^4)_\infty}. \tag{D}
$$

Propositions 3.1–3.5 give examples of evaluations of F at special points of $F(z,a,q)$ and $F(z,aq,q)$. We can also find recurrences in a, which show that if we can evaluate $F(z,a,q)$ and $F(z,aq,q)$, then we can find $F(z,aq^n,q)$ for any integer n.

Proposition 3.6. *Let $G_n = F(z,aq^n,q)/z^n$. Then G_n has the property*

$$
G_{n+1} = -p_{n-1}(x;aq,q)G_0 + p_n(x;a,q)G_1,
$$

or equivalently,

$$
\begin{aligned}
F(z,aq^{n+1},q) &= -p_{n-1}(x;aq,q)z^{n+1}F(z,a,q) \\
&\qquad +p_n(x;a,q)z^nF(z,aq,q).
\end{aligned} \tag{3.8}
$$

Proof. A routine calculation shows that

$$
F(z,aq^{n+1},q) = (1+z^2+aq^{n-1}z)F(z,aq^n,q) - z^2F(z,aq^{n-1},q) \tag{3.9}
$$

which implies that $\{G_n\}$ satisfies the recurrence

$$
y_{n+1} = (2x+aq^{n-1})y_n - y_{n-1}.
$$

Since $\{p_n(x;a,q), p_{n-1}(x;aq,q)\}$ is a basis of solutions of the above recurrence, G_n must be a linear combination of $p_n(x;a,q)$ and $p_{n-1}(x;aq,q)$ and the proposition follows. □

Note that the way $\{p_n\}$ appears in (3.9) is similar to the way Lommel polynomials arise from Bessel functions, see [19].

4 Continued fractions

Let

$$\begin{aligned} C_n(x;a,q) =& \frac{p_n^*(x;a,q)}{2p_n(x;a,q)} \\ =& \cfrac{1}{2x+a-\cfrac{1}{2x+aq-\cfrac{1}{2x+aq^2-\cfrac{1}{\ddots-\cfrac{1}{2x+aq^{n-1}}}}}}. \end{aligned}$$

In this section we apply the asymptotic evaluations of §3 to evaluate

$$C_\infty(x;a,q) = \lim_{n\to\infty} C_n(x;a;q).$$

When there is only one term in the asymptotic expansion of Proposition 2.1, the n dependence is simple, and $C_\infty(x;a,q)$ exists. If x is chosen so that two terms exist in Proposition 2.1, $C_\infty(x;a,q)$ may not exist. However, restricting n to a congruence class, when x is suitably chosen, will yield a limit.

First we consider the single term case. As before we set $x = \cos\theta$, $z = e^{i\theta}$, and $|e^{-i\theta}| < |e^{i\theta}|$, and it follows from Proposition 2.1 that

$$C_\infty(x;a,q) = \lim_{n\to\infty} \frac{p_n^*(x;a,q)}{2p_n(x;a,q)} = \frac{F(z,aq,q)}{z\,F(z,a,q)}. \tag{4.1}$$

Moreover

$$C_\infty(x;aq^m,q) = \lim_{n\to\infty} \frac{p_n^*(x;aq^m,q)}{2p_n(x;aq^m,q)} = \frac{F(z,aq^{m+1},q)}{z\,F(z,aq^m,q)},$$

hence

$$\begin{aligned} &C_\infty(x;aq^m,q) = \\ &\frac{-p_{m-1}(x;aq,q)z^{m+1}F(z,a,q)+p_m(x;a,q)z^mF(z,aq,q)}{-p_{m-2}(x;a,q)z^{m+1}F(z,a,q)+p_{m-1}(x;a,q)z^mF(z,aq,q)}, \end{aligned} \tag{4.2}$$

holds for $x \notin [-1,1]$.

For our first theorem we put $q = p^2$, $2x = p^{1/2}+p^{-1/2}$, $a = p^{3/2}$, and use Propositions 3.2 and 3.3 to evaluate $F(p^{-1/2}, p^{3/2}, p^2)$ and $F(p^{-1/2}, p^{7/2}, p^2)$.

Theorem 4.1. *If $0 < p < 1$, then*

$$\cfrac{1}{1+p+p^2-\cfrac{p}{1+p+p^4-\cfrac{p}{1+p+p^6-\cfrac{p}{\ddots-\cfrac{p}{1+p+p^{2n}-\ddots}}}}}$$

$$=\frac{1}{p}-\frac{1}{p}\frac{(p,p^9;p^{10})_\infty(p^8,p^{12};p^{20})_\infty}{(p^3,p^7;p^{10})_\infty(p^4,p^{16};p^{20})_\infty}$$

Theorem 4.2 below follows from the parameter identifications $q = p^2$, $2x = p^{1/2} + p^{-1/2}$, $a = p^{5/2}$, and the use of Propositions 3.2 and 3.3 to evaluate $F(p^{-1/2}, p^{5/2}, p^2)$ and $F(p^{-1/2}, p^{9/2}, p^2)$.

Theorem 4.2. *If $0 < p < 1$, then*

$$\cfrac{1}{1+p+p^3-\cfrac{p}{1+p+p^5-\cfrac{p}{1+p+p^7-\cfrac{p}{\ddots-\cfrac{p}{1+p+p^{2n+1}-\cfrac{p}{\ddots}}}}}}$$

$$=1+\frac{1}{p}-\frac{1}{p}\frac{(p^2,p^8;p^{10})_\infty\,(p^6,p^{14};p^{20})_\infty}{(p^4,p^6;p^{10})_\infty\,(p^2,p^{18};p^{20})_\infty}$$

To prove Theorem 4.3 we set $q = p^2$, $2x = p^{3/2} + p^{-3/2}$, $a = p^{1/2}$, and apply Proposition 3.4 to evaluate $F(p^{-3/2}, p^{1/2}, p^2)$ and $F(p^{-3/2}, p^{5/2}, p^2)$.

Theorem 4.3. *If $0 < p < 1$, then*

$$\cfrac{1}{1+p^3+p^2-\cfrac{p^3}{1+p^3+p^4-\cfrac{p^3}{1+p^3+p^4-\cfrac{p^3}{\ddots-\cfrac{p^3}{1+p^3+p^{2n}-\cfrac{p^3}{\ddots}}}}}}$$

$$=\frac{1}{p^3}-\frac{(p^3,p^7;p^{10})_\infty(p^4,p^{16};p^{20})_\infty}{p^3\left((p,p^9;p^{10})_\infty(p^8,p^{12};p^{20})_\infty+p(p^3,p^7;p^{10})_\infty(p^4,p^{16};p^{20})_\infty\right)}$$

For Theorem 4.4 we put $q = p^2$, $2x = p^{3/2} + p^{-3/2}$, $a = p^{5/2}$, and use Proposition 3.4 to evaluate $F(p^{-3/2}, p^{5/2}, p^2)$ and $F(p^{-3/2}, p^{9/2}, p^2)$.

Theorem 4.4. *If* $0 < p < 1$, *then*

$$\cfrac{1}{1+p^3+p^4-\cfrac{p^3}{1+p^3+p^6-\cfrac{p^3}{1+p^3+p^8-\cfrac{p^3}{\ddots-\cfrac{p^3}{1+p^3+p^{2n+2}-\cfrac{p^3}{\ddots}}}}}}$$

$$= \frac{(p^2+1)(p,p^9;p^{10})_\infty(p^8,p^{12};p^{20})_\infty - (p^2-p+1)(p^3,p^7;p^{10})_\infty(p^4,p^{16};p^{20})_\infty}{p^3\left[(p,p^9;p^{10})_\infty(p^8,p^{12};p^{20})_\infty + (p-1)(p^3,p^7;p^{10})_\infty(p^4,p^{16};p^{20})_\infty\right]}$$

Next we turn to evaluations which correspond to two terms in the asymptotic formula of Proposition 2.1. The first example has $x = 0$, $z = i$, so that

$$2p_n(0;a,q^2) \approx i^n F(i,a,q^2) + i^{-n}F(-i,a,q^2).$$

There are two possible limits, for n even and n odd. Proposition 3.5 evaluates the sums and differences at $a = iq$ and $a = iq^3$, implying the next two theorems.

Theorem 4.5. *If* $0 < p < 1$, *then*

$$\lim_{n\to\infty} \cfrac{1}{p-\cfrac{1}{-p^3-\cfrac{1}{p^5-\cfrac{1}{\ddots-\cfrac{1}{-p^{4n+3}}}}}} = \frac{p^3(p^{16},p^{64};p^{80})_\infty}{(p^{32},p^{48};p^{80})_\infty}$$

Theorem 4.6. *If* $0 < p < 1$, *then*

$$\lim_{n\to\infty} \cfrac{1}{p-\cfrac{1}{-p^3-\cfrac{1}{p^5-\cfrac{1}{\ddots-\cfrac{1}{p^{4n+1}}}}}} = \frac{(p^4,p^{36};p^{40})_\infty(p^{32},p^{48};p^{80})_\infty}{p(p^{12},p^{28};p^{40})_\infty(p^{16},p^{64};p^{80})_\infty}$$

The evaluation of Andrews et al [4] chooses $z = e^{i\pi/3}$, $2x = 2\cos(\pi/3) = 1$, and $a = 1$ and uses Proposition 3.1. In this case Proposition 2.1 contains two terms for the asymptotics of $p_n(x;a,q)$, and one must restrict n to congruence classes modulo 3 to get a limiting result. For example, if $\epsilon = 0, 1$, or -1, and $z = e^{i\pi/3}$, then

$$\lim_{N\to\infty} C_{3N+\epsilon+1}(1/2;1,q) = \lim_{N\to\infty} \frac{p^*_{3N+\epsilon+1}(1/2;1,q)}{2p_{3N+\epsilon+1}(1/2;1,q)}.$$

Therefore

$$\begin{aligned}(4.3)\qquad &\lim_{N\to\infty} C_{3N+\epsilon+1}(1/2;1,q)\\ &= \frac{z^{1+\epsilon}F(e^{i\pi/3},q,q)-z^{-1-\epsilon}F(e^{-i\pi/3},q,q)}{z^{2+\epsilon}F(e^{i\pi/3},1,q)-z^{-2-\epsilon}F(e^{-i\pi/3},1,q)}.\end{aligned}$$

In view of Proposition 3.1, we have evaluated the continued fraction "$C_\infty(1/2;1,q)$" with three different limits. This is equivalent to Theorem 2.8 in [4], which is stated below.

Theorem 4.7. *Let* $\epsilon = 0, 1$ *or* -1.

$$\lim_{N\to\infty} \cfrac{1}{1-\cfrac{1}{1+q-\cfrac{1}{1+q^2-\cfrac{1}{\ddots-\cfrac{1}{1+q^{3N+\epsilon}}}}}} =$$

$$= -\omega^2\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty}\frac{(q\omega^2;q)_\infty-\omega^{\epsilon+1}(q\omega;q)_\infty}{(q\omega^2;q)_\infty-\omega^{\epsilon-1}(q\omega;q)_\infty}$$

In fact it is straightforward to prove the following generalization of (4.3), or Theorem 4.7, to the case of k different limits.

Theorem 4.8. *If* $0\le s\le k-1$, *and* $\omega := e^{2\pi i/k}$ *then*

$$\begin{aligned}(4.4)\qquad &\lim_{N\to\infty} C_{Nk+s}(\cos(j\pi/k);a,q)\\ &= e^{ij\pi/k}\frac{\omega^{js}F(e^{ij\pi/k},aq,q)-F(e^{-ij\pi/k},aq,q)}{\omega^{js+s}F(e^{ij\pi/k},a,q)-F(e^{-ij\pi/k},a,q)},\end{aligned}$$

holds for $1\le j<k$.

5 Generalizations

Assume that $\{p_n(x)\}$ is generated by

$$\begin{aligned}(5.1)\qquad &p_0(x):=1,\quad p_1(x)=(x-b_0)/a_1,\\ &xp_n(x)=a_{n+1}p_{n+1}(x)+b_np_n(x)+a_np_{n-1}(x),\quad n>0,\end{aligned}$$

where $b_n\in\mathbb{R}$ and $a_n>0$ for all n. The spectral theorem for orthogonal polynomials asserts that there is a probability measure μ such that $\{p_n(x)\}$ is orthonormal with respect to μ.

Theorem 5.1. ([14, Theorem 40, p. 143]) *Assume that*

$$(5.2)\qquad \sum_{n=1}^{\infty}\{\,|b_n|+|a_n-1/2|\,\}\text{ converges.}$$

Then the orthogonality measure μ has the decomposition $d\mu = \mu'(x)dx + d\nu$ where μ' is continuous and positive on $(-1, 1)$ and its support is $[-1, 1]$. Moreover the function ν is a jump function (a step function with possibly infinitely many jumps), which is constant on $(-1, 1)$. Furthermore

$$\lim_{n\to\infty} \left\{ \sin\theta\, p_n(\cos\theta) - \sqrt{\frac{2}{\pi}\frac{\sin\theta}{\mu'(\cos\theta)}}\, \sin[(n+1)\theta - \phi(\theta)] \right\} = 0, \tag{5.3}$$

uniformly on every compact subinterval of $(0, \pi)$. Here ϕ may depend on θ but not on n.

The angle $\phi = \phi(\theta)$ is called the phase shift. For more information on asymptotics of general orthogonal polynomials the interested reader may consult Nevai's survey article [15]. It is worth mentioning that the polynomials of §2 correspond to the case $a_n = 1/2$ and $b_n = aq^n/2$.

Assume that $\{p_n(x)\}$ satisfies (5.1) and the condition (5.2). Let $\{p_n^*(x)\}$ be a solution to the recursion in (5.1) with the initial conditions

$$p_0^*(x) := 0, \quad p_1^*(x) := 1/a_1.$$

The polynomials $\{p_{n+1}^*(x) : n = 0, \dots\}$ are orthogonal polynomials and $p_{n+1}^*(x)$ is of exact degree n. If the recursion coefficients in (5.1) satisfy the condition (5.2) then $\{a_1 p_{n+1}^*(x)\}$ is also a system of orthonormal polynomials and (5.3) holds for its measure of orthogonality.

Theorem 5.2. *Assume that the convergence condition (5.2) holds and that μ and μ^* are the probability measures with respect to which $\{p_n(x)\}$ and $\{p_{n+1}^*(x)\}$ are orthogonal. Let ϕ and ϕ^* be the phase shifts in (5.3) corresponding to $\{p_n(x)\}$ and $\{p_{n+1}^*(x)\}$. Denote the nth convergent of the infinite continued fraction by*

$$\cfrac{1}{x - b_0 - \cfrac{a_1^2}{x - b_1 - \cfrac{a_2^2}{x - b_2 - \cfrac{a_3^2}{\ddots}}}},$$

by $C_n(x), n = 1, 2, \dots$. Then for $s = 0, 1, \dots, k-1$ we have

$$\begin{aligned} &\lim_{N\to\infty} C_{Nk+s}(\cos(j\pi/k)) \\ &= \frac{1}{a_1}\sqrt{\frac{\mu'(\cos(j\pi/k))}{(\mu^*)'(\cos(j\pi/k))}}\, \frac{\sin(j\pi s/k - \phi^*(j\pi/k))}{\sin(j\pi(s+1)/k - \phi(j\pi/k))}, \end{aligned} \tag{5.4}$$

where $1 \le j < k$.

Proof. Apply (5.3) to p_n and $a_1 p_{n+1}^*$ then use $C_{Nk+s}(x) = p_{Nk+s}^*(x)/p_{Nk+s}(x)$, with $x = \cos(j\pi/k)$. □

Theorem 3.1 of Andrews et al. [4] proved a version of Theorem 5.2 when $a_n = 1, j = 1, k = 3$, but their b_n was allowed to be complex.

Theorem 5.3. ([14, Theorem 42, p. 145]) *Assume that*

$$\sum_{n=1}^{\infty} n\,\{\,|b_n| + |a_n - 1/2|\,\}\ \text{converges.} \tag{5.5}$$

With ρ as defined in (2.4)*, the following limiting relation*

$$\lim_{n\to\infty} \frac{p_n(x)}{\{\rho(x)\}^{n+1}} = \frac{2}{\sqrt{x^2-1}}\varphi(1/\rho(x)), \tag{5.6}$$

holds uniformly for $|\rho(x)| \ge R > 1$ where $\varphi(1/\rho(x))$ is a function analytic in the domain $|\rho(x)| > 1$.

The function φ is defined by

$$\varphi(z) = \lim_{n\to\infty} \varphi_{2n}(z), \tag{5.7}$$

and $\varphi_{2n}(z)$ is a polynomial in z $(= e^{i\theta})$ of exact degree $2n$ and is defined through

$$2i\sin\theta\, p_n(\cos\theta) = e^{i(n+1)\theta}\varphi_{2n}(e^{-i\theta}) - e^{-i(n+1)\theta}\varphi_{2n}(e^{i\theta}), \tag{5.8}$$

see [18]. The existence of φ is guaranteed under (5.2), which is weaker than (5.5).

Theorem 5.4. *Assume that* (5.5) *holds and let $\varphi^*_{2n}(z)$ be the polynomials in* (5.8) *corresponding to $a_1 p^*_{n+1}(\cos\theta)$ and let $\varphi^*(z) = \lim_{n\to\infty}\varphi^*_{2n}(z)$. Then*

$$\cfrac{1}{x-b_0-\cfrac{a_1^2}{x-b_1-\cfrac{a_2^2}{x-b_2-\cfrac{a_3^2}{\ddots}}}} = \frac{\rho(x)}{a_1}\,\frac{\varphi^*(1/\rho(x))}{\varphi(1/\rho(x))}, \tag{5.9}$$

for $x \notin [-1,1]$. Moreover the convergence is uniform for $|\rho(x)| \ge R > 1$.

Ramanujan also considered continued fractions where

$$C_N(x,c) = \cfrac{1}{x-b_0-\cfrac{a_1^2}{x-b_1-\cfrac{a_2^2}{x-b_2-\cfrac{a_3^2}{\ddots-\cfrac{a_{N-1}^2}{x-b_{N-1}+c}}}}} \tag{5.10}$$

Let $\{p_n(x)\}$ and $\{p_n^*(x)\}$ be the numerators and denominators of the continued fraction (5.5) which corresponds to $c = 0$. Ramanujan probably knew that

$$C_N(x,c) = \frac{p_N^*(x) + cp_{N-1}^*(x)}{p_N(x) + cp_{N-1}(x)}, \tag{5.11}$$

a fact crucial in Marcel Riesz's treatment of the moment problem, [17, p. x, p. 47]. The polynomials $\{p_N(x) + cp_{N-1}(x)\}$ are called quasi orthogonal polynomials, , [17]. Moreover $\{p_N^*(x) + cp_{N-1}^*(x)\}$ are their numerators. Shohat and Tamarkin us $-\tau$ instead of c ant τ has a geometric interpretation.

The next two theorems are the versions of Theorems 5.2 and 5.4 for these continued fractions.

Theorem 5.5. *If* (5.5) *holds, then*

$$\lim_{N\to\infty} C_N(x,c) = \frac{\rho(x)}{a_1} \frac{\varphi^*(1/\rho(x))}{\varphi(1/\rho(x))},$$

holds for $x \notin [-1,1]$.

Proof. Apply (5.6) to $p_N(x)$ and $p_N^*(x)$ and use (5.11). □

Theorem 5.6. *If* (5.2) *holds, then*

$$\begin{aligned}\lim_{N\to\infty} & C_{Nk+s}(\cos(j\pi/k), c) \\ &= \frac{1}{a_1}\sqrt{\frac{\mu'(\cos(j\pi/k))}{(\mu^*)'(\cos(j\pi/k))}} \\ &\times\frac{\sin(j\pi s/k - \phi^*(j\pi/k)) + c\,\sin(j\pi(s-1)/k - \phi^*(j\pi/k))}{\sin(j\pi(s+1)/k - \phi(j\pi/k)) + c\,\sin(j\pi s/k - \phi(j\pi/k))},\end{aligned}$$

where $1 \le j < k$.

The proof follows from Theorem 5.2 and (5.11).

Remark 5.7. *When* $\{a_n\}$ *and* $\{b_n\}$ *are bounded sequences then the measure of orthogonality is unique. On the other hand, for* $x \notin \mathbb{R}$, $C_N(x,c)$ *converges to the same value for all real* c *if and only if the measure of orthogonality is unique,* [17].

Recall the Rogers-Ramanujan continued fraction

$$\lim_{n\to\infty} \cfrac{1}{1+\cfrac{p}{1+\cfrac{p^2}{1+\cfrac{p^3}{\ddots\, 1+\cfrac{p^n}{1}}}}} = \frac{(p^1,p^4;p^5)_\infty}{(p^2,p^3;p^5)_\infty} \tag{5.12}$$

Denote the continued fraction on the left-hand side of (5.12) by $RR(p)$.

Theorem 5.8. *If* $0 < p < 1$, *then*

$$\lim_{n\to\infty} \cfrac{1}{p - \cfrac{1}{-p^3 - \cfrac{1}{p^5 - \cfrac{1}{\ddots - \cfrac{1}{-p^{4n+3} - 1}}}}} = \frac{(p^1,p^4;p^5)_\infty}{(p^2,p^3;p^5)_\infty} = RR(p)$$

Theorem 5.9. *If* $0 < p < 1$, *then*

$$\lim_{n\to\infty} \cfrac{1}{p - \cfrac{1}{-p^3 - \cfrac{1}{p^5 - \cfrac{1}{\ddots - \cfrac{1}{p^{4n+1} - 1}}}}} = -RR(-p).$$

Proof. We will only sketch the proofs of Theorems 5.8 and 5.9. As in the proofs of Theorems 4.5 and 4.6 we need to find

$$\frac{i^{N-2}(i-1)F(i,-q^3,-q^2) - i^{2-N}(1-1/i)F(-i,-q^3,-q^2)}{\frac{1}{2}i^{N-1}(i-1)F(i,q,-q^2) - \frac{1}{2}i^{1-N}(1-1/i)F(-i,q,-q^2)}$$
$$= \begin{cases} \frac{-1}{2}\,\frac{iF(i,-q^3,-q^2)+F(-i,-q^3,-q^2)}{F(i,q,-q^2)+iF(-i,q,-q^2)} & \text{if} N \text{ is odd,} \\ \frac{-1}{2}\,\frac{iF(i,-q^3,-q^2)-F(-i,-q^3,-q^2)}{F(i,q,-q^2)-iF(-i,q,-q^2)} & \text{if } N \text{is even.} \end{cases}$$

We next evaluate these numerators and denominators as infinite products. For the denominators let

$$H(q) = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q^4;q^4)_k} = \frac{(q^2;q^4)_\infty}{(q,q^4;q^5)_\infty},$$

By formula (20) on Slater's list, we conclude that

$$\begin{aligned} &F(i,q,-q^2) - iF(-i,q,-q^2) \\ &= H(q/i) - iH(qi) = (1-i)H(q), \end{aligned} \tag{5.13}$$

The last equality in (5.13) follows by checking the real and imaginary parts and the even and odd terms of the sum. Similarly

$$\begin{aligned} &F(i,q,-q^2) + iF(-i,q,-q^2) \\ &= H(q/i) + iH(qi) = (1+i)H(-q), \end{aligned} \tag{5.14}$$

which follows from (5.13) by replacing i by $-i$ and q by $-q$.

For the numerators apply (16) on Slater's list to get

$$H_2(q) = \sum_{k=0}^{\infty} \frac{q^{k^2+2k}}{(q^4;q^4)_k} = \frac{(q^2;q^4)_\infty}{(q^2,q^3;q^5)_\infty}.$$

Therefore we have established

$$(5.15)\qquad \begin{aligned} &iF(i,-q^3,-q^2)-F(-i,-q^3,-q^2)\\ &=iH_2(q/i)-H_2(qi)=(i-1)H_2(q), \end{aligned}$$

and

$$(5.16)\qquad \begin{aligned} &iF(i,-q^3,-q^2)+F(-i,-q^3,-q^2)\\ &=iH_2(q/i)+H_2(qi)=(i+1)H_2(-q), \end{aligned}$$

which complete the evaluation of the numerators and denominators. □

6 Al-Salam-Chihara polynomials

The Al-Salam-Chihara polynomials $\{Q_n(x;a,b|q)\}$ are [5], [13]

$$(6.1)\qquad \begin{aligned} Q_n(\cos\theta) &= Q_n(\cos\theta;a,b|q)\\ &:= \frac{(ab;q)_n}{a^n}\,{}_3\phi_2\left(\begin{matrix} q^{-n},ae^{i\theta},ae^{-i\theta}\\ ab,0\end{matrix}\,\middle|\, q,\ q\right)\\ &= (ae^{i\theta};q)_n\,{}_2\phi_1\left(\begin{matrix} q^{-n},be^{-i\theta}\\ q^{1-n}e^{-i\theta}/a\end{matrix}\,\middle|\, q,\ \frac{q}{a}e^{i\theta}\right), \end{aligned}$$

They satisfy the three term recurrence relation

$$(6.2)\qquad \begin{aligned} 2xQ_n(x) &= Q_{n+1}(x)+(a+b)q^nQ_n(x)\\ &\quad +(1-q^n)(1-abq^{n-1})Q_{n-1}(x), \end{aligned}$$

and the initial conditions $Q_0(x)=1$, $Q_1(x)=2x-a-b$. On the other hand the initial conditions for $Q_n^*(x)$ are $Q_0^*(x)=0$, $Q_1^*(x)=2$. The Al-Salam-Chihara polynomials satisfy the condition (5.2) with

$$a_n=\sqrt{(1-q^n)(1-abq^{n-1})}/2, \quad b_n=(a+b)q^n/2.$$

In this section we shall explicitly evaluate the continued fractions of §5 for the Al-Salam-Chihara polynomials using asymptotic results.

The polynomials $\{Q_n(x)\}$ and $\{Q_n^*(x)\}$ have the generating functions

$$(6.3)\qquad \sum_{n=0}^{\infty}\frac{Q_n(\cos\theta;a,b|q)}{(q;q)_n}\,t^n=\frac{(at,bt;q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty},$$

and

$$(6.4)\qquad \sum_{n=0}^{\infty}\frac{Q_n^*(\cos\theta;a,b|q)}{(q;q)_n}\,t^n=2t\sum_{n=0}^{\infty}\frac{(at,bt;q)_n}{(te^{i\theta},te^{-i\theta};q)_{n+1}}\,q^n,$$

respectively, [5]. The polynomials $\{Q_n(x)\}$ and $\{Q_n^*(x)\}$ have the symmetry property

$$\begin{aligned} Q_n(-x,a,b) &= (-1)^nQ_n(x,-a,-b),\\ Q_n^*(-x,a,b) &= (-1)^{n-1}Q_n^*(x,-a,-b), \end{aligned}$$

which follow from (6.3) and (6.4). Thus we only consider the case $0 < x < 1$, or when $x = \cos\theta$ we may restrict ourselves to $0 < \theta < \pi/2$. With $z = e^{i\theta}$, and $0 < \theta < \pi$ the asymptotic formulas

$$
\begin{aligned}
&\frac{Q_n(\cos\theta; a, b|q)}{(q;q)_n} \\
&= \left[\frac{(a/z, b/z; q)_\infty}{(q, z^{-2}; q)_\infty} z^n + \frac{(az, bz; q)_\infty}{(q, z^2; q)_\infty} z^{-n}\right](1 + o(1)),
\end{aligned}
\tag{6.5}
$$

and

$$
\begin{aligned}
\frac{Q_n^*(\cos\theta; a, b|q)}{(q;q)_n} = &\left[\frac{e^{in\theta}}{i\sin\theta}\,{}_2\phi_1\left(\begin{matrix} ae^{-i\theta}, be^{-i\theta} \\ qe^{-2i\theta}\end{matrix}\,\middle|\, q, q\right)\right. \\
&\left. - \frac{e^{-in\theta}}{i\sin\theta}\,{}_2\phi_1\left(\begin{matrix} ae^{i\theta}, be^{i\theta} \\ qe^{2i\theta}\end{matrix}\,\middle|\, q, q\right)\right] + o(1)
\end{aligned}
\tag{6.6}
$$

follow from (6.3) and (6.4), respectively.

Let $G_n(x; a, b, q)$ denote the nth convergent of the continued fraction

$$
G(x; a, b) := \cfrac{2}{2x - (a+b) - \cfrac{(1-q)(1-ab)}{2x - (a+b)q - \cfrac{(1-q^2)(1-abq)}{\ddots}}}.
\tag{6.7}
$$

Observe that in general, with $z = e^{i\theta}$, $0 < \theta < \pi$, then

$$
\begin{aligned}
\frac{Q_n^*(\cos\theta; a, b|q)}{Q_n(\cos\theta; a, b|q)} = 2z &\left[\frac{(a/z, b/z; q)_\infty}{(q, qz^{-2}; q)_\infty} z^{2n+2} - \frac{(az, bz; q)_\infty}{(q, qz^2; q)_\infty}\right]^{-1} \\
\times &\left[z^{2n}\,{}_2\phi_1\left(\begin{matrix} a/z, b/z \\ q/z^2\end{matrix}\,\middle|\, q, q\right)\right. \\
&\left. - {}_2\phi_1\left(\begin{matrix} az, bz \\ qz^2\end{matrix}\,\middle|\, q, q\right)\right](1 + o(1))
\end{aligned}
$$

Apply (III.31) in Gasper and Rahman [9] with their parameters a, b, c chosen as $a/z, b/z, q/z^2$ to see that

$$
\begin{aligned}
{}_2\phi_1\left(\begin{matrix} a/z, b/z \\ q/z^2\end{matrix}\,\middle|\, q, q\right) = &\frac{(ab, z^2; q)_\infty}{(az, bz; q)_\infty} \\
&- \frac{(z^2, a/z, b/z; q)_\infty}{(z^{-2}, az, bz; q)_\infty}\,{}_2\phi_1\left(\begin{matrix} az, bz \\ qz^2\end{matrix}\,\middle|\, q, q\right)
\end{aligned}
$$

which follows from the q-binomial theorem. Therefore

$$\begin{aligned}
&G_n(\cos\theta;a,b,q)\\
&= -2z\left[\frac{(a/z,b/z;q)_\infty}{(q,qz^{-2};q)_\infty}z^{2n+2}-\frac{(az,bz;q)_\infty}{(q,qz^2;q)_\infty}\right]^{-1}\\
&\times\left[{}_2\phi_1\left(\begin{matrix}az,bz\\ qz^2\end{matrix}\,\middle|\,q,q\right)\left\{1+z^{2n}\frac{(a/z,b/z,z^2;q)_\infty}{(az,bz,z^{-2};q)_\infty}\right\}\right.\\
&\qquad\left.-z^{2n}\frac{(ab,z^2;q)_\infty}{(az,bz;q)_\infty}\right]+o(1),
\end{aligned}\tag{6.8}$$

where $z=e^{i\theta}$.

Theorem 6.1. *Let k be an odd positive integer and $\omega_k := e^{2\pi i/k}$. With $r = 0,1,\ldots,k-1$, then*

$$\begin{aligned}
&\lim_{N\to\infty} G_{kN+r}(\cos(\pi/k);a,b,q) = 2\omega_k^{(k-1)/2}\\
&\times\left[\frac{(-a\omega_k^{(k-1)/2},-b\omega_k^{(k-1)/2};q)_\infty}{(q,q\omega_k^{k-1};q)_\infty}\omega_k^{r+1}-\frac{(-a\omega_k^{(k+1)/2},-b\omega_k^{(k+1)/2};q)_\infty}{(q,q\omega_k;q)_\infty}\right]^{-1}\\
&\times\left[{}_2\phi_1\left(\begin{matrix}-a\omega_k^{(k+1)/2},-b\omega_k^{(k+1)/2}\\ q\omega_k\end{matrix}\,\middle|\,q,q\right)\right.\\
&\times\left\{1+\frac{\omega_k^r(-a\omega_k^{(k-1)/2},-b\omega_k^{(k-1)/2},\omega_k;q)_\infty}{(-a\omega_k^{(k+1)/2},-b\omega_k^{(k+1)/2},\omega_k^{k-1};q)_\infty}\right\}\\
&\left.-\omega_k^r\,\frac{(ab,\omega_k;q)_\infty}{(-a\omega_k^{(k+1)/2},-b\omega_k^{(k+1)/2};q)_\infty}\right].
\end{aligned}$$

The proof follows from (6.8). A similar result can be proved for $x = \cos(j\pi/k)$, $j=1,2,\ldots,k-1$. The special case $k=3$ is worth noting.

Theorem 6.2. *With $\omega = e^{2\pi i/3}$ we have*

$$\begin{aligned}
&\lim_{N\to\infty} G_{3N+\epsilon-1}(1/2;a,b,q)\\
&=\frac{2\omega(q^3;q^3)_\infty}{(-a\omega,-b\omega,q\omega;q)_\infty\omega^\epsilon-(-a\omega^2,-b\omega^2,q\omega^2;q)_\infty}\\
&\times\left[\left\{1+\frac{\omega^{\epsilon-1}(-a\omega,-b\omega,\omega;q)_\infty}{(-a\omega^2,-b\omega^2,\omega^2;q)_\infty}\right\}{}_2\phi_1\left(\begin{matrix}-a\omega^2,-b\omega^2\\ q\omega\end{matrix}\,\middle|\,q,q\right)\right.\\
&\qquad\left.-\omega^{\epsilon-1}\,\frac{(ab,\omega;q)_\infty}{(-a\omega^2,-b\omega^2;q)_\infty}\right].
\end{aligned}$$

For the q-ultraspherical polynomials $\{Q_n(\cos\theta;\beta e^{i\theta},\beta e^{-i\theta})\}$ Theorem 6.2 gives

Theorem 6.3.

$$\lim_{N\to\infty} G_{3N+\epsilon-1}(1/2;\beta e^{i\pi/3},\beta e^{-i\pi/3},q)$$
$$= \frac{2\omega(q^3;q^3)_\infty/(\beta;q)_\infty}{(\beta\omega^2,q\omega;q)_\infty\omega^\epsilon-(\beta\omega,q\omega^2;q)_\infty}$$
$$\times\left[\left\{1+\frac{\omega^{\epsilon-1}\,(\beta\omega^2,\omega;q)_\infty}{(\beta\omega,\omega^2;q)_\infty}\right\}{}_2\phi_1\left(\begin{matrix}\beta\omega,\beta\\ q\omega\end{matrix}\,\middle|\,q,q\right)\right.$$
$$\left.-\omega^{\epsilon-1}\,\frac{(\beta^2,\omega;q)_\infty}{(\beta,\beta\omega;q)_\infty}\right].$$

Finally we consider the continued fraction when $x \notin [-1,1]$.

Theorem 6.4. (Askey and Ismail [5]) *For $x=(z+1/z)/2$ with $|z|>1$ we have*

$$\lim_{n\to\infty} G_n(x;a,b,q) = G(x;a,b)$$
$$= \frac{2(q,qz^{-2};q)_\infty}{z(a/z,b/z;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}a/z,b/z\\ qz^{-2}\end{matrix}\,\middle|\,q,q\right). \tag{6.9}$$

In particular with $z=q/a$

$$G((a^2+q^2)/(2aq);a,b) = \frac{2a}{q-ab}, \tag{6.10}$$

for $q>|a|$, follows from (6.9) and the q-binomial theorem. Similarly

$$G((b^2+q^2)/(2bq);a,b) = \frac{2b}{q-ab}$$

holds when for $q>|b|$.

Theorem 6.5. *Let N be the largest integer such that $q^{N+1}>|a|$. Then*

$$G(x_m;a,b) = \frac{2a(-a^2)^m(q^{m+2}/a^2;q)_m}{(q^{m+1}-ab)q^{3\binom{m+1}{2}}}$$
$$\times{}_2\phi_2\left(\begin{matrix}q^{-m},abq^{-m-1}\\ abq^{-m},a^2q^{-2m-1}\end{matrix}\,\middle|\,q,a^2q^{-m}\right) \tag{6.11}$$

where

$$x_m = \frac{q^{2m+2}+a^2}{2aq^{m+1}},\quad m=1,2,\ldots,N.$$

Proof. Choose $z=q^{m+1}/a$ then apply the ${}_2\phi_1$ to ${}_2\phi_2$ transformation (III.4) in [9] to the ${}_2\phi_1$ in (6.9). The result is (6.11). □

A similar theorem can be proved with a and b interchanged.

7 Remarks

In this section we briefly describe the difference between our work and [7]. One main difference is that we introduce a continuous variable x into the continued fraction and formulate the convergence of the continued fraction in terms of the asymptotics of orthogonal polynomials. This forces the coefficients b_n and λ_n in (1.1) to satisfy $b_n \in \mathbb{R}, n \geq 0$ and $\lambda_n > 0, n > 0$. The results in [7] allow b_n and λ_n to be complex in a way that relates λ_n to x. Our results evaluate the mod k convergence of the continued fractions in terms of the measures of orthogonality of the numerator and denominator polynomials, a topic not considered in [7].

To further illustrate our approach we consider the associated continuous q-ultraspherical polynomials where

$$b_n = 0, \quad \lambda_n = \frac{(1-\alpha q^n)(1-\alpha\beta q^{n-1})}{4(1-\alpha\beta q^n)(1-\alpha\beta q^{n-1})}. \tag{7.1}$$

Bustoz and Ismail [8] proved that the large degree asymptotics of the monic polynomials $\{P_n(\cos\theta;\beta,\alpha)\}$ is given by

$$\begin{aligned} P_n(\cos\theta;\beta,\alpha) = &\frac{(\alpha;q)_\infty}{2^n(\alpha\beta;q)_\infty}\left|{}_2\phi_1(\beta,\beta e^{2i\theta};qe^{2i\theta};q,\alpha)\right| \\ &\times\frac{\sin[(n+1)\theta+\phi(\theta)]}{\sin\theta}[1+o(1)], \end{aligned} \tag{7.2}$$

where $\phi(\theta)$ is the argument of ${}_2\phi_1(\beta,\beta e^{-2i\theta};qe^{-2i\theta};q,\alpha)$. Moreover

$$P_n^*(x;\beta,\alpha) = P_{n-1}(x;\beta,\alpha+1). \tag{7.3}$$

If $\{P_n(\cos\theta;\beta,\alpha)\}$ are orthogonal with respect to a probability measure $\mu(x;\beta,\alpha)$ then $\mu'(x;\beta,\alpha)$ is supported on $[-1,1]$ and

$$\mu'(\cos\theta;\beta,\alpha) = \frac{2}{\pi}\,\frac{(1-\alpha\beta)(\alpha\beta^2;q)_\infty}{(1-\alpha)(\alpha;q)_\infty}\left|{}_2\phi_1\left(\begin{matrix}\beta,\beta e^{2i\theta}\\ qe^{2i\theta}\end{matrix}\,\middle|\, q,\alpha\right)\right|^2, \tag{7.4}$$

so that $((\mu^*)'(\cos\theta;\beta,\alpha) = \mu'(\cos\theta;\beta,\alpha+1)$.

References

[1] W. A. Al-Salam and M. E. H. Ismail, Orthogonal polynomials associated with the Rogers-Ramanujan continued fraction, Pacific J. Math. **105** (1983), 269–283

[2] G. E. Andrews, A page from Ramanujan's lost notebook, Indian J. Math. **32** (1990), 207–216.

[3] G. E. Andrews, B. C. Berndt, J. Sohn, A. J. Yee, and A. Zaharescu, On Ramanujan's continued fraction for $(q^2;q^3)_\infty/(q;q^3)_\infty$, Trans. Amer. Math. Soc. **365** (2003), 2397–2411.

[4] G. E. Andrews, B. C. Berndt, J. Sohn, A. J. Yee, and A. Zaharescu, Continued fractions with three limit points, Advances in Math. **192** (2005), 231–258.

[5] R. A. Askey and M. E. H. Ismail, Recurrence relations, continued fractions and orthogonal polynomials, Memoirs Amer. Math. Soc. Number **300** (1984).

[6] B. C. Berndt and J. Sohn, Asymptotic formulas for two continued fractions in Ramanujan's Lost Notebook, J. London, Math. Soc., **65** (2002), 271–284.

[7] D. Bowman and J. McLaughlin, Continued fractions with multiple limits, to appear.

[8] J. Bustoz and M. E. H. Ismail, The associated ultraspherical polynomials and their q-analogues, Canadian J. Math. **34** (1982), 718–736.

[9] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, second edition Cambridge University Press, Cambridge, 2004.

[10] M. E. H. Ismail and X. Li, Ramanujan's q-continued fractions, a preprint.

[11] M. E. H. Ismail and F. S. Mulla, on the generalized Chebyshev polynomials, SIAM J. Math. Anal. **18**, (1987), 243–258.

[12] W. B. Jones and W. Thron, Continued Fractions: Analytic Theory and Applications, Cambridge University Press, Cambridge, 1980.

[13] R. Koekoek and R. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogues, Reports of the Faculty of Technical Mathematics and Informatics no. 98-17, Delft University of Technology, Delft, 1998.

[14] P. Nevai, Orthogonal polynomials. Mem. Amer. Math. Soc. No. 213, 1979, 213 pages.

[15] P. Nevai, Géza Freud, orthogonal polynomials and Christoffel functions. A case study, J. Approx. Theory **48** (1986), 3–167.

[16] L. J. Slater, Further identities of the Rogers-Ramanujan type, Proc. London Math. Soc. (2) **54**, (1952). 147–167.

[17] J. Shohat and J. D. Tamarkin, *The Problem of Moments*, revised edition, American Mathematical Society, Providence, 1950.

[18] G. Szegő, *Orthogonal Polynomials*, Fourth Edition, Amer. Math. Soc., Providence, 1975.

[19] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, second edition, Cambridge University Press, Cambridge, 1944.