MORE MONOTONICITY THEOREMS FOR PARTITIONS

J. Friedman, J. T. Joichi, and D. Stanton[1]

Abstract. Consider the collection of all integer partitions, whose part sizes lie in a given set. Such a set is called monotone if the generating function has weakly increasing coefficients. The monotone subsets are classified, assuming an open conjecture.

1. Introduction.

Suppose P is a set of positive integers. It is well known [1] that the generating function for integer partitions, whose part sizes lie in P, is

$$F_P(q) = \sum_{n=0}^{\infty} a_n q^n = \prod_{p\in P} \frac{1}{1-q^p}.$$

Bateman and Erdös [2] found necessary and sufficient conditions on P so that the kth difference of the sequence a_n is asymptotically positive. In this paper we consider $k = 1$. We seek a stronger conclusion, that all of the coefficients in $(1-q)F_P(q)$ past $-q$ are non-negative. We call such a set of positive integers *monotone*.

Clearly any P containing 1 is monotone, so we can assume that if n is the smallest element of P, then $n > 1$. If the coefficient of q^{n+1} in $(1-q)F_P(q)$ is non-negative, then we must have $n+1 \in P$ for a monotone P. In this way it is easy to see that $\{n, \ldots, 2n-1\} \subset P$. In Theorems 1 and 2 we classify all monotone P whose minimum value n satisfies $n \geq 6$, assuming Conjecture 1 below.

A set P is called *asymptotically alternating*, if there exists a large enough k so that the kth differences of a_n alternate in sign. We classify the asymptotically alternating sets P in Theorem 5.

We shall let NN denote non-negative terms and $SP(\text{past } q^a)$ denote strictly positive terms past q^a in a power series $F(q)$. For example $-q + \frac{1}{1-q^2} = 1 - q + NN$, $q^3/(1-q) = SP(\text{past } q^2)$.

2. The conjecture.

In this section we concentrate on properties of the function

$$f_{n,m}(q) = \frac{(1-q)}{\prod_{i=n}^{m}(1-q^i)}.$$

We formulate a conjecture on $f_{n,m}(q)$ below, and use it to classify monotone sets P in §3. We are particularly interested in the values of m for which $f_{n,m}(q) = 1 - q + NN$, because for any of these m, $P = \{n, \cdots, m\}$ is monotone.

---

[1]This work was supported by NSF grant DMS-9001195.

Proposition 1. *We have* $f_{n,\infty}(q) = 1 - q + NN = 1 - q + NN + SP(\text{past } q^{3n+1})$.

Proof. Applying the q-binomial theorem [1], we obtain

$$\begin{aligned} f_{n,\infty}(q) =& (1-q)\sum_{k=0}^{\infty} q^{nk}/\prod_{i=1}^{k}(1-q^i) \\ =& 1 - q + q^n + q^{2n}/(1-q^2) + \sum_{k=3}^{\infty} q^{nk}/\prod_{i=2}^{k}(1-q^i) \\ =& 1 - q + NN = 1 - q + NN + SP(\text{past } q^{3n+1}). \end{aligned} \tag{2.1}$$

□

Conjecture 1. *For an odd positive integer* $n > 1$, $f_{n,2n-1}(q) = 1 - q + NN$. *If, in addition,* $n \geq 7$, *then* $f_{n,2n-1}(q) = 1 - q + NN + SP(\text{past } q^{3n+4})$. *If* $n > 1$ *is even, then* $f_{n,2n+1}(q) = 1 - q + NN + SP(\text{past } q^{3n+7})$.

It is easy to see the even part of Conjecture 1 follows from the odd part. If $h_{n,m}(q) = f_{n,m}(q) - 1 + q = NN$, then

$$h_{n,2n+1}(q) = \frac{1}{1-q^n}(h_{n+1,2n+1}(q) + q^n - q^{n+1}). \tag{2.2}$$

If n is even, $h_{n+1,2n+1}(q)$ contains q^{n+1}, so $h_{n,2n+1}(q) = NN$, and $= NN + SP(\text{past } q^{3n+7})$ for $n \geq 6$. The cases $n = 2$ and $n = 4$ can be proven separately.

A natural way to prove Conjecture 1 for a given n, is to use the asymptotics to verify the large coefficients, and check the small coefficients separately. For this one needs an effective bound for the positivity of the large coefficients. In turns out that a recurrence relation can find this effective bound empirically, using Mathematica or a programming language.

Proposition 2. *Conjecture 1 holds for* $n \leq 37$.

Proof. We verify the n odd case. Let $a_k(n, n+i)$ denote the number of partitions of k into parts of size $n, \cdots, 2n-1$, whose largest part is $n+i$, $0 \leq i \leq n-1$. We must show

$$\delta(k) = \sum_{i=0}^{n-1}(a_k(n, n+i) - a_{k-1}(n, n+i)) \geq 0, \text{for } k \geq 2.$$

By removing this largest part, we have

$$a_k(n, n+i) = \sum_{j=0}^{i} a_{k-n-i}(n, n+j). \tag{2.3}$$

Suppose that by applying (2.3) recursively to $\delta(k)$, we can obtain

$$\delta(k) = \delta(k-t) + \text{a non-negative linear combination of } a_j(n, n+i)\text{'s}.$$

If we verify that $\delta(2), \cdots, \delta(t+1) \geq 0$, then Conjecture 1 holds.

For example, if $n = 3$,

$$\delta(k) = \delta(k-20) + a_{k-21}(3,3),$$

and we check that $\delta(2), \cdots, \delta(21) \geq 0$. This is feasible as long as $t = t(n)$ does not grow too rapidly with n. Empirically, we find $t(n) = (2n-2)(2n-1)$. If the smallest part is used to generate a recurrence analogous to (2.3), the empirical result is $t(n) = n(n+1)$ if 3 does not divide n, otherwise $t(n) = 4n(n+1)$. In this way Conjecture 1 was verified for $n \leq 37$.

We shall need the lemma below in the next section.

Lemma 1. *For an odd positive integer $n \geq 7$, the coefficient of q^{6n+1} in $f_{n,2n-1}(q)$ is at least 2.*

Proof. The q-binomial theorem [1] implies that

$$f_{n,2n-1}(q) = 1 - q + \sum_{k=1}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q (1-q)q^{nk}. \tag{2.4}$$

The only terms in (2.4) which contribute to q^{6n+1} are $k = 4$ and $k = 5$. It is easy to see that this coefficient is equal to the coefficient of q^{2n+2} in

$$\frac{q}{(1-q^2)(1-q^3)(1-q^4)} + \frac{1}{(1-q^4)(1-q^6)(1-q^8)(1-q^{10})} - \frac{1}{(1-q^2)(1-q^4)(1-q^6)}.$$

An elementary injection shows that this coefficient is at least 2, for $n \geq 7$, n odd. We do not give the details. $\square$

3. Monotone subsets.

In this section we use Conjecture 1 to classify the monotone subsets P in Theorems 1 and 2. For most of this section we shall assume that n is odd. Basically we need a method to change the set P from an interval $\{n, \cdots, 2n-1\}$ to a larger class of sets. The lemma below accomplishes this.

Lemma 2. *Suppose that $H(q) = 1 - q + NN + SP(\text{past } q^a)$, and*

$$H(S,q) = H(q) / \prod_{s \in S} (1 - q^s),$$

where S is any set of positive integers. If $\min\{s : s \in S\} \geq a$, then $H(S,q) = 1 - q + NN$.

Proof. We must show that $g(S,q) = H(S,q) - 1 + q = NN$. An easily verified recurrence for $b \notin S$ is

$$g(S \cup \{b\}, q) = \frac{1}{1-q^b}(g(S,q) + q^b - q^{b+1}). \tag{3.1}$$

if $S = \varnothing$, $g(\varnothing, q)$ is positive past q^a, thus positive at q^{b+1}. So (3.1) implies $g(\{b\}, q) = NN + SP(\text{past } q^{b+1})$, and the argument follows for finite S by induction on $|S|$. If S is infinite, to check that the coefficient of q^j is non-negative, we apply the finite part this lemma for the finite set $S \cap \{x : x \leq j\}$. $\square$

We next find monotone sets P from Lemma 2 and Conjecture 1.

Proposition 3. *Assuming Conjecture 1, $P = \{n, \cdots, 2n-1\} \cup Q$, is monotone, where $n \geq 7$ is odd, and Q is any subset of $\{3n+4, 3n+5, \cdots, \}$.*

Suppose that $\{n, ..., 2n-1\} \subset P \subset \{n, ..., 3n-2\}$ is monotone. It is easy to see that if an even number $e \geq 2n$ is in P, then $e+1$ must also be in P. The next proposition shows that this condition characterizes such monotone sets P.

Proposition 4. *Assuming Conjecture 1, $P = \{n, \cdots, 2n-1\} \cup E \cup O$ is monotone, where $n \geq 7$ is odd, and E (O) is any set of even (odd) integers in $\{2n, \cdots, 3n-2\}$ such that $E^+ = \{e+1 : e \in E\} \subset O$.*

Proof. Suppose $n \geq 7$ is odd, and put, for any subset $S \subset \{2n, 2n+1, \cdots\}$,

$$g(n, S) = f_{n,2n-1}(q) / \prod_{s \in S} (1 - q^s) - 1 + q.$$

The coefficients of $f_{n,2n-1}(q)$ up to q^{3n+4} can be explicitly found, so that Conjecture 1 implies

$$g(n, \varnothing) = q^n + \sum_{i=0}^{(n-3)/2} q^{2n+2+2i} + q^{3n+3} + SP(\text{past } q^{3n+4}). \tag{3.2}$$

Suppose that $E \cup O = \{a_1 < \cdots < a_k\}$. To go from $g(n, \varnothing) = NN$ to $g(n, E \cup O) = NN$ we add either a single odd a_i, or a consecutive pair a_i (even), $a_{i+1} = a_i + 1$ (odd). For these two cases, we have

$$g(n, S \cup \{a\}) = (q^a - q^{a+1} + g(n, S))/(1 - q^a), \text{ for } a \notin S, \tag{3.3}$$

(3.4)

$$g(n, S \cup \{a, a+1\}) = \frac{q^a}{1 - q^a} + (q^{2a+2} - q^{a+2} + g(n, S))/[(1-q^a)(1-q^{a+1})], \text{ for } a, a+1 \notin S.$$

We see from (3.2) that $g(n, \varnothing)$ contains all of the even powers of q from $2n+2$ to $3n-1$. If a_1 is odd, then (3.3) implies that $g(n, \{a_1\})$ is non-negative and contains all of the even powers of q from $a_1 + 2$ to $3n - 1$. If a_1 is even, then (3.4) implies that $g(n, \{a_1, a_1 + 1\})$ is non-negative and contains all of the even powers of q from $a_1 + 3$ to $3n - 1$. We continue by induction on i, noting that the single negative term in (3.3) and (3.4) are even powers past the new term which is added, thus is always cancelled. We obtain

$$g(n, E \cup O) = NN + SP(\text{past } q^{3n+4}), \text{ for } n \geq 7 \text{ odd.} \tag{3.5}$$

□

Proposition 5. *Assuming Conjecture 1, if $n \geq 7$ is odd, then $P = \{n, \cdots, 2n-1\} \cup E \cup O \cup Q$ is monotone, where E, O, and Q are chosen as in Propositions 3 and 4.*

Proof. This follows from (3.5) and Lemma 2. □

Suppose that we generalize Proposition 5 to $P = \{n, \cdots, 2n-1\} \cup E \cup O \cup A$, and $n \geq 7$ is odd, where $A \subset \{3n-1, 3n, 3n+1, 3n+2, 3n+3, 3n+4\}$. From

(3.5) we know that $g(n, E \cup O) = NN + SP(\text{past } q^{3n+4})$. We will use (3.3), (3.4), and analogous versions for three and four in a row (see (3.6) and (3.7) below), to conclude that $g(n, E \cup O \cup A) = NN + SP(\text{past } q^{3n+4})$, for the appropriate sets A.

From (3.2) there is exactly one term in $g(n, \varnothing)$ from q^{3n} to q^{3n+4}, namely $+q^{3n+3}$. For $g(n, E \cup O \cup A)$, the possible new partitions in this range, whose differences we must take, are

(1) $\{3n-1\}$,
(2) $\{3n\}, \{n, 2n\}$,
(3) $\{3n+1\}, \{n, 2n+1\}, \{n+1, 2n\}$
(4) $\{3n+2\}, \{n, 2n+2\}, \{n+1, 2n+1\}, \{n+2, 2n\}$
(5) $\{3n+3\}, \{n, 2n+3\}, \{n+1, 2n+2\}, \{n+2, 2n+1\}, \{n+3, 2n\}$
(6) $\{3n+4\}, \{n, 2n+4\}, \{n+1, 2n+3\}, \{n+2, 2n+2\}, \{n+3, 2n+1\}, \{n+4, 2n\}$.

If $3n-1 \in A$, for non-negativity we must have either $2n \in E$ or $3n \in A$. If $2n \in E$ we use (3.3) with $a = 3n-1$ to get strict positivity past $3n+4$. The case $3n-1 \notin A$, $3n \in A$ is done by the same argument. If $3n-1, 3n \in A$ we use (3.4), and must check the coefficient of q^{3n+1} in $g(n, E \cup O \cup \{3n-1, 3n\})$. Again we must have either $3n+1 \in A$ or $2n+1 \in O$. If $2n+1 \in O$, then (3.4) gives strict positivity past $3n+4$. Otherwise $3n-1, 3n, 3n+1 \in A$, and we use

$$\begin{aligned} g(n, E \cup O \cup \{a, a+1, a+2\}) &= \frac{q^a}{(1-q^a)(1-q^{a+2})} + \frac{q^{2a+4}}{(1-q^{a+1})(1-q^{a+2})} \\ &+ (q^{3a+3} - q^{a+3} + g(n,, E \cup O))/[(1-q^a)(1-q^{a+1})(1-q^{a+2})] \end{aligned} \tag{3.6}$$

for $a = 3n-1$. If $2n+2 \in E$, then the term q^{3n+2} appears in $g(n, E \cup O)$, and (3.6) implies $g(n, E \cup O \cup \{3n-1, 3n, 3n+1\}) = NN + SP(\text{past } q^{3n+4})$. If $2n+2 \notin E$, then clearly we must have $3n-1, 3n, 3n+1, 3n+2 \in A$. This time we use

(3.7)

$$\begin{aligned} g(n, E \cup O \cup \{a, a+1, a+2, a+3\}) &= \frac{q^a}{(1-q^a)(1-q^{a+2})(1-q^{a+3})} \\ + \frac{q^{2a+4} + q^{3a+3}}{(1-q^a)(1-q^{a+1})(1-q^{a+2})} &+ (q^{2a+6} + q^{4a+7} - q^{a+4} - q^{2a+3} + g(n, E \cup O)) \\ /[(1-q^a)(1-q^{a+1})(1-q^{a+2})(1-q^{a+3})] \end{aligned}$$

for $a = 3n-1$. Since $g(n, \varnothing)$ contains q^{3n+3}, the q^{a+4} term is cancelled in (3.7). From Lemma 1, $g(n, \varnothing)$ contains at least $+2q^{6n+1}$, so (3.7) implies that $g(n, E \cup O \cup \{3n-1, 3n, 3n+1, 3n+2\}) = NN + SP(\text{past } q^{3n+4})$.

Finally, if $3n+3 \in A$, we must have either $2n+4 \in E$ or $3n+4 \in A$, and we retain strict positivity past q^{3n+4}.

Applying Lemma 2, we have proven the main theorem.

Theorem 1. *Assuming Conjecture 1, the monotone subsets P whose minimum value n is odd, $n \geq 7$, are $P = \{n, \cdots, 2n-1\} \cup E \cup O \cup A \cup Q$, where*

(1) *E is any set of even integers from $\{2n, \cdots, 3n-2\}$,*
(2) *O is a set of odd integers from $\{2n, \cdots, 3n-2\}$ such that $E^+ \subset O$,*
(3) *A is a subset of $\{3n-1, \cdots, 3n+3\}$ such that if $3n+i \in A$ for $i \neq 2$, then either $3n+i+1 \in P$ or $2n+i+1 \in P$,*
(4) *Q is any subset of $\{3n+4, 3n+5, \cdots\}$.*

For even values of n, the coefficient of q^{3n+1} easily implies $2n+1 \in P$. The analog of (3.2) for $n \geq 6$, which follows from Conjecture 1, is

(3.8)
$$f_{n,2n-1}(q)/(1-q^{2n+1}) = 1-q+q^n+q^{2n+1}+\sum_{i=0}^{n/2} q^{2n+4+2i}+2q^{3n+6}+SP(\text{past } q^{3n+7}).$$

A complicated injection proves that (3.8) also holds for $n = 4$. Completely analogous arguments, based upon (3.8) yield the next theorem. We do not need an even version of Lemma 1, because the largest gap in (3.8) from $3n+2$ to $3n+6$ has width 2, not width 4, as in (3.2).

Theorem 2. *Assuming Conjecture 1, the monotone subsets P whose minimum value n is even, and $n \geq 4$ are $P = \{n, \cdots, 2n-1, 2n+1\} \cup E \cup O \cup A \cup Q$, where*

1. *E is any set of even integers from $\{2n+2, \cdots, 3n+1\}$,*
2. *O is a set of odd integers from $\{2n+2, \cdots, 3n+1\}$ such that $E^+ \subset O$,*
3. *A is a subset of $\{3n+2, \cdots, 3n+6\}$ such that if $3n+2i \in A$ then either $3n+2i+1 \in P$, or $2n+2i+1 \in P$*
4. *Q is any subset of $\{2n, 3n+7, 3n+8, \cdots\}$.*

Propositions 1, 3 and 4 imply that $f_{n,m}(q) = 1-q+NN$ if

1. $m \in \{2n-1, 2n+1, 2n+3, \cdots, 3n-2, 3n-1, 3n, \cdots, \infty\}$, for n odd
2. $m \in \{2n+1, 2n+3, 2n+5, \cdots, 3n+1, 3n+2, 3n+3, \cdots, \infty\}$, for n even.

We cannot prove a weaker version, namely for any fixed n, there is some finite $m(n)$ for which $f_{n,m}(q) = 1-q+NN$. Nonetheless, we do have the following result.

Proposition 6. *Suppose n is odd. If $f_{n,m}(q) = 1-q+NN$ for some $m = m_0 \geq 3n-2$, then $f_{n,m}(q) = 1-q+NN$ for all $m > m_0$.*

Proof. Upon adding m_0+1, from (3.1), we need only show that the coefficient of q^{m_0+2} in $f_{n,m_0}(q)-1+q$ is ≥ 1. But this coefficient equals the coefficient of q^{m_0+2} in $f_{n,\infty}(q)$. From (2.1) we have

$$f_{n,\infty}(q) = (1-q)+q^n+q^{2n}/(1-q^2)+q^{3n}/((1-q^2)(1-q^3))+NN,$$

and any term past q^{3n+1} appears with coefficient ≥ 1 in the fourth term. Since n is odd, q^{3n+1} and q^{3n-1} appear in the third term. Clearly q^{3n} appears in the fourth term. So $m_0 \geq 3n-3$ is sufficient. However, we already know that $m_0 = 3n-3$ fails, so $m_0 \geq 3n-2$. $\square$

4. Injections.

The most natural proof of Conjecture 1 would be an injection from partitions of $k-1$ into partitions of k. We have found such injections for $n \leq 9$, but not for general n. One may also change Conjecture 1 to an equivalent injection on larger sets by using the q-binomial theorem. For example,

$$f_{n,m}(q) = 1-q+q^n+\sum_{k=2}^{\infty}\frac{q^{kn}}{\prod_{i=2}^{k}(1-q^i)}-\sum_{k=1}^{\infty}\frac{q^{k(m+1)}}{\prod_{i=2}^{k}(1-q^i)\prod_{j=n}^{m}(1-q^j)}. \tag{4.1}$$

An injection from the set representing the second sum to the set for the first sum is equivalent to Conjecture 1. One may hope that large values of m would make an injection easier to find.

One may also begin with

$$f_{n,n+1}(q) = \frac{1}{1-q^n} - \frac{q}{1-q^{n+1}},$$

all of whose terms are known, and try adding $n+2, n+3, \cdots, 2n-1$, to reach Conjecture 1. We use an injection to completely classify the negative terms for $f_{n,n+2}(q)$.

Theorem 3. *Suppose $n = 2l+1 > 1$ is odd. All of the coefficients in $f_{n,n+2}(q)$ are at least -1. Moreover the coefficient of q^k is -1 if, and only if, $k = an+b(n+2)+1$, where $0 \le b = nj+r$, $0 \le r \le l-1$, $0 \le a \le a+j \le r$.*

Proof. For example, if $n = 5$, then $r = 0$ gives $k = 1$, and $r = 1$ gives $k = 8, 13, 43$ as the four negative terms of $f_{5,7}(q)$.

We construct an injection from partitions of $k-1$ whose parts are from $\{n, n+1, n+2\}$, to partitions of k whose parts are from $\{n, n+1, n+2\}$.

First if $n+1$ is a part, add 1 to it to create a part of size $n+2$. So we assume $n+1$ is not part, and the partition is $n^a(n+2)^b$. We need a partition of $an+b(n+2)+1$ into parts of size n and $n+1$.

Let $b = nj+r$, where $0 \le r \le n-1$. Define the injection by

$$an + b(n+2) + 1 \to \begin{cases} n^{a+j+n-r}(n+1)^{nj+2(r-l)}, & \text{if } r \ge l \\ n^{a+j-r-1}(n+1)^{nj+2r+1}, & \text{if } 0 \le r \le l-1. \end{cases}$$

It is routine to check the map is an injection where it is well defined. It is not well defined if, and only if, the multiplicity of n in the second case is negative. These are the coefficients stated in Theorem 1, because they yield distinct integers for $an + b(n+2) + 1$. □

5. Related questions.

It is natural to ask when $f_{n,n+m}(q)/(1-q)$ is strictly positive.

Theorem 4. *There is an integer partition of k into parts of size $\{n, n+1, \cdots, n+m\}$ for all $k \ge n[\frac{n+m-2}{m}]$. Moreover, this bound is best possible.*

Proof. The q-binomial theorem implies, in terms of q-binomial coefficients [1],

$$f_{n,n+m}(q)/(1-q) = \sum_{i=0}^{\infty} \begin{bmatrix} m+i \\ i \end{bmatrix}_q q^{in}.$$

If $i \ge [\frac{n+m-2}{m}]$, then the degree of the q-binomial coefficient is at least $n-1$, so all terms between q^{in} and $q^{(i+1)n-1}$ appear. □

Friedman and Zeilberger [3] proved that $f_{2n,2n+2j}(q)(1-q)^j$ alternates in sign, thus $P = \{2n, \cdots, 2n+2j\}$ is asymptotically alternating. The next theorem classifies the asymptotically alternating sets P.

Theorem 5. *$P = \{a_1, \cdots, a_n\}$ is asymptotically alternating, if, and only if, $\alpha_2 \ge \alpha_j$ for all j, where $\alpha_j = |\{i : j \text{ divides } a_i\}|$.*

Proof. We shall use the following fact from [4]. If $p(q)$ is a real polynomial such that $p(0) = 1$, and $p(q) > 0$ for $q < 0$, then there exists an integer a such that $(1-q)^a p(q)$ alternates in sign.

First assume that $\alpha_2 \geq \alpha_j$ for all j. Let

$$p(q) = \frac{(1-q)^{n-\alpha_2}(1-q^{a_1 a_2 \cdots a_n})^{\alpha_2}}{\prod_{i=1}^{n}(1-q^{a_i})}.$$

It is easy to check using cyclotomic polynomials that $p(q)$ is a polynomial in q with $p(0)=1$, and that $p(q)$ has no negative real roots. Thus, there exists an integer $a>0$ such that $(1-q)^a p(q)$ is alternating, or equivalently $(1+q)^a p(-q)$ has non-negative coefficients. Since $a_1 a_2 \cdots a_n$ is even, we see that

$$\frac{(1+q)^{n-\alpha_2+a}}{\prod_{i=1}^{n}(1-(-q)^{a_i})}$$

has non-negative coefficients. Replacing q by $-q$ gives the first part of the theorem.

Next, suppose that $\alpha_2 < \alpha_j$ for some j. We can assume that α_j is maximized, so that j must be odd. We show that the coefficients cannot be alternating, by showing that the leading terms in the asymptotic expansion for the coefficients are not alternating for a large.

The leading term in the partial fractions decomposition for the rational function is

$$A/(1-\omega q)^{\alpha_j},$$

where ω is primitive jth root of 1, and

$$A = \frac{(1-\omega^{-1})^a}{\prod_{j \text{ divides } a_i} a_i \prod_{j \text{ does not divide } a_i}(1-\omega^{-a_i})}.$$

The absolute value of the coefficient of q^k is a polynomial in k, whose leading term is

$$|\frac{A\omega^k k^{\alpha_j - 1}}{(\alpha_j - 1)!}|. \tag{5.1}$$

We first determine which primitive jth roots ω maximize (5.1). Putting $\omega = exp(2\pi i m/j)$, we find

$$|A| = c(2sin(\pi m/j))^a, \tag{5.2}$$

where c is a constant independent of a. If a is large enough, the largest value of $|A|$ occurs if $m=(j\pm 1)/2$, which is primitive. If there are many values of j which maximize α_j, the largest such j with $m=(j\pm 1)/2$ gives the largest value of $|A|$. Let J denote the largest of these values of j.

Adding these two terms, we see that the sign of the coefficient of q^k, for large k, is the same as

$$cos(\frac{\pi k(J-1)}{J} + \phi),$$

where ϕ is an angle independent of k. This implies that the sign behavior of the large coefficients is determined modulo J, not modulo 2. $\square$

There is also a version of Theorem 5 which allows numerator factors. Odlyzko [5] proved that the kth difference for $P=\{1,2\cdots\}$ is initially alternating, and then immediately non-negative, for all large values of k.

References

1. G. E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics and Its Applications, Vol. 2 (G.-C. Rota ed.), Addison-Wesley, Reading, Mass., 1976, (reissued by Cambridge Univ. Press, London and New York, 1985).
2. P. Bateman and P. Erdös, *Montonicity of partition functions*, Mathematika **3** (1956), 1–14.
3. J. Friedman and D. Zeilberger, *A generalization of Odlyzko's conjecture: the coefficients of* $(1-q)^j/((1-q^{2n})...(1-q^{2n+2j})$ *alternate in sign*, Proc. AMS **118** (1993), 1013.
4. G. Hardy, J. Littlewood, and G. Polya, *Inequalities*, Cambridge University Press, 1934.
5. A. Odlyzko, *Differences of the partition function*, Acta Arith. **49** (1988), 237–254.

Department of Mathematics, University of San Diego, San Diego, CA 92110.

School of Mathematics, University of Minnesota, Minneapolis, MN 55455.