

NOTES FOR NEIGHORLY PARTITIONS

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ABSTRACT. A proof of the first Rogers-Ramanujan identity is given using admissible neighborly partitions. This completes a program initiated by Mohsen and Mourtada. The admissible neighborly partitions involve an unusual mod 3 condition on the parts.

1. INTRODUCTION

Using commutative algebra, Mohsen and Mourtada [4] gave combinatorial interpretations of the numerator infinite products of the Rogers-Ramanujan identities [1, p. 104]

$$\sum_{k=0}^{\infty} \frac{q^{k^2}}{(q; q)_k} = \frac{(q^2, q^3, q^5; q^5)_{\infty}}{(q; q)_{\infty}}, \quad \sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q; q)_k} = \frac{(q^1, q^4, q^5; q^5)_{\infty}}{(q; q)_{\infty}}.$$

To do so they defined a set of integer partitions λ , called *neighborly*, a related set of graphs H_{λ} , and a *signature* for each graph $G \in H_{\lambda}$.

Theorem 1.1. [4] *Assuming the first Rogers-Ramanujan identity, the numerator infinite product is*

$$\sum_{\lambda \in \text{Neighborly}} q^{|\lambda|} \sum_{G \in H_{\lambda}} \text{signature}(G) = (q^2, q^3, q^5; q^5)_{\infty} = 1 + \sum_{k=1}^{\infty} (-1)^k q^{5k^2-k/2} (1 + q^k).$$

They ask [4, p. 3] for a proof of Theorem 1.1 without assuming the Rogers-Ramanujan identities. The purpose of this note is twofold:

- (1) to provide such as proof (see Theorem 4.3),
- (2) to simplify the double sum in Theorem 1.1 to a single sum of signed admissible neighborly partitions (see Proposition 3.4).

Along the way we give a combinatorial interpretation for the classical generalization Theorem 4.3 of Theorem 1.1.

We use the standard notation for q -series found in [1] and [3], and write the parts of an integer partition in increasing order.

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2. NEIGHBORLY PARTITIONS

Definition 2.1. A **neighborly partition** $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$ has all multiplicities at most 2, and for any part λ_i , there is a part λ_j , $j \neq i$, such that $|\lambda_i - \lambda_j| \leq 1$.

A neighborly partition λ can be considered as an ordered pair of partitions: $\lambda = (\mu_1, \mu_2)$, a distinct partition μ_1 and another distinct partition μ_2 whose parts are a subset of the parts of μ_1 .

Example 2.2. If the neighborly partition is $\lambda = (1, 2, 3, 3, 6, 6, 8, 8, 9, 9, 14, 14)$, then

$$\lambda = ((1, 2, 3, 6, 8, 9, 14), (3, 6, 8, 9, 14)) = (\mu_1, \mu_2).$$

The partition μ_1 consists of some runs, with singletons possible. In the example

$$\mu_1 = (1, 2, 3, 6, 8, 9, 14),$$

the runs are $1 \leftrightarrow 2 \leftrightarrow 3, 6, 8 \leftrightarrow 9$, and 14. Note that if x is a singleton in μ_1 , then x must appear in μ_2 .

Mohsen and Mourtada defined a *signature* on a graph G_λ defined by a neighborly partition λ .

Definition 2.3. The graph G_λ of a neighborly partition λ has vertices which are the parts of λ , and edges from the consecutive parts in runs of μ_1 , called the **backbone**, along with edges between equal parts, called **hanging edges**.

Example 2.4. If $\lambda = ((1, 2, 3, 6, 8, 9, 14), (3, 6, 8, 9, 14))$ the backbone of G_λ is

$$1 \leftrightarrow 2 \leftrightarrow 3 \quad 6 \quad 8 \leftrightarrow 9 \quad 14$$

with hanging edges

$$\begin{array}{cccccc}
 3 \leftrightarrow 3 & 6 \leftrightarrow 6 & 8 \leftrightarrow 8 & 9 \leftrightarrow 9 & 14 \leftrightarrow 14, & \text{or} \\
 1 \leftrightarrow 2 \leftrightarrow 3 & 6 & 8 \leftrightarrow 9 & 14 & & \\
 G_\lambda = & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 & & 3 & 6 & 8 & 9 & 14
 \end{array}$$

Definition 2.5. The **signature** of the graph G_λ is the signed sum over all vertex spanning forests H of G_λ

$$\text{signature}(G_\lambda) = \sum_{H \in \text{VS}(G_\lambda)} (-1)^{\#\text{edges in } H}.$$

Example 2.6. If $\lambda = (1, 2, 3, 4)$, $G_\lambda = 1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4$, there are two vertex spanning forests: the entire graph, or the graph with the edge $2 \leftrightarrow 3$ deleted. So $\text{signature}(G_\lambda) = (-1)^3 + (-1)^2 = 0$.

First we compute the signature on a backbone in the shape of a chain, i.e. without hanging edges.

Definition 2.7. Let $\lambda_n = (1, 2, 3, \dots, n+1)$ so that G_{λ_n} is a chain with n edges.

Proposition 2.8. The signature of a chain with n edges is

$$B_n = \text{signature}(G_{\lambda_n}) = \begin{cases} -1 & \text{if } n \equiv 1 \pmod{3}, \\ 1 & \text{if } n \equiv 2 \pmod{3}, \\ 0 & \text{if } n \equiv 3 \pmod{3}. \end{cases}$$

Example 2.9. If $n = 5$, let

$$e_1 = 1 \leftrightarrow 2, \quad e_2 = 2 \leftrightarrow 3, \quad e_3 = 3 \leftrightarrow 4, \quad e_4 = 4 \leftrightarrow 5, \quad e_5 = 5 \leftrightarrow 6.$$

The vertex spanning subgraphs are

$$\{e_1, e_2, e_3, e_4, e_5\}, \{e_1, e_3, e_4, e_5\}, \{e_1, e_2, e_4, e_5\}, \{e_1, e_2, e_3, e_5\}, \{e_1, e_3, e_5\},$$

so $B_5 = 1$.

Proof. Let $B_n(x)$ be the generating function for vertex spanning forests H of G_{λ_n} according to the number of edges,

$$B_n(x) = \sum_{H \in VS(G_{\lambda_n})} x^{\#\text{ edges in } H},$$

By counting the number of edges in connected components from left to right, the coefficient of x^{n-k} in $B_n(x)$ is the number of compositions of $n - k$ into $k + 1$ parts. So

$$B_n(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-k-1}{k} x^{n-k}.$$

The generating function of $B_n(x)$ is

$$(2.1) \quad \sum_{n=1}^{\infty} B_n(x)t^n = xt/(1 - xt - xt^2),$$

so

$$\sum_{n=1}^{\infty} B_n(-1)t^n = -t/(1 + t + t^2) = -t \frac{1-t}{1-t^3},$$

which proves the mod 3 behavior of B_n . □

Remark 2.10. One can also see the generating function as compositions of 1's and 2's (Fibonacci numbers), by counting the number of new vertices each successive edge gives. So one would see (2.1) almost immediately.

We now consider the case when G_λ of a neighborly partition λ has one connected component.

Proposition 2.11. Suppose $\lambda = (\mu_1, \mu_2)$ where $\mu_1 = (1, 2, \dots, n)$, and $\mu_2 = (a_1, a_2, \dots, a_s)$, $s \geq 1$. Then

$$\text{signature}(G_\lambda) = (-1)^s B_{a_1} B_{n-a_s+1} \prod_{k=2}^s B_{a_k - a_{k-1} + 2}.$$

Thus the signature of any connected component of any G_λ is $+1, -1$ or 0 .

Proof. The hanging edges must be in any vertex spanning forest H . Thus we need spanning forests for the chains

$$\begin{aligned} 1 &\leftrightarrow 2 \leftrightarrow \dots \leftrightarrow a_1 \leftrightarrow a_1, \\ a_1 &\leftrightarrow a_1 \leftrightarrow a_1 + 1 \leftrightarrow \dots \leftrightarrow a_2 \leftrightarrow a_2, \dots \\ a_s &\leftrightarrow a_s \leftrightarrow a_s + 1 \dots \leftrightarrow n, \end{aligned}$$

which have respectively

$$a_1, a_2 - a_1 + 2, a_3 - a_2 + 2, \dots, n - a_s + 1 \quad \text{edges.}$$

The choices for spanning forests in these smaller chains may be done independently. Each hanging edge has been used twice, so the factor $(-1)^s$ compensates. \square

Example 2.12. If $\lambda = ((1, 2, 3, 4, 5, 6, 7), (3, 6, 7))$, $\text{signature}(G_\lambda) = (-1)^3 B_3 B_5 B_3 B_1 = 0$.

Finally we need to keep track of the signatures of the connected components of G_λ .

Definition 2.13. Let λ be a neighborly partition. The **signature multiset** $SIG(c)$ of a connected component

$$c = ((k, k+1, k+2, \dots, n), (a_1, a_2, \dots, a_s)), \quad s \geq 1$$

of G_λ is the multiset

$$SIG(c) = \{a_1 - k + 1, a_2 - a_1 + 2, a_3 - a_2 + 2, \dots, a_s - a_{s-1} + 2, n - a_s + 1\}.$$

If $s = 0$ then

$$SIG(c) = \{n - k\}.$$

The signature multiset $SIG(G_\lambda)$ for a general neighborly partition λ is the multiset union over all connected components of the individual signature multisets.

Example 2.14. If $\lambda = ((2, 4, 5, 6, 7, 10, 12, 13, 14), (2, 4, 6, 10, 14))$,

$$G_\lambda = \begin{array}{cccccccc} 2 & 4 \leftrightarrow 5 \leftrightarrow 6 \leftrightarrow 7 & 10 & 12 \leftrightarrow 13 \leftrightarrow 14 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 2 & 4 & 6 & 10 & 14 \end{array}$$

the connected components are

$$2 \leftrightarrow 2, \quad 4 \leftrightarrow 4 \leftrightarrow 5 \leftrightarrow 6 \leftrightarrow 6 \leftrightarrow 7, \quad 10 \leftrightarrow 10, \quad 12 \leftrightarrow 13 \leftrightarrow 14 \leftrightarrow 14.$$

Because the signature is independent of labels, Proposition 2.11 can be applied to each connected component.

$$SIG(G_\lambda) = \{1, 1, 1, 4, 2, 1, 1, 3, 1\} = \{1, 1\} \cup \{1, 4, 2\} \cup \{1, 1\} \cup \{3, 1\}.$$

Remark 2.15. One may find the signature multiset by counting the edges in the chains that the parts of μ_2 cut in the runs of μ_1 .

The signature of any neighborly partition is always 0, 1, or -1 .

Theorem 2.16. Let $\lambda = (\mu_1, \mu_2)$ be a neighborly partition. Then $\text{signature}(G_\lambda) = 0$ exactly when $SIG(G_\lambda)$ contains an element $x \equiv 0 \pmod{3}$. Otherwise,

$$\text{signature}(G_\lambda) = (-1)^{t+s}$$

where t is the number of elements $x \in SIG(G_\lambda)$ such that $x \equiv 1 \pmod{3}$, and s is the number of parts of μ_2 .

Remark 2.17. If $\lambda = (\mu_1, \mu_2)$ is neighborly and $\text{signature}(G_\lambda) \neq 0$, then μ_2 does not contain consecutive parts, and thus μ_2 is a difference 2 partition.

3. ADMISSIBLE NEIGHBORLY PARTITIONS

Theorem 2.16 shows that $\text{signature}(B_\lambda)$ is ± 1 or 0 for any neighborly partition. Thus we can eliminate the inner sum in Theorem 1.1, and replace the set of neighborly partitions by the smaller set of partitions when $\text{signature}(B_\lambda) \neq 0$. These are admissible neighborly partitions.

Definition 3.1. A neighborly partition $\lambda = (\mu_1, \mu_2)$ is **admissible** if $\text{SIG}(B_\lambda)$ contains no elements which are congruent to 0 modulo 3.

Example 3.2. The neighborly partition λ in Example 2.14 is not admissible. A chain with n edges is admissible if 3 does not divide n .

Since admissible neighborly partitions have signature ± 1 , we may rename the signature by the sign.

Definition 3.3. The **sign** of an admissible neighborly partition $\lambda = (\mu_1, \mu_2)$ is

$$\text{sign}(\lambda) = (-1)^{t+s}$$

where t is the number of elements $x \in \text{SIG}(G_\lambda)$ such that $x \equiv 1 \pmod{3}$, and s is the number of parts of μ_2 .

Then Theorem 1.1 is equivalent to the following propositions.

Proposition 3.4. The generating function for all signed admissible neighborly partitions λ is

$$\sum_{\lambda \in \text{AdmNeighborly}} \text{sign}(\lambda) q^{|\lambda|} = \prod_{k=0}^{\infty} (1 - q^{5k+2})(1 - q^{5k+3})(1 - q^{5k+5}) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(5k+1)/2}$$

Example 3.5. There are 4 admissible partitions of $n = 8$, two positive and two negative, so the coefficient of q^8 in Proposition 3.4 is 0.

positive : $\lambda = ((2, 3), (3))$, $\text{SIG}(\lambda) = \{2, 1\}$, $\lambda = ((1, 3), (1, 3))$, $\text{SIG}(\lambda) = \{1, 1, 1, 1\}$,
negative : $\lambda = ((4), (4))$, $\text{SIG}(\lambda) = \{1, 1\}$, $\lambda = ((1, 2, 3), (2))$, $\text{SIG}(\lambda) = \{2, 2\}$.

4. GENERATING FUNCTIONS

In this section we use generating functions to prove the Main Theorem 4.3. It accomplishes goal (1) of the Introduction by choosing $x = 1$.

Definition 4.1. Let $GF_n(q)$ denote the generating function for all signed admissible neighborly partitions with exactly n parts,

$$GF_n(q) = \sum_{\lambda \in \text{Admissible Neighborly with } n \text{ parts}} \text{sign}(\lambda) q^{|\lambda|}.$$

We shall later prove the following recurrence.

Proposition 4.2. The generating function $GF_n(q)$ satisfies the recurrence

$$(1 - q^n)GF_n(q) = -(q^{2n-2} + q^{3n-3})GF_{n-2}(q) + (q^{2n-2} + q^{3n-4} + q^{3n-3})GF_{n-3}(q) - q^{3n-4}GF_{n-4}(q).$$

Our main result is the generating function for admissible neighborly partitions according to number of parts and the sum of the parts.

Theorem 4.3. *The generating function for all signed admissible neighborly partitions is*

$$GF(x, q) = \sum_{n=0}^{\infty} GF_n(q)x^n = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{(q; q)_k} q^{(5k^2-k)/2} (xq; q)_{k-1} (1 - xq^{2k}).$$

Proof. Let $H(x)$ be the right side in Theorem 4.3. Then $H(x)$ has a well-known functional equation [5],

$$\frac{H(x)}{(xq; q)_{\infty}} - \frac{H(xq)}{(xq^2; q)_{\infty}} = qx \frac{H(xq^2)}{(xq^3; q)_{\infty}}.$$

This implies that H_n , the coefficient of x^n in $H(x)$, satisfies

$$(4.1) \quad (1 - q^n)H_n = -q^n(1 - q^{n-1})H_{n-1} - (q^{2n-2} + q^{2n-1})H_{n-2} + q^{2n-2}H_{n-3}.$$

Iterating (4.1) on $(1 - q^{n-1})H_{n-1}$ gives the same recurrence as in Proposition 4.2, so $H_n = GF_n(q)$. \square

5. ANOTHER REALIZATION OF $sign(\lambda)$

In order to prove Proposition 4.2, we need to simplify the graphs G_{λ} , keeping the same vertices and labels, but defining a new sign. This will be done by deleting edges in the chains cut out by the hanging edges to obtain a new graph G'_{λ} , so that $sign(\lambda)$ is now just $(-1)^{\#edges(G'_{\lambda})}$.

Via Theorem 2.16, an admissible neighborly partition λ has $SIG(G_{\lambda})$ with no elements that are multiples of 3. The elements of $SIG(G_{\lambda})$ are the lengths of chains cut out by the hanging edges. We will delete edges in G_{λ} on each subchain by the following rule, always preserving the hanging edges.

If a chain has n edges $\{e_1, e_2, \dots, e_n\}$ from left to right,

- (1) delete edges $\{e_3, e_6, \dots, e_{n-2}\}$ if $n \equiv 2 \pmod{3}$
- (2) delete edges $\{e_3, e_6, \dots, e_{3m}\} \cup \{e_{3m+2}, \dots, e_{6m-1}\}$ if $n = 6m + 1$,
- (3) delete edges $\{e_3, e_6, \dots, e_{3m}\} \cup \{e_{3m+2}, \dots, e_{6m+2}\}$ if $n = 6m + 4$,

Definition 5.1. *Let G'_{λ} denote the graph G_{λ} with these edges deleted.*

Example 5.2. *If $\lambda = ((1, 2, 3, 4, 5, 6, 7), (1, 3, 6))$,*

$$G_{\lambda} = \begin{array}{cccccccc} 1 \leftrightarrow & 2 \leftrightarrow & 3 \leftrightarrow & 4 \leftrightarrow & 5 \leftrightarrow & 6 \leftrightarrow & 7 \\ \updownarrow & & \updownarrow & & & \updownarrow & \\ 1 & & 3 & & & 6 & \end{array}$$

In the chain $1 \leftrightarrow 1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 3$ we delete the third edge $2 \leftrightarrow 3$ to obtain $1 \leftrightarrow 1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 3$. In the chain $3 \leftrightarrow 3 \leftrightarrow 4 \leftrightarrow 5 \leftrightarrow 6 \leftrightarrow 6$ we delete the third edge $4 \leftrightarrow 5$ to obtain $3 \leftrightarrow 3 \leftrightarrow 4 \leftrightarrow 5 \leftrightarrow 6 \leftrightarrow 6$, so

$$G'_{\lambda} = \begin{array}{cccccccc} 1 \leftrightarrow & 2 & 3 \leftrightarrow & 4 & 5 \leftrightarrow & 6 \leftrightarrow & 7 \\ \updownarrow & & \updownarrow & & & \updownarrow & \\ 1 & & 3 & & & 6 & \end{array}$$

Note that the third edge is deleted, along with every next third edge, except for the middle, and the initial and final edges are preserved. Thus all hanging edges and vertex labels are preserved. We need to see how the sign can be preserved.

Proposition 5.3. *For any admissible neighborly partition λ*

$$\text{sign}(\lambda) = (-1)^{\#\text{edges}(G'_\lambda)}.$$

Proof. Let's first check that any chain in G_λ with $3m + 1$ edges has an odd number of edges in G'_λ , while chains in G_λ with $3m + 2$ edges have an even number of edges in G'_λ .

In the second case, if $n = 3m + 2$, the sign is $+1$ and the number of edges is $n - (n - 2)/3 = 2m + 2$ which is even. For the first case, if $n = 6m + 1$, the sign is -1 and the number of edges is $n - 2m = 4m + 1$ which is odd. For the first case, if $n = 6m + 4$, the sign is -1 and the number of edges is $n - 2m - 1 = 4m + 3$ which is odd. Finally, $\text{sign}(\lambda)$ in Theorem 2.16 also includes a factor of $(-1)^s$, where s is the number of hanging edges. Each hanging edge occurs in 2 chains, so this factor compensates for double counting these edges. \square

Since we are deleting every third edge from G_λ to obtain G'_λ , the connected components of G'_λ are small and limited.

Proposition 5.4. *For any admissible neighborly partition λ , the connected components of G'_λ are one of six types*

$$\begin{aligned} a \leftrightarrow a, \quad a \leftrightarrow a + 1, \quad a \leftrightarrow a \leftrightarrow a + 1, \quad a \leftrightarrow a + 1 \leftrightarrow a + 1, \\ a \leftrightarrow a + 1 \leftrightarrow a + 2, \quad a \leftrightarrow a + 1 \leftrightarrow a + 1 \leftrightarrow a + 2. \end{aligned}$$

Finally we use these six possible connected components to prove Proposition 4.2.

Proof of Proposition 4.2. Since $q^n GF_n(q)$ is the signed generating function with n parts and no 1, $(1 - q^n)GF_n(q)$ is the generating function for signed admissible neighborly partitions with n parts that include a part of size 1. The first connected component in any G'_λ must contain a 1 and be one of the six graphs in Proposition 5.4.

- (1) If the first component is $1 \leftrightarrow 1$, the remaining $n - 2$ vertices have labels at least 3, and the signed generating function is $-q^2 q^{2(n-2)} GF_{n-2}(q)$.
- (2) If the first component is $1 \leftrightarrow 2$, the remaining $n - 2$ vertices have labels at least 4, and the signed generating function is $-q^3 q^{3(n-2)} GF_{n-2}(q)$.
- (3) If the first component is $1 \leftrightarrow 1 \leftrightarrow 2$, the remaining $n - 3$ vertices have labels at least 3, and the signed generating function is $q^4 q^{2(n-3)} GF_{n-3}(q)$. This is because deleting $1 \leftrightarrow 1 \leftrightarrow 2$ removes 3 vertices, possibly from the first chain, so its mod 3 value is unchanged, and remains admissible
- (4) If the first component is $1 \leftrightarrow 2 \leftrightarrow 2$, the remaining $n - 3$ vertices have labels at least 4, and the signed generating function is $q^5 q^{3(n-3)} GF_{n-3}(q)$.

- (5) If the first component is $1 \leftrightarrow 2 \leftrightarrow 3$, the remaining $n - 3$ vertices have labels at least 4, and the signed generating function is $q^6 q^{3(n-3)} GF_{n-3}(q)$. As before we are deleting 3 vertices, so admissibility is preserved.
- (6) If the first component is $1 \leftrightarrow 2 \leftrightarrow 2 \leftrightarrow 3$, the remaining $n - 4$ vertices have labels at least 4, and the signed generating function is $-q^8 q^{3(n-4)} GF_{n-4}(q)$.

These are the six terms in Proposition 4.2. □

6. REMARKS

A topological explanation of Proposition 2.8 via an Euler characteristic is given in [2, Cor. 6.3].

The second Rogers-Ramanujan identity has a similar interpretation.

Proposition 6.1. *The signed generating function for all admissible neighborly partitions γ without a part of size 1 is*

$$\begin{aligned} \sum_{\gamma} \text{sign}(\gamma) q^{|\gamma|} &= \prod_{k=0}^{\infty} (1 - q^{5k+4})(1 - q^{5k+5})(1 - q^{5k+6}) \\ &= \sum_{k=0}^{\infty} (-1)^k q^{k(5k+3)/2} (1 + q + \cdots + q^{2k}). \end{aligned}$$

One may use a version of Proposition 4.2 which counts edges in G'_{λ} to prove the next proposition.

Proposition 6.2. *The generating function for signed admissible partitions λ such that G'_{λ}*

- (1) *has $2n$ vertices and $n + j$ edges is*

$$(-1)^{n+j} \frac{(-q; q^2)_{n-j} (q^{2n-2j-1}; q^{-2})_j}{(q^2; q^2)_{2j} (q^2; q^2)_{n-2j}} q^{2(n-j)^2 + 4j^2 + 2j},$$

- (2) *has $2n + 1$ vertices and $n + j + 1$ edges is*

$$(-1)^{n+j} \frac{(-q; q^2)_{n-j} (q^{2n-2j-1}; q^{-2})_j}{(q^2; q^2)_{2j+1} (q^2; q^2)_{n-2j-1}} q^{2(n-j)^2 + 4j^2 + 6j + 2}.$$

We do not know a proof of Theorem 4.3 using Proposition 6.2.

It is classically known [5] that

$$(6.1) \quad GF(x, q) = (xq; q)_{\infty} \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q; q)_k} x^k = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{(q; q)_k} q^{(5k^2-k)/2} (xq; q)_{k-1} (1 - xq^{2k})$$

also satisfies Proposition 4.2.

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