OCTABASIC LAGUERRE POLYNOMIALS AND PERMUTATION STATISTICS

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Abstract. A set of orthogonal polynomials with 8 independent "q's" is defined which generalizes the Laguerre polynomials. The moments of the measure for these polynomials are the generating functions for permutations according to eight different statistics. Specializing these statistics gives many other well-known sets of combinatorial objects and relevant statistics. The specializations are studied, with applications to classical orthogonal polynomials and equidistribution theorems for statistics.

1. Introduction.

In this paper we study a set of orthogonal polynomials which generalize the Laguerre polynomials. There are many possible ways to do this, for example by considering any of the polynomials above the Laguerre on the "Askey tableau" [As-Wl], or the q-Laguerre polynomials [Mo]. Here we take a family of polynomials which have "8 independent q's", and thus infinitely many possible "q-analogs." We refer to these polynomials as "octabasic Laguerre" polynomials.

The advantage of the 8 q's is that they can be specialized in many different ways, to obtain other sets of polynomials, for example Charlier, Chebyshev, and Hermite. Combinatorially, the specialization to Charlier polynomials is equivalent to embedding set partitions inside permutations, and the specialization to Chebyshev polynomials is equivalent to embedding non-crossing set partitions inside permutations. Our model for the octabasic Laguerre polynomials provides simultaneously these embeddings, as well as embeddings of other combinatorial objects (e.g., involutions) inside permutations.

Our setting also allows for a uniform study of statistics on permutations, set partitions, non-crossing partitions, involutions, and other families of combinatorial objects.

The octabasic Laguerre polynomials are given by the three term recurrence relation

$$p_{n+1}(x) = (x - b_n)p_n(x) - \lambda_n p_{n-1}(x), \quad p_0(x) = 1, \ p_{-1}(x) = 0,$$

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where

$$b_n = a[n+1]_{r,s} + b[n]_{t,u}, \quad \lambda_n = ab[n]_{p,q}[n]_{v,w}, \tag{1.1}$$

and

$$[n]_q = \frac{1-q^n}{1-q}, \quad [n]_{r,s} = \frac{r^n - s^n}{r-s} = r^{n-1} + r^{n-2}s + \cdots + s^{n-1}.$$

It is clear that by rescaling x in (1.1), we could take $b = 1$.

The 8 q's are r, s, t, u, p, q, v, and w.

If we replace each occurrence of $[n]_{c,d}$ in (1.1) by n, and put $a = b = 1$, then we recover the recurrence relation for the Laguerre polynomials $L_n^0(x)$. So the octabasic Laguerre polynomials are multi-q versions of $L_n^0(x)$. It is possible to define octabasic versions of $L_n^\alpha(x)$, see (11.4).

2. Moments.

For the Laguerre polynomials, there are explicit formulas for the polynomials, measure, and moments. The octabasic Laguerre polynomials are too general to have such explicit facts known. In this section we use the Viennot theory [V1, V2] to give a combinatorial version of the moments in Theorem 2.1 and Theorem 2.2.

The n^{th} moment for $L_n^0(x)$ is $\mu_n = n!$. So one would expect the moments of the octabasic Laguerre polynomials to be generating functions for permutations counted according to certain statistics. This is indeed the case and the precise statement appears in Theorems 2.1 and 2.2. Specializations of Theorem 2.1, which reduce the moments to $n!_q$, are given in [Si-St].

For the definition of the statistics, it is convenient to represent a permutation σ as a word $\sigma(1)\sigma(2)\cdots\sigma(n)$ consisting of increasing runs, separated by the descents of the permutation. For example, the permutation $\sigma = 26|357|4|189$ has 4 runs separated by 3 descents. The runs of length 2 or more will be called *proper runs* and those of length 1 will be called *singleton runs*. We write $\mathrm{run}(\sigma)$ for the total number of runs in σ.

The elements $\sigma(i)$ of σ fall into four classes: the elements which begin proper runs (openers), the elements which close proper runs (closers), the elements which form singleton runs (singletons), and the elements which continue runs (continuators). We shall abbreviate these classes of elements "op", "clos", "sing", and "cont" respectively. In the example, $\mathrm{op}(\sigma) = \{2, 3, 1\}$, $\mathrm{clos}(\sigma) = \{6, 7, 9\}$, $\mathrm{sing}(\sigma) = \{4\}$, and $\mathrm{cont}(\sigma) = \{5, 8\}$.

Definition 2.1. *For $\sigma \in S_n$, the statistics lsg(σ) and rsg(σ) are defined by*

$$lsg(\sigma) = \sum_{i=1}^{n} lsg(i), \qquad rsg(\sigma) = \sum_{i=1}^{n} rsg(i),$$

where lsg(i) = the number of runs of σ strictly to the left of i which contain elements smaller and greater than i, and rsg(i) = the number of runs of σ strictly to the right of i which contain elements smaller and greater than i.

We also define lsg and rsg on the openers of σ

$$lsg(op)(\sigma) = \sum_{i \in \mathrm{op}(\sigma)} lsg(i), \qquad rsg(op)(\sigma) = \sum_{i \in \mathrm{op}(\sigma)} rsg(i).$$

Each of the statistics on the remaining three classes of elements have analogous definitions.

For example, if $\sigma = 26|357|4|189$, then $\mathrm{lsg}(7) = 0$, $\mathrm{rsg}(7) = 1$, $\mathrm{lsg}(\mathrm{op})(\sigma) = 0{+}1{+}0 = 1$, $\mathrm{rsg}(\mathrm{op})(\sigma) = 1{+}1{+}0 = 2$, $\mathrm{lsg}(\mathrm{clos})(\sigma) = 0$, $\mathrm{rsg}(\mathrm{clos})(\sigma) = 2{+}1{+}0 = 3$, etc.

Theorem 2.1. *(* [Si-St]*, Thm. 1) The n^{th} moment μ_n for the octabasic Laguerre polynomials is*

$$\mu_n = \sum_{\sigma \in S_n} r^{\mathrm{lsg}(\mathrm{sing})(\sigma)} s^{\mathrm{rsg}(\mathrm{sing})(\sigma)} t^{\mathrm{lsg}(\mathrm{cont})(\sigma)} u^{\mathrm{rsg}(\mathrm{cont})(\sigma)} p^{\mathrm{lsg}(\mathrm{op})(\sigma)} q^{\mathrm{rsg}(\mathrm{op})(\sigma)}$$
$$v^{\mathrm{lsg}(\mathrm{clos})(\sigma)} w^{\mathrm{rsg}(\mathrm{clos})(\sigma)} a^{\mathrm{run}(\sigma)} b^{n-\mathrm{run}(\sigma)}.$$

The proof of Theorem 2.1 which appears in [Si-St] uses a bijection from weighted Motzkin paths to permutations. Biane [Bi] gave another bijection for the same Motzkin paths. It leads to another eight variable generating function for the moments, which we give in Theorem 2.2. This time we consider the cycle decomposition of σ. Elements i lie in four disjoint sets: *peaks* $(\sigma^{-1}(i) < i > \sigma(i))$, *valleys* $(\sigma^{-1}(i) > i < \sigma(i))$, *double descents* $(\sigma^{-1}(i) > i > \sigma(i))$, and *double ascents* $(\sigma^{-1}(i) < i < \sigma(i))$. We consider a fixed point $(i = \sigma(i))$ as a degenerate double descent. We use the abbreviations "pe", "va", "dd", and "da" for these four sets. For example, if $\sigma = (1\ 7\ 2)(3)(4\ 9\ 8\ 6)(5)(10\ 11)$, then $\mathrm{pe} = \{7, 9, 11\}$, $\mathrm{va} = \{1, 4, 10\}$, $\mathrm{dd} = \{2, 3, 5, 6, 8\}$, $\mathrm{da} = \varnothing$. Strictly speaking, the peaks, valleys, etc. considered here are *cyclic* peaks, valleys, etc. (they are defined in terms of the cycle decomposition). Their definition differs from that of the same terms occurring elsewhere in the literature, where they might be called *linear* peaks, valleys, etc. (defined in terms of the 2-line representation of a permutation; e.g., $\sigma(i-1) < \sigma(i) > \sigma(i+1)$ for linear peaks).

We need another pair of statistics, replacing lsg and rsg. Given i let

$$l1(i, \sigma) = |\{j : j < i, j < \sigma(i), \sigma^{-1}(j) > i\}|,$$

and

$$l2(i, \sigma) = |\{j : j \le i, j > \sigma(i), \sigma^{-1}(j) > i\}|.$$

For a set $S \subseteq [n]$ and a permutation ρ, we define $l1(S, \rho) := \sum_{i \in S} l1(i, \rho)$ and $l2(S, \rho) := \sum_{i \in S} l2(i, \rho)$.

Theorem 2.2. *The n^{th} moment μ_n for the octabasic Laguerre polynomials is*

$$\mu_n = \sum_{\sigma \in S_n} r^{\mathrm{l1}(\mathrm{dd}(\sigma),\sigma)} s^{\mathrm{l2}(\mathrm{dd}(\sigma),\sigma)} t^{\mathrm{l1}(\mathrm{da}(\sigma),\sigma^{-1})} u^{\mathrm{l2}(\mathrm{da}(\sigma),\sigma^{-1})} p^{\mathrm{l1}(\mathrm{pe}(\sigma),\sigma)} q^{\mathrm{l2}(\mathrm{pe}(\sigma),\sigma)}$$
$$v^{\mathrm{l1}(\mathrm{pe}(\sigma),\sigma^{-1})} w^{\mathrm{l2}(\mathrm{pe}(\sigma),\sigma^{-1})} a^{\mathrm{pe}(\sigma)+\mathrm{dd}(\sigma)} b^{\mathrm{pe}(\sigma)+\mathrm{da}(\sigma)}.$$

3. Set partitions.

We are interested in specializations of the parameters p, q, r, s, t, u, v, w in the octabasic Laguerre polynomials. Of particular interest are those specializations which will reduce the moments to generating functions for various types of (unordered) set partitions.

Before discussing such specializations, we devote this section to the elucidation of the relation, useful for our purposes, between permutations and set partitions. We begin with some definitions and convenient notation.

As usual, a *partition* of the set $[n]:=\{1,2,\ldots,n\}$ is a collection of non-empty pairwise disjoint sets, called *blocks*, whose union is $[n]$. Neither the order of the elements inside a block, nor the ordering among blocks is relevant in a partition, and, as is done generally, we will assume that the blocks $B_1, B_2, \ldots, B_k$ of a partition are indexed in increasing order of their minimum elements. Using this *standard indexing* of the blocks, a set partition of $[n]$ can also be represented by its *restricted growth function* [Wa-Wh], [Mi], "RG function", which for our purposes is the word $w = w_1 w_2 \ldots w_n$ in which w_i is the index of the block containing the element i. For example, the set partition 1 6 8 / 2 3 5 / 4 / 7 has RG function 1 2 2 3 2 1 4 1. The terminology reflects the property of RG functions (due to the standard indexing of the blocks) that $w_i \leq 1 + max\{w_j : j < i\}$ for all i.

In contrast with a set partition, a *set composition* of $[n]$ is an *ordered* collection of non-empty pairwise disjoint sets whose union is $[n]$. There are, of course, $k!$ set compositions having the same underlying partition into k blocks. We extend the idea of a RG function to that of *block indexing function*, "BI function", defined for set compositions. Thus, the set composition 4 / 1 6 8 / 7 / 2 3 5 has BI function 2 4 4 1 4 2 3 2.

Obviously, if $S(n,k)$ denotes the Stirling number of the second kind, we have: the total number of partitions of $[n]$ is $\sum_k S(n,k)$, the n^{th} Bell number, and the total number of set compositions of $[n]$ is $\sum_k k!S(n,k)$. Clearly, the number of set compositions of $[n]$ is larger than the number of permutations of $[n]$. We will define an equivalence relation on set compositions which has $n!$ equivalence classes, and then a bijection between a set of class representatives and permutations. This bijection will permit us to translate the statistics lsg and rsg defined on permutations, into the language of set compositions. Consequently, specializations of Theorem 2.1 will be expressible in terms of set compositions with suitable side conditions and, in turn, many of these specializations will give rise to results about various types of set partitions.

On the set $SC[n]$ of compositions of $[n]$ we consider the relation $\sim$ defined by: $\pi \sim \rho$ if $\pi = \rho$ or if ρ can be obtained from π through interchanges of adjacent blocks which are "separated." By *separated blocks* we mean that all elements of one block are larger than all elements of the other. For example, 1 6 7 / 2 3 5 / 4 /8 $\sim$ 1 6 7 / 2 3 5 / 8 / 4 $\sim$ 1 6 7 / 8 / 2 3 5 / 4 $\sim$ 8 / 1 6 7 / 2 3 5 / 4. It is easy to verify that $\sim$ is an equivalence relation.

Lemma 3.1. *Each equivalence class of the relation $\sim$ contains exactly one canonical set composition, that is, a set composition with the property that each block's maximum element is larger than the minimum of the next block (if the latter exists).*

Proof. Let $\pi = B_1, B_2, \ldots, B_k$ be a set composition of $[n]$. Suppose that $max\{a \in$

$B_i\} > min\{b \in B_{i+1}\}$ for every $1 \leq i < j$. If $j = k$, then π itself is canonical. On the other hand, if $max\{a \in B_j\} < min\{b \in B_{j+1}\}$, then B_j and B_{j+1} are separated. Let l be the minimum index such that for every $l \leq t \leq j$ we have B_{j+1} separated from B_t and $max\{a \in B_{j+1}\} > min\{a \in B_t\}$. Then $\pi \sim \pi' := B_1, B_2, \ldots, B_{l-1}, B_{j+1}, B_l, B_{l+1}, \ldots, B_j, B_{j+2}, \ldots, B_k$. Furthermore, in π' the maximum of the i^{th} block is larger than the minimum of the $(i+1)^{st}$ block for $1 \leq i < j+1$. This follows from the assumptions on j and l, together with the observation that if two blocks are *not* separated, then the maximum of each is larger than the minimum of the other. Now replace π with π', recompute j (which will now be larger than its previous value), and repeat until — after at most k iterations — a canonical set composition c is reached such that $\pi \sim \pi' \sim \cdots \sim c$. Thus, every set composition is $\sim$ equivalent with some canonical composition.

It remains to show that no two canonical set compositions are in the same $\sim$ class. Let c be canonical and define a partial order on the blocks of c, namely the transitive closure of $B < C$ if B precedes C in c and they are non-separated blocks. Denote this poset as $P(c)$. Suppose now that $c \sim c'$, where c' is canonical as well. Then c and c' must have the same underlying partition and $P(c) = P(c')$. Moreover, c and c' correspond to linear extensions of this partial order, having the property that the maximum of each block is larger than the minimum of the following block. We claim that the poset admits only one such extension, hence, $c = c'$.

Specifically, this (unique) extension $B_1, B_2, \ldots, B_k$ has the property that for each i, B_i is the minimal element of $P(c;i) := P(c) - \{B_1, B_2, \ldots, B_{i-1}\}$ which has the largest minimum element. For suppose that B and C are two minimal elements of $P(c;i)$. This implies that they are separated blocks and, say, all elements of B are smaller than all elements of C. Suppose that $B_i = B$ and $B_j = C$ for some $j > i$. We may and shall assume that none of $B_{i+1}, B_{i+2}, \ldots, B_{j-1}$ is a minimal element in $P(c;i)$. Since we are constructing a linear extension, $B_{i+1}, \ldots, B_{j-1}$ must be incomparable to $B_j = C$, that is, they must be separated from C as blocks. However, since in our linear extension the maximum of a block must be larger than the minimum of the next block, there is a smallest l, $i+1 \leq l \leq j-1$, such that the maximum of B_l is larger than the minimum of $B_j = C$. But B_l and C must be separated, so B_l must lie entirely to the right of C, and – by the choice of B_l – we must have B_{l-1} entirely to the left of C. This means that the maximum of B_{l-1} fails to be larger than the minimum of B_l, contradicting the condition necessary for the linear extension to be a canonical set composition. □

In view of Lemma 3.1, the canonical set compositions are distinct representatives of the equivalence classes of $\sim$, and we will refer to them as *canonical representatives.*

Observe that if $\pi = B_1, B_2, \ldots, B_k$ is a partition of $[n]$, then the pairs (f_i, l_i), $i = 1, 2, \ldots, k$, where $f_i := \min\{a \in B_i\}$ and $l_i := \max\{a \in B_i\}$, determine the number of $\sim$ classes for the set compositions having π as their underlying set partition. In fact, if we define a directed graph having $v_i = (f_i, l_i)$ as vertices, with an edge from v_i to v_j if and only if $l_i > f_j$, then each hamiltonian path in this graph gives the ordering of the blocks for a different canonical representative, and conversely. While this description of canonical representatives is not computationally useful for large n, it can prove helpful in certain arguments.

Proposition 3.2. *There is a bijection between the equivalence classes $SC[n]/\sim$ and permutations in S_n.*

Proof. Given a canonical representative of a $\sim$ class, $c = B_1, B_2, \ldots, B_k$, write the elements in each block in an increasing sequence, and let $\sigma \in S_n$ be the permutation $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)$ obtained by concatenating these sequences in the same order as the order of the blocks. By Lemma 3.1, each block constitutes a run in σ and, conversely, if we form a block out of the elements in each run of a permutation then the resulting set composition is a canonical representative of a class in $SC[n]/\sim$. $\square$

We can clearly use Proposition 3.2 and Theorem 2.1 to interpret the moments of our octabasic polynomials as generating functions for statistics on the BI functions of canonical representatives of set compositions. The notions of opener, closer, singleton, and continuation elements defined for permutations apply also to the BI functions. They correspond, respectively, to the first occurrence of a repeated letter, the last occurrence of a repeated letter, a letter which occurs only once, and a repeated letter which is neither the first nor the last of its kind. For example, in the BI function $w = 2\ 3\ 3\ 4\ 3\ 2\ 2\ 1$ of the set composition $8\ /\ 1\ 6\ 7\ /\ 2\ 3\ 5\ /\ 4$, $w_2 = 3$ is an opener, $w_3 = 3$ is a continuator, and $w_5 = 3$ is a closer. These three 3's correspond to the permutation run 2 3 5.

The statistics lsg and rsg on permutations become lrs (left right smaller) and lrg (left right greater) on the BI function for canonical representatives of set compositions. More precisely, if w is a BI function for a set composition of $[n]$, then

$$lrs(i) := |\{I < w_i \colon\ \exists\ j < i < k\ s.t.\ w_j = w_k = I\}|,$$

$$lrg(i) := |\{I > w_i \colon\ \exists\ j < i < k\ s.t.\ w_j = w_k = I\}|,$$

and

$$lrs(w) := \sum_{i=1}^{n} lrs(i), \qquad lrg(w) := \sum_{i=1}^{n} lrg(i).$$

A run which contributes to $lsg(i)$ for some element i of a permutation σ becomes, via our bijection, a block of smaller index than that containing i and which has elements both to the left and to the right of i, thus contributing to $lrs(i)$ on the BI function. Similarly, $rsg(i) = lrg(i)$. So, if $\sigma \in S_n$ corresponds with the BI function w of a canonical representative, then

$$lsg(\sigma) = lrs(w), \qquad rsg(\sigma) = lrg(w).$$

Analogous equalities hold for the statistics on openers, closers, singletons, and continuators.

The remainder of this section consists of preliminaries to our discussion of specializations of the octabasic Laguerre polynomials obtained from setting subsets of the q's equal to zero. The results of the following propositions provide conditions, in terms of our statistics, which lead from canonical set compositions to set partitions.

Proposition 3.3. *Let $c = B_1, B_2, \ldots, B_k$ be a canonical set composition of $[n]$ and let $w = w_1 w_2 \ldots w_n$ be its BI function.*

(1) *For w, $lrg(clos) = lrg(sing) = 0$ if and only if the reverse of w, $w^{rev} := w_n w_{n-1} \ldots w_1$, is an RG function.*
(2) *For w, $lrs(op) = lrs(sing) = 0$ if and only if the complement of w, $w^c := (n+1-w_1)(n+1-w_2)\ldots(n+1-w_n)$, is an RG function.*

Proof. To prove (1), we begin by showing that if i is the largest element of the m^{th} block of c, $m > 1$, then there is some $j > i$ such that $w_j = m-1$. Since c is canonical, the maximum of B_{m-1} is larger than the minimum of B_m. But if $lrg(clos) = lrg(sing) = 0$, then the maximum of B_{m-1} cannot lie between the minimum and maximum of B_m, so it must be larger than the maximum of B_m. Thus, there is $w_j = m-1$ for some $j > i$, that is, w^{rev} is an RG function. Conversely, suppose w^{rev} is an RG function. Then in w, the last (rightmost) occurrence of each $m \in [k]$ must be of the form $w_{i_m} = m$ with $1 \le i_k < i_{k-1} < \cdots < i_1 \le n$. Hence, in w, we have $lrg(i_m) = 0$ for all m, i.e., $lrg(clos) = lrg(sing) = 0$ for w.

The proof of (2) is similar. We need to show that if i is the smallest element of B_m, $m < k$, then there is some $j < i$ such that $w_j = m+1$. Again, since c is canonical, the maximum of B_m is larger than the minimum of B_{m+1}. But $lrs(op) = lrs(sing) = 0$, so the minimum of B_{m+1} must in fact be smaller than i. So there is $w_j = m+1$ with $j < i$. Conversely, if w^c is an RG function, then the first (leftmost) occurrence of each $m \in [k]$, must be of the form $w_{i_m} = m$ with $1 \le i_k < i_{k-1} < \cdots < i_1 \le n$. Consequently, in w, $lrs(i_m) = 0$ for every m, so $lrs(op) = lrs(sing) = 0$ for w. □

One of the specializations considered in the next section gives rise to non-crossing partitions (see [Si1] for references). A partition of $[n]$ is *non-crossing* if for every four elements $1 \le a < b < c < d \le n$ the following condition is satisfied: if a, c are in the same block and b, d are in the same block, then all four elements are in the same block. It is well-known that the number of non-crossing partitions of $[n]$ is the n^{th} Catalan number, $C_n = \frac{1}{n+1}\binom{2n}{n}$. An RG function w represents a non-crossing partition if and only if whenever $w_i = w_j$ for some $1 \le i < j \le n$, we have $w_r \ge w_i$ for all $i < r < j$. This is easy to verify using the observation that a crossing between two blocks B_a and B_b is equivalent to the presence of a subword $abab$ or $baba$ in the RG function.

Proposition 3.4. *Let w be the BI function of an arbitrary set composition. If $lrg(w) = 0$ or $lrs(w) = 0$, then the underlying partition is non-crossing.*

Proof. If the underlying partition has two crossing blocks, then any set composition of it will contain $ijij$ as a subword in its BI function w, for some $i \ne j$. We will therefore have $lrs(w) > 0$ and $lrg(w) > 0$ independently of whether $i < j$ or $i > j$. □

Proposition 3.5. *Each non-crossing partition has precisely two canonical compositions whose BI functions w^s and w^g satisfy $lrs(w^s) = 0$ and $lrg(w^g) = 0$.*

Proof. If two blocks of a non-crossing partition are not separated, then all elements of one lie "nested" between two successive elements of the other. If we seek a canonical set composition c for which lrs is null, then its associated poset $P(c)$ (as introduced in the proof of Lemma 3.1) is completely determined by the fact that each block must be smaller in $P(c)$ than all blocks in which it is nested. Then w^s is the BI function of the set composition resulting from the unique linear extension of $P(c)$ which gives a canonical set composition. The canonical set composition with BI function w^g follows from the dual of the poset above, since this time we wish to give each block a larger index than that of every block in which it is nested. $\square$

The next Proposition will make it easy to understand several of the zero specializations of the octabasic Laguerre polynomials.

Proposition 3.6. *Let w be the BI function of a canonical set composition. Then $lrs(clos) = lrg(clos) = 0$ if and only if every two non-singleton blocks are separated. Also, $lrs(op) = lrg(op) = 0$ if and only if every two non-singleton blocks are separated.*

Proof. Suppose that $lrs(j) > 0$ for some element $j \in op(w) \cup cont(w)$, and, say, $j \in B_b$. Then there exists a block B_a and $1 \leq i < j < l \leq n$ such that $w_i = w_l = a$, and $a < b$. We may assume that i and l are the minimum and maximum of B_a. Now, the closer which is the maximum of the block B_b cannot be larger than l, otherwise $lrg(l) > 0$ contradicting $lrg(clos) = 0$, nor can it be smaller than l because then it would lie between j and l, hence between i and l, and would cause $lrs(clos) > 0$. Similarly, we reach a contradiction if we suppose that $lrg(j) > 0$ for $j \in op(w) \cup cont(w)$. Thus, $lrs(j) > 0$ or $lrg(j) > 0$ implies $j \in sing(w)$. From this, it follows easily that every two non-singleton blocks must be separated. The converse is trivial. The second statement has an entirely similar proof which we omit. $\square$

An additional class of set partitions which will arise from our zero-specializations is that of *non-overlapping set partitions*. A partition is non-overlapping ("NOP") if for every two blocks B and B' which are not separated, if $min\{a \in B\} < min\{a \in B'\}$, then $max\{a \in B\} > max\{a \in B'\}$. The term "non-overlapping set partition" is adopted from Flajolet [Fl-Sch] where asymptotic results are established. It turns out that NOP's can be characterized via our statistics.

Proposition 3.7. *If the BI function w of a set composition satisfies $lrs(op) = lrs(clos) = 0$, then the underlying partition is non-overlapping. Each non-overlapping partition has exactly one indexing of its blocks which gives a canonical set composition.*

Proof. If two blocks, B_i and B_j, $i < j$, of a canonical set composition are *not* separated, then $min\{a \in B_j\} < min\{a \in B_i\} < max\{a \in B_i\} < max\{a \in B_j\}$ is the only relation on their minima and maxima allowed by the condition $lrs(op) = lrs(clos) = 0$. Hence, the underlying partition is indeed non-overlapping.

As in the proof of Proposition 3.5, a NOP has a unique indexing of its blocks which produces a canonical set composition, namely, the blocks must be indexed in decreasing order of their minimum elements. $\square$

Another class of set partitions which will arise from our zero-specializations is that of partitions in which every pair of non-separated blocks, B_a and B_b, satisfies one of the following two conditions: (i) if $minB_a < minB_b \leq maxB_b < maxB_a$, then B_a, B_b do not cross and $|B_b| \leq 2$; or (ii) if $minB_a < minB_b < maxB_a <$

$maxB_b$, then $i \in B_b$, $i < maxB_a$ imply $i = minB_b$, and $i \in B_a$, $i > minB_b$ imply $i = maxB_a$. We call such a set partition *minimally non-separated*.

Proposition 3.8. *If the BI function w of a set composition satisfies $lrs(sing) = lrs(op) = 0$ and $lrs(cont) = lrg(cont) = 0$, then the underlying partition is minimally non-separated. Each minimally non-separated partition has exactly one indexing of its blocks which gives a canonical set composition with $lrs(sing) = lrs(op) = 0$.*

Proof. The condition $lrs(op) = lrs(sing) = 0$ implies, by Proposition 3.3(2), that w^c, the complement of w, is an RG function. Since $lrs(cont) = lrg(cont) = 0$, w^c satisfies the conditions defining the RG function for a minimally non-separated partition.

If we seek a canonical set composition with a given minimally non-separated underlying partition and whose BI function has $lrs(op) = lrs(sing) = 0$, then the only allowable indexing of the blocks is in decreasing order of the block minima. □

A similar characterization can be given for *"not under"* set partitions ("NU"). These are set partitions whose RG function contains no subword of the form *abba* with $a < b$.

Proposition 3.9. *If the BI function w of a set composition satisfies $lrs(sing) = lrs(op) = 0$ and $lrg(cont) = lrg(clos) = 0$, then the underlying partition is NU. Each NU partition has exactly one indexing of its blocks which gives a canonical set composition with $lrs(sing) = lrs(op) = 0$.*

Proof. As in the preceding proof, $lrs(sing) = lrs(op) = 0$ implies that w^c is an RG function. If, in addition, $lrg(cont) = lrg(clos) = 0$ for w, then $lrs(cont) = lrs(clos) = 0$ for w^c, so w^c contains no subword *abba* with $a < b$. □

4. Moments of the zero specializations.

We are particularly interested in the specializations of the octabasic Laguerre polynomials which arise from putting a subset of the q's equal to 0. A priori there are $2^8 = 256$ such specializations. It is clear, however, from the recurrence relation (1.1), that the polynomials are symmetric under each interchange of bases belonging to the same pair (for example, the interchange of r and s), and under the interchange of the pairs $\{p, q\}$ and $\{v, w\}$. These interchanges generate a group G of order 32 which acts on the boolean algebra of 8 elements. There are 54 orbits of this action. Each orbit is described by a four-tuple of integers, whose entries are 0, 1, or 2. These indicate, for each of the four pairs of bases, whether neither, one, or both bases are set equal to zero. For example, 1020 denotes the orbit where the subset of null bases has 1 element from $\{r, s\}$, none from $\{t, u\}$, 2 from $\{p, q\}$, and none from $\{v, w\}$. So $r = p = q = 0$ belongs to the orbit 1020.

In this section combinatorial interpretations of the moments of the specializations are given.

One way to understand the 54 orbits is to put the non-zero q's equal to 1, put $a = b = 1$, and identify combinatorially the resulting moments. For example, if none of the q's are put equal to 0, then the moments are 1, 1, 2, 6, 24, 120, 720,..., just the number of permutations of $[n]$.

The 54 orbits fall into two categories. There are 27 orbits which give $\lambda_n = 0$ for $n > 1$, so they do not correspond to orthogonal polynomials with positive measures. Nonetheless, the "moments" have combinatorial interpretations which we give here. In the quotient of the boolean algebra by the group G, these 27 orbits fall into three intervals: [0020, 0222], [1020,1222], and [2020,2222]. Each of these intervals contains 9 orbits. The remaining 27 orbits correspond to true orthogonal polynomials.

The 9 orbits in the interval [0020,0222] specialize the moments to 1, 1, 2, 6, 20, 68, 232, 792, ... From Proposition 3.6 applied to 0020, we see that μ_n is the number of $\sim$ classes of set compositions of $[n]$ whose non-singleton blocks are separated. For the canonical representatives of such compositions, the statistics lrs and lrg are null on continuing elements and on closers. Hence, all orbits in the interval [0020, 0222] give indeed the same sequence of moments. This sequence appears in [Ri]. The explicit formula for μ_n is

$$\mu_n = a^n + \sum_{j=1}^{n/2} \sum_{k=0}^{n-2j} \binom{k+j}{j} \binom{n-j-k-1}{j-1} a^{k+j} (1 + a(r+s))^{n-2j-k}. \tag{4.1}$$

The generating function for μ_n is

$$\sum_{n=0}^{\infty} \mu_n z^n = \frac{1 - z - a(r+s)z}{1 - z - az - a(r+s)z + a^2(r+s)z^2}. \tag{4.2}$$

Based on Proposition 3.6, a routine counting argument verifies that (4.1) follows from Theorem 2.1, and the Lagrange inversion formula proves (4.2) from (4.1). By partial fractions, with t_1, t_2 being the roots of the denominator in (4.2), we obtain

$$\mu_n = \frac{1}{a^2(r+s)} \Big[\frac{t_1^{-n-1} - t_2^{-n-1}}{t_2 - t_1} - (1 + a(r+s)) \frac{t_1^{-n} - t_2^{-n}}{t_2 - t_1} \Big].$$

Riordan [Ri] gave the explicit formula

$$\mu_n(r = s = a = 1) = \sum_{\substack{p=-n \\ p \equiv 0,3 \mod 8}}^{n} \binom{2n}{n-p} - \sum_{\substack{p=-n \\ p \equiv 1,4 \mod 8}}^{n} \binom{2n}{n-p}. \tag{4.3}$$

It is easy to prove (4.3) from (4.2).

The second interval of 9 orbits is [1020, 1222], with moments 1, 1, 2, 5, 13, 34, 89, ...: the Fibonacci numbers of even index. Indeed, the moments for the orbit 1020 count (by Propositions 3.3(2) and 3.6) the set *partitions* whose non-singleton blocks are separated. With this interpretation, it is easy to establish that the moments satisfy the recurrence $\mu_n = 3\mu_{n-1} - \mu_{n-2}$, $\mu_0 = \mu_1 = 1$, which implies that for $n > 0$, $\mu_n = F_{2n}$. The explicit formula for the μ_n can be found by putting $r = 0$ in (4.1). As in the preceding case, all 9 orbits in the interval [1020,1222] lead to the same moments.

The third interval of 9 orbits is [2020, 2222], with moments 1, 1, 2, 4, 8, 16, 32, ... The combinatorial objects associated with the moments in this case are set partitions whose blocks are intervals (alternatively, compositions of the integer n). This interpretation follows from Proposition 3.6 which implies that every two non-singleton blocks must be separated, together with the observation that $lrs(sing) = lrg(sing) = 0$ forces all singletons to be separated from other blocks as well. As in the previous two cases, all orbits in the interval [2020, 2222] yield the same moments, and the n^{th} moment has the expression

$$\mu_n = a(1+a)^{n-1}.$$

We now turn to the remaining 27 orbits, all of which correspond to true orthogonal polynomials ($\lambda_n > 0$). Directly from Theorem 2.1, the orbit 0000 corresponds to permutations.

One of the orbits ($r = p = 0$), corresponds, by Proposition 3.3, to set partitions.

(1) 1010: 1, 1, 2, 5, 15, 52, ..., set partitions.

There are 11 orbits above 1010 in the quotient of the boolean algebra, whose moments we discuss next. Four of these orbits give moments which count specialized types of set partitions:

(2) 1011: 1, 1, 2, 5, 14, 43, 143, 509, ..., non-overlapping set partitions (by Proposition 3.7).

(3) 1111: 1, 1, 2, 5, 14, 42, 132, 429, ..., non-crossing set partitions (by Propositions 3.4 and 3.5).

(4) 1210: 1, 1, 2, 5, 15, 48, 163, 571, ..., minimally non-separated set partitions (by Proposition 3.8).

(5) 1211: 1, 1, 2, 5, 14, 41, 123, 374, ..., non-crossing partitions such that if block B is nested inside block A, then $|B| \leq 2$. (This is an easily obtained strengthening of the conditions from the case 1111 or 1210).

Each of these orbits has a counterpart designated by a four-tuple whose first entry is 2 instead of 1. Thus, we get 5 further orbits which lie above 1010 and whose moments count set partitions – unrestricted set partitions and set partitions of the four other types listed above – with the supplementary condition that singleton blocks are separated from all other blocks.

(6) 2010: 1, 1, 2, 4, 10, 28, 90, 326, ..., set partitions with singletons separated from all blocks.

(7) 2011: 1, 1, 2, 4, 9, 22, 58, 164, ..., non-overlapping set partitions with the singletons separated from all blocks.

(8) 2111: 1, 1, 2, 4, 9, 21, 51, 127, ..., non-crossing set partitions with singletons separated from all blocks.

(9) 2210: 1, 1, 2, 4, 10, 24, 66, 172, ..., minimally non-separated set partitions with singletons separated from all blocks.

(10) 2211: 1, 1, 2, 4, 9, 20, 46, 105, ..., non-crossing set partitions such that if block B is nested inside block A, then $|B| = 2$.

The moments of the orbits 2111, 2210, and 2211 lend themselves to yet another combinatorial interpretation, in terms of (partial) matchings. This will be described below, following the discussion of the orbit 2110.

(11) 2110: 1, 1, 2, 4, 10, 26, 76, 232, ...

If we consider $r = s = t = p = 0$ in Theorem 2.1, we see that the n^{th} moment enumerates set partitions of $[n]$ with the property that if two blocks B_a

and B_b are not separated and $minB_b < minB_a$, then all the elements in the set $(minB_a, min\{maxB_a, maxB_b\}) \cap (B_a \cup B_b)$ must belong to B_a (this is to ensure that $lrs(cont) = 0$). Therefore, there is a bijection between the set partitions of this type and (partial) matchings of the set $[n]$. Specifically, a (partial) matching of $[n]$ is a partition of $[n]$ into blocks of cardinality at most 2. Each matching corresponds bijectively with a set partition counted by the moments when $r = s = t = p = 0$, by first indexing the pairs of the matching in decreasing order of their minima, and then adjoining each non-separated singleton block to the block of least index from which the singleton is not separated. For example, the matching 1 / 2 8 / 3 / 4 / 5 13 / 6 / 7 / 9 11 / 10 / 12 / 14 /15 16 of [16], corresponds to the set partition 1 / 2 3 4 8 / 5 6 7 12 13 / 9 10 11 / 14 / 15 16.

The singletons of a partial matching are singletons in the sense of set partitions as well. Of two points in $[n]$ which are matched, the smaller is an opener and the larger is a closer. Two non-separated non-singleton blocks are two pairs of the matching which either cross, or else one is nested under the other. Let $C(m)$ be the number of crossings and $U(m)$ be the number of nestings (as described above) in a matching m. It is then easy to see that for a matching, $lrg(op) = C(m) + U(m)$ and $lrg(clos) = U(m)$. The statistics $C(m)$ and $U(m)$ appear again in (5.4).

In view of the interpretation in terms of matchings for the moments of 2110, we can obtain simple alternative interpretations for the moments of three of the orbits discussed earlier:

(8') 2111: 1, 1, 2, 4, 9, 21, 51, 127, ..., non-crossing (partial) matchings.

(9') 2210: 1, 1, 2, 4, 10, 24, 66, 172, ..., (partial) matchings such that every singleton is separated from all but at most one pair of the matching.

(10') 2211: 1, 1, 2, 4, 9, 20, 46, 105, ..., non-crossing (partial) matchings in which every singleton is separated from all but at most one pair of the matching.

The remaining orbit above 1010 is

(12) 1110: 1, 1, 2, 5, 15, 50, 181, 697, ..., set partitions in which if A and B are blocks satisfying $min\{a \in A\} < min\{b \in B\}$, then $A \cap [min\{b \in B\}, max\{b \in B\}] \subseteq \{max\{a \in A\}\}$.

The remaining 15 orbits will be given as permutations with restrictions. We use Theorem 2.1. Alternative descriptions can be given from Theorem 2.2. By means of the canonical composition corresponding to a permutation, we can refer to runs as being "separated," "non-separated," "minimally non-separated," etc., if they form blocks with these properties.

(13) 1000: 1, 1, 2, 5, 17, 70, 349, 2017, ..., permutations in which any singleton run must be a left-to-right minimum.

(14) 0100: 1, 1, 2, 6, 24, 116, 652, 4156, ..., permutations with $lsg(cont) = 0$, i.e., if run A precedes run B, then $[min\{a \in A\}, max\{a \in A\}] \cap B \subseteq \{min\{b \in B\}, max\{b \in B\}\}$.

(15) 0010: 1, 1, 2, 6, 22, 94, 460, 2537, ..., permutations in which each opener is a left-to-right minimum.

(16) 2000: 1, 1, 2, 4, 12, 40, 180, 924, ..., permutations in which every singleton run must be a left-to-right minimum and a right-to-left maximum.

(17) 1100: 1, 1, 2, 5, 17, 66, 305, 1545, ..., permutations in which each singleton run is a left-to-right minimum and if run A precedes run B, then $[min\{a \in A\}, max\{a \in A\}] \cap B \subseteq \{min\{b \in B\}, max\{b \in B\}\}$.

(18) 0200: 1, 1, 2, 6, 24, 112, 592, 3468, ..., permutations in which every two non-separated runs are minimally non-separated.

(19) 0110: 1, 1, 2, 6, 22, 92, 426, 2146, ..., the specialization $u = p = 0$ yields permutations in which the openers are left-to-right minima and where, if A, B are non-separated runs with $min\{a \in A\} > min\{b \in B\}$, then $B \cap [min\{a \in A\}, \max\{a \in A\}] \subseteq \{min\{b \in B\}, max\{b \in B\}\}$.

(20) 0011: 1, 1, 2, 6, 21, 81, 343, 1591, ..., the specialization $p = w = 0$ gives permutations in which the openers are left-to-right minima and the closers are right-to-left maxima.

(21) 0111: 1, 1, 2, 6, 21, 80, 326, 1408, ..., combining cases (14) and (20), we obtain permutations in which the openers are left-to-right minima, the closers are right-to-left maxima, and every two non-separated runs A, B with $min\{a \in A\} > min\{b \in B\}$ satisfy $B \cap [min\{a \in A\}, \max\{a \in A\}] \subseteq \{min\{b \in B\}, max\{b \in B\}\}$.

(22) 0210: 1, 1, 2, 6, 22, 90, 396, 1846, ..., permutations in which the openers are left-to-right minima and non-separated runs are minimally non-separated.

(23) 1200: 1, 1, 2, 5, 17, 62, 269, 1205, ..., a strengthening of the condition in (18): permutations in which the singleton runs are left-to-right minima and every two non-separated runs are minimally non-separated.

(24) 2100: 1, 1, 2, 4, 12, 36, 152, 624, ..., combining the conditions in (14) and (16), permutations in which the singleton runs are left-to-right minima and right-to-left maxima, and in which if run A precedes run B then $[min\{a \in A\}, max\{a \in A\}] \cap B \subseteq \{min\{b \in B\}, max\{b \in B\}\}$.

(25) 2200: 1, 1, 2, 4, 12, 32, 132, 416, ..., permutations in which the singleton runs are left-to-right minima and right-to-left maxima, and where every two non-separated runs are minimally non-separated.

(26) 0211: 1, 1, 2, 6, 21, 79, 311, 1266, ..., combining the conditions in (18) and (20), we have permutations in which the openers are left-to-right minima, the closers are right-to-left maxima, and non-separated runs are minimally non-separated.

5. Explicit formulas for the moments.

In §4 we gave the combinatorial interpretations for the moments of the 54 zero specializations of the octabasic Laguerre polynomials. We also gave an explicit formula for the moments in 27 of these cases. In general, Theorems 2.1 and 2.2 are the best that can be said for the moments. However, some specializations do give classical sequences and their q-analogs. In this section we give these examples.

First we note that the generating function of μ_n is always given by the continued fraction [C1], [Fl],

$$\sum_{n=0}^{\infty} \mu_n t^n = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{\ddots}}}. \tag{5.1}$$

Each of the 54 cases has such an expansion. For example, orbit (1), 1010, gives a continued fraction for the generating function of a 6-q version of the Bell numbers.

There are 6 cases, after all of the remaining parameters are set equal to 1, for which μ_n is clearly a well known sequence (see [Fl-Sch] for the Bessel numbers)

(1) (orbit (0)) $n!$,
(2) (orbit (1)) Bell numbers B_n,
(3) (orbit (2)) Bessel numbers B_n^*,
(4) (orbit (3)) Catalan numbers C_n,
(5) (orbit (8)) Motzkin numbers M_n,
(6) (orbit (11)) involution numbers I_n.

They correspond, respectively, to permutations, set partitions, non-overlapping set partitions, non-crossing set partitions, non-crossing (partial) matchings, and (partial) matchings. We now give the specializations for a single q-analog of each of these cases.

Three specializations giving $n!_q$ are given in [Si-St].

The q-Bell numbers arise from the q-Stirling numbers of the second kind [Mi], [Wa-Wh]. This is the specialization $r = p = 0$, $s = q$, $b = 1$, $\{t, u\} = \{v, w\} = \{1, q\}$, giving the q-Charlier polynomials (see [deM-St-Wh]).

There are many choices for q-Catalan numbers (see [Fu-Ho], [Bo-Sh-Si], [Si2]). A well-known one is given by the recurrence

$$C_n(q) = \sum_{k=0}^{n-1} C_k(q)C_{n-1-k}(q)q^k, \quad C_0(q) = C_1(q) = 1. \tag{5.2}$$

We have $\mu_n = C_n(q)$ if $r = t = p = v = 0$, $a = 1$, $b = q$, $s = u = q^2$, $w = q^3$. The polynomials are a family of q-Chebyshev.

The number of (partial) matchings on $[n]$ is

$$I_n = \sum_{k=0}^{n/2} \binom{n}{2k} (2k-1)(2k-3)\cdots 1.$$

If $r = s = t = p = 0$, $u = v = a = b = 1$, $w = q$, then we have

$$\mu_n = \sum_{k=0}^{n/2} \binom{n}{2k} [2k-1]_q[2k-3]_q \cdots [1]_q. \tag{5.3}$$

It is also easy to see that

$$\mu_n = \sum_{m \in M(n)} q^{C(m)+2U(m)}, \tag{5.4}$$

where $M(n)$ is the collection of matchings of $[n]$ and $C(m), U(m)$ were defined in the discussion of orbit (11) in §4. The polynomials are discrete q-Hermite.

The Motzkin numbers are given by

$$M_n = \sum_{k=0}^{n/2} \binom{n}{2k} C_k.$$

If $r = s = t = p = v = 0$, $u = w = a = b = 1$, then we have

$$\mu_n = \sum_{k=0}^{n/2} \binom{n}{2k} C_k(q). \tag{5.5}$$

The polynomials are q-Chebyshev.

6. The quadrabasic Laguerre polynomials.

The monic Laguerre polynomials have the explicit formula

$$\hat{L}_n^0(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k!} (-1)^{n-k} x^k \tag{6.1}$$

and the coefficient of x^k has a simple combinatorial interpretation (see [Fo-St], (6.6) and (6.7) below). In the octabasic Laguerre polynomials, the coefficient of x^k is a polynomial in a and the "8 q's," whose own coefficients do not necessarily have constant sign (such is the case, for example, with the coefficient of x in $p_3(x)$). This means that any combinatorial interpretation of the octabasic polynomials will be complex.

In this section we specialize the octabasic Laguerre polynomials to a "quadrabasic" set, for which the coefficient of x^k is a polynomial in a and "4 q's" whose coefficients have constant sign. We obtain in Theorem 6.2 a weighted version of the combinatorial interpretation for Laguerre polynomials. We remark that the Viennot theory does not give this result. Our choice of specialization will be determined from the "even-odd" polynomials associated with the Laguerre polynomials. We begin with a brief review of general facts about "even-odd" polynomials.

Suppose that $p_n(x)$ is a set of orthogonal polynomials whose three term recurrence has coefficients $b_n^{'} = 0$ and $\lambda_n^{'} > 0$ for all n. This implies that $p_{2n}(x) = e_n(x^2)$ and $p_{2n+1}(x) = x\, o_n(x^2)$, for some polynomials $e_n(x)$ and $o_n(x)$. It turns out that both $\{e_n\}_n$ and $\{o_n\}_n$ are sequences of orthogonal polynomials (see [C1]). The coefficients of their three-term recurrence relations are

$$\begin{aligned} b_n(e) &= \lambda_{2n+1}^{'} + \lambda_{2n}^{'} \\ \lambda_n(e) &= \lambda_{2n-1}^{'}\lambda_{2n}^{'} \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} b_n(o) &= \lambda_{2n+2}^{'} + \lambda_{2n+1}^{'} \\ \lambda_n(o) &= \lambda_{2n+1}^{'}\lambda_{2n}^{'}. \end{aligned} \tag{6.3}$$

Moreover, the moments for the polynomials p_n, e_n, and o_n satisfy

$$\begin{aligned} \mu_n(e) &= \mu_{2n}(p) \\ \mu_n(o) &= \mu_{n+1}(e)/\mu_1(e). \end{aligned} \tag{6.4}$$

Now let us consider $\lambda_{2n+1}^{'} = n+1$ and $\lambda_{2n}^{'} = n$. Then (6.2) and (1.1) imply that the specialization of the octabasic Laguerre polynomials obtained by setting all 8 q's, a, and b equal to 1, satisfies

$$p_{2n}(x) = \hat{L}_n^0(x^2). \tag{6.5a}$$

It is not hard to see that the "odd" polynomials for this specialization are

$$p_{2n+1}(x) = x\hat{L}_n^1(x^2) = \sum_{k=0}^{n} \binom{n}{k} \frac{(n+1)!}{(k+1)!} (-1)^{n-k} x^{2k+1}. \tag{6.5b}$$

From (6.1) and (6.5b) it is easy to see that $\hat{L}_n^0(x)$ and $\hat{L}_n^1(x)$ have simple combinatorial interpretations:

$$\hat{L}_n^0(x) = \sum_{A\subseteq[n], f:A\to[n]} x^{n-|A|}(-1)^{|A|}, \tag{6.6}$$

and

$$\hat{L}_n^1(x) = \sum_{A\subseteq[n], f:A\to[n+1]} x^{n-|A|}(-1)^{|A|}, \tag{6.7}$$

where, in both cases, the function f is an injection.

We give in Theorem 6.2 quadrabasic versions of (6.6) and (6.7). To do this, we specialize the octabasic Laguerre polynomials (1.1) so that (6.2) occurs. The most natural choice is $\{t,u\}=\{p,q\}$ and $\{r,s\}=\{v,w\}$, so that

$$\lambda'_{2n+1} = a[n+1]_{r,s}, \qquad \lambda'_{2n} = [n]_{t,u}. \tag{6.8}$$

However, it is easy to see that the same specialization, $\{t,u\}=\{p,q\}$ and $\{r,s\}=\{v,w\}$, also gives (6.3) for the odd polynomials if

$$\lambda'_{2n+2} = a[n+1]_{r,s}, \qquad \lambda'_{2n+1} = [n]_{t,u}. \tag{6.9}$$

Thus we concentrate on these "quadrabasic" Laguerre polynomials which come from $\{t,u\}=\{p,q\}$ and $\{r,s\}=\{v,w\}$.

For Theorem 6.2 we need statistics on the pairs (A,f) which involve a and the four remaining q's: r,s,t,u. The rather technical definition of these statistics is in terms of the cycles and paths in the functional digraph of f.

The functional digraph of an injection $f\colon A\to[n]$ consists of disjoint paths and cycles. Each path P is of the form $a_0(P)\to a_1(P)\to\cdots\to a_l(P)$, where $f(a_j(P)) = a_{j+1}(P)$ for $0\le j<l$, with $f^{-1}(a_0(P))$ empty, and $a_l(P)\in[n]-A$. For simplicity in notation, we will write a_j for $a_j(P)$. We also put $last(P)=a_l$ and if $i=a_k\in P$ we write $ind(i,P)=k$ for the index of i on the path P.

The following quantities computed on points and paths of the digraph will be helpful in describing the combinatorial interpretation of our quadrabasic Laguerre polynomials. For any path P in the digraph and two integers $i<j$, we put

$$n_P(i,j) = |\{a\in P\colon\ i<a<j\}|.$$

For $p\in P$ and two integers $i<j$, we put

$$m_P(p;i,j) = |\{a\in P\colon\ i<a<j, ind(p,P)<ind(a,P)\}|,$$

that is, the number of points on the path "to the right" of p, whose values are strictly between i and j. And finally, for $i\in A$, we denote by $F(i)$ the "first forward iterate" of f which is smaller than i,

$$F(i) = \begin{cases} f^p(i), \text{ where } p=min\{m\ge1, f^m(i)<i\} \text{ if such } m \text{ exists,}\\ i, \text{ if } \{m\ge 1, f^m(i)<i\} \text{ is empty.}\end{cases}$$

For example, suppose that the path $P = \ 2\to7\to1\to5\to3\ $ is a connected component of the functional digraph of f. Then $n_P(1,4)=|\{2,3\}|=2$, $m_P(7;1,4)=|\{3\}|=1$, and $F(2)=F(7)=1$, $F(1)=1$, and $F(5)=3$.

Definition 6.1. *Let $A \subseteq [n]$ and let f be an injection $f : A \to [n]$. We define the weight $w(A, f)$ of (A, f) by $w(A, f) = \prod_{i\in[n]} w(i)$, where*

$$w(k) = \begin{cases} t^{\alpha}u^{\beta} \text{ if } k \in P \text{ and } k < last(P) \text{ for some path } P \text{ of } f, \\ ar^{\gamma}s^{\delta} \text{ if } k \in P \text{ and } k > last(P) \text{ for some path } P \text{ of } f, \\ ar^{\gamma}s^{\delta} \text{if } k \text{ lies on a cycle of } f, \\ 1 \quad \text{if } k \notin A, \end{cases}$$

where the exponents $\alpha, \beta, \gamma, \delta$ are defined below, using the notation established previously, and with Q ranging over paths in the functional digraph of f:

$$\alpha = k - 1 - m_P(k; 0, k) - \sum_{last(Q)>last(P)} n_Q(0, k),$$

$$\beta = last(P) - k - m_P(k; k, last(P)) - \sum_{last(Q)>last(P)} n_Q(k, last(P)),$$

$$\gamma = F(k) - 1 - \sum_{last(Q)>k} n_Q(0, F(k)),$$

$$\delta = k - F(k) - \sum_{last(Q)>k} n_Q(F(k), k).$$

Theorem 6.2. *The quadrabasic Laguerre polynomials obtained by setting $\{t, u\} = \{p, q\}$ and $\{r, s\} = \{v, w\}$ in (1.1) have the following interpretation:*

$$\sum_{A\subseteq[n], f:A\to[n]} x^{n-|A|}(-1)^{|A|}w(A, f),$$

where f is injective and $w(A, f)$ is the monomial in r, s, t, u, a in Definition 6.1.

Corollary 6.3. *For all k, the coefficient of x^k in the quadrabasic Laguerre polynomials obtained by setting $\{t, u\} = \{p, q\}$ and $\{r, s\} = \{v, w\}$ in (1.1), is a polynomial (in a, r, s, t, u) all of whose coefficients have the same sign.*

Proof of Theorem 6.2. Assuming (6.8) we have

(6.10a) $$p_{2n+1}(x) = xp_{2n}(x) - [n]_{t,u}p_{2n-1}(x)$$

(6.10b) $$p_{2n}(x) = xp_{2n-1}(x) - a[n]_{r,s}p_{2n-2}(x).$$

If we set $p_{2n}(x) = \sum_k E_{n,k}x^{2k}(-1)^{n-k}$, and $p_{2n+1}(x) = \sum_k O_{n,k}x^{2k+1}(-1)^{n-k}$, then (6.10) is equivalent to

(6.11a) $$O_{n,k} = E_{n,k} + [n]_{t,u}O_{n-1,k}$$

(6.11b) $$E_{n,k} = O_{n-1,k-1} + a[n]_{r,s}E_{n-1,k}.$$

It is clear that (6.11) proves Corollary 6.3, as well as a version of Corollary 6.3 for the quadrabasic version of $\hat{L}^1_n(x)$. What remains is to use (6.11) to produce the statistics in Definition 6.1. In order to show that

$$E_{n,k} = \sum_{\substack{A\subseteq[n],|A|=n-k \\ f:A\to[n]}} w(A, f)$$

and

$$O_{n,k} = \sum_{\substack{A\subseteq[n],|A|=n-k\\ f:A\to[n+1]}} w(A,f)$$

for all n,k, we will define recursively a weight $\overline{w}(A,f)$ on the ordered pairs (A,f) in (6.6) and (6.7), such that $\overline{w}$ satisfies (6.11), and then prove that $\overline{w}$ agrees with the weight w of Definition 6.1.

For (6.11a), let $A \subseteq [n]$, $|A| = n-k$, and let $f\colon A \to [n+1]$ be an injection. If $n+1 \notin Im(f)$, just delete $n+1$ and let $A' = A$ and $f'\colon A' \to [n]$, with f' given by the function f when viewed as $f : A \to [n]$. This gives a pair (A', f') contributing to the term $E_{n,k}$ in (6.11a). We set $\overline{w}(A', f') = \overline{w}(A, f)$. If $n+1 \in Im(f)$, then $n+1$ must be the last point of a path P of f, since $A \subseteq [n]$. Define the pairing $i \leftrightarrow i'$ between $i \in [n+1] - \{f^{-1}(n+1)\}$ and $i' \in [n]$ via

$$i' = \begin{cases} i \text{ if } i < f^{-1}(n+1) \\ i-1 \text{ if } i > f^{-1}(n+1). \end{cases}$$

Now let $A' = \{i' : i \in A - f^{-1}(n+1)\}$, and $f'\colon A' \to [n]$ defined by $f'(i') = [f(i)]'$. Then the pair (A', f') corresponds to $O_{n-1,k}$ in (6.11a). We set

$$\overline{w}(A,f) = t^{f^{-1}(n+1)-1}u^{n-f^{-1}(n+1)}\overline{w}(A',f'),$$

with the monomial $t^{f^{-1}(n+1)-1}u^{n-f^{-1}(n+1)}$ from $[n]_{t,u}$, so $\overline{w}(A,f)$ corresponds to the term $[n]_{t,u}O_{n-1,k}$ in (6.11a).

Similarly, for (6.11b) consider $A \subseteq [n]$, $|A| = n-k$, and an injection $f\colon A \to [n]$. If $n \notin A$, then let A' be identical to A, but viewed as a subset of $[n-1]$, and $f'\colon A' \to [n]$ pointwise equal to f. We set $\overline{w}(A', f') = \overline{w}(A, f)$, and the pairs (A', f') arising from this case correspond to the $O_{n-1,k-1}$ term in (6.11b). If $n \in A$, we let $A' = A - \{n\}$, and obtain an injection $f'\colon A' \to [n-1]$ as follows: delete n from the functional digraph of f, and connect $f^{-1}(n)$ (if it exists) to $f(n)$. The pair (A', f') corresponds to $E_{n-1,k}$ in (6.11b). We set

$$\overline{w}(A,f) = ar^{f(n)-1}s^{n-f(n)}\overline{w}(A',f')$$

and thus we get the term $a[n]_{r,s}E_{n-1,k}$.

Thus, each $\overline{w}$-weighted pair (A,f) contributing to the left hand side of (6.11) corresponds to a $\overline{w}$-weighted pair (A',f') contributing to the right hand side. We will regard this correspondence as a "reduction" of (A,f) obtained by "processing" the largest element. Effectively, with the exception of the first case under (6.11b), in order to construct (A',f') we reduce the digraph for (A,f) by one point. The reduction consists of removing the largest element currently in the digraph, or its preimage. The latter type of reduction occurs only in the second case of (6.11a), and entails the "compression" of the remaining values to an initial interval of the positive integers. The compression preserves the order relation among values, and the value of an element decreases by one unit for each reduction in which a smaller element is deleted.

We claim that $\overline{w}(A,f)$ obtained upon applying the above reduction rules as long as the domain of the injection is non-empty, agrees with $w(A,f)$ of Definition

6.1. To verify this, first observe the following property of the reduction process. If at some stage the maximum element is $last(P)$ for some path in the digraph of the function, then its preimage is eliminated and the values of the elements are compressed. The maximum value is reduced by one unit, but at the next reduction it will be again the maximum element and still last in its path. Therefore, once the last point of a path has the maximum value of all points in the digraph, the path will be completely eliminated from the digraph through a sequence of consecutive reductions. Second, if the maximum is not last on a path of the digraph, then it is simply deleted.

Now let us consider a point in A. If its original value is k, let k^* be its value at the time when it is deleted. If k belonged to a path P of the original digraph and $k < last(P)$, then

$$k^* = k - \sum_{last(Q)>last(P)} |\{a \in Q\colon a < k\}| - |\{b \in P\colon b < k, ind(k,P) < ind(b,P)\}|.$$

The difference $k-k^*$ is the number of previous reductions when values smaller than k were deleted, thereby causing compressions which diminished the original label k. At this same stage, the label of the last element on the path containing k^* is not necessarily $last(P)$ but $last(P)^*$,

$$\begin{aligned} last(P)^* = last(P) - &\sum_{last(Q)>last(P)} |\{a \in Q\colon a < last(P)\}| \\ &- |\{b \in P\colon b < last(P), ind(k,P) < ind(b,P)\}|, \end{aligned}$$

and at this stage $last(P)^*$ is the image of k^*. By the second case of (6.11a) above, the ratio $\overline{w}(A,f)/\overline{w}(A',f')$ for this reduction is equal to $t^{k^*-1}u^{last(P)^*-k^*}$. After minor simplifications, we recognize this expression as $w(k) = t^\alpha u^\beta$ from Definition 6.1.

A simpler situation is the first type of reduction described above for (6.11a). In this case, the element deleted from the digraph derives from an original point k which was not in the domain of the injection. Hence, the value 1 of the ratio $\overline{w}(A,f)/\overline{w}(A',f')$ for such reductions agrees with $w(k)$ in Definition 6.1. The first type of reduction under (6.11b) does not contribute to the $\overline{w}$-weight and no point is removed from the digraph.

Finally, suppose that in the original digraph the point k belongs to a path P and $k > last(P)$, or k is on a cycle. Then when this point is deleted from the digraph, its current value k^* is the largest value present in the current digraph and is equal to

$$k^* = k - \sum_{last(Q)>k} |\{a \in Q\colon a < k\}|.$$

Note also that at this stage, the image of k^* is $F(k)^*$. That is, the current value of the original "first forward iterate" of k. Indeed, the previous reductions have eliminated all values derived from elements larger than k. We have

$$F(k)^* = F(k) - \sum_{last(Q)>k} |\{a \in Q\colon a < F(k)\}|.$$

Thus, the ratio $\overline{w}(A,f)/\overline{w}(A',f')$ for such a reduction is $ar^{F(k)^*-1}s^{k^*-F(k)^*}$. Again, we recognize this as $w(k) = ar^\gamma s^\delta$ from Definition 6.1.

Hence, $\overline{w}(A,f) = \prod_k w(k) = w(A,f)$ and the proof is completed. $\square$

7. More even-odd polynomials.

The first result of this section is an alternative interpretation of the orbit (19) from §4. There, the moments of the specializations falling in the orbit 0110 were interpreted in terms of permutations with side conditions. Here, the moments for the orbit 0110 will arise in the context of the quadrabasic and even-odd polynomials discussed in §6, by setting $t = p = 0, r = v, s = w, u = q, b = 1$. This specialization gives the following coefficients for the three-term recurrence:

$$b_0 = a,\ b_n = a[n+1]_{r,s} + u^{n-1},\ \text{if } n > 0, \quad \lambda_n = a[n]_{r,s}u^{n-1}. \tag{7.1}$$

These are precisely the coefficients for the even polynomials (see (6.2)) for $\lambda'_{2n} = u^{n-1}$, $\lambda'_{2n+1} = a[n+1]_{r,s}$, and so $\mu_n(e)$ is the nth moment for the orbit 0110 of §4. The coefficients of the three-term recurrence for the corresponding odd polynomials are

$$b_n(o) = u^n + a[n+1]_{r,s}, \quad \lambda_n(o) = a[n+1]_{r,s}u^{n-1}. \tag{7.2}$$

If we rescale the odd polynomials to $\tilde{o}_n(x) := a^n o_n(x/a)$, the coefficients

$$b_n(\tilde{o}) = [n+1]_{r,s} + \frac{u^n}{a}, \qquad \lambda_n(\tilde{o}) = [n+1]_{r,s}\frac{u^{n-1}}{a}$$

are reminiscent of r, s, u versions of the coefficients for Charlier polynomials and suggest that the nth moment, $\mu_n(\tilde{o})$, is related to the enumeration of certain partitions. More precisely, these coefficients would give the correct weights on the steps of the associated Motzkin paths (see [V1], [Si-St]) if the level k steps were shifted to level $k+1$. We can accomplish this by adding an initial NE step and a final SE step. The resulting Motzkin paths counted by $\mu_n(\tilde{o})$ have length $n+2$ and do not touch the x-axis except at their initial and final points. That is, these paths correspond to partitions of the set $[n+2]$ such that no interval $[i]$ is a union of blocks, for any $i < n+2$. We call such partitions *connected set partitions*, abbreviated $ConnSP$. From these observations and (6.4) we obtain

$$\mu_n(e) = \mu_1(e)\mu_{n-1}(o) = a \cdot a^{n-1}\mu_{n-1}(\tilde{o}),$$

the reciprocal polynomial (in terms of a) of $\frac{1}{a}ConnSP(n+1)$, enumerating connected set partitions of $[n+1]$. With $a = r = s = u = 1$, the moments from §4, orbit 0110 arise as the sequence $\mu_n(e) = |ConnSP(n+1)|$ for $n \geq 1$: 1, 2, 6, 22, 92, 426, ... We have:

Theorem 7.1. *Let b_n and λ_n be given by (7.1). The moments are given by*

$$\mu_n = \sum_{\pi \in ConnSP(n+1)} r^{lrs(cont)(\pi)+lrs(clos)(\pi)} s^{lrg(cont)(\pi)+lrg(clos)(\pi)} u^{lrs(sing)(\pi)+lrs(op)(\pi)-blocks(\pi)+1} a^{n+1-blocks(\pi)}.$$

We remark that Theorem 7.1 admits the following alternative formulation:

Theorem 7.1′. *Let b_n and λ_n be given by (7.1). The moments are given by*

$$\mu_n = \sum_{\pi \in ConnSP(n+1)} r^{rs(cont)(\pi)+rs(clos)(\pi)} s^{lsg(cont)(\pi)+lrg(clos)(\pi)} \\ w^{rs(sing)(\pi)+lr(clos)(\pi)-blocks(\pi)+1} a^{n+1-blocks(\pi)}.$$

The notation rs ("right smaller") is an abbreviation of lrs permitted by the standard indexing of the blocks in a partition (the blocks are indexed in increasing order of their minima), while $lr = lrs+lrg$. The latter arises from a modification of the weights on the Motzkin paths: assign weight 1 to the NE steps, and a monomial from $\lambda_k(o)$ to level k SE steps, and it turns out that $lrs(op) = lr(clos)$. Other versions can be formulated for further modifications of how the combined weight $\lambda_k(o)$ is distributed between NE steps ending at level k and SE steps beginning at level k.

Next we consider the general quadrabasic polynomials of §6. Assuming (6.4), we see that $\mu_n(o)$ is an (r, s, t, u, a)-version of $(n + 1)!$. The remainder of this section is devoted to obtaining a combinatorial description of these moments in terms of suitable permutation statistics (Theorem 7.2). To begin with, from (6.3) we have

$$\begin{aligned} b_n(o) =& a[n + 1]_{r,s} + [n + 1]_{t,u} \\ \lambda_n(o) =& a[n + 1]_{r,s}[n]_{t,u}. \end{aligned} \tag{7.3}$$

As in the proof of Theorem 2.1 (see [Si-St]), $\mu_n(o)$ enumerates weighted Motzkin paths of length n which, in turn, correspond to permutations in S_{n+1} counted according to certain combinatorial statistics. We will obtain the correspondence in two stages: first we describe a bijection φ from the paths to a subset of S_{n+2}, and then a bijection ψ from this subset to S_{n+1}.

Let P be a path counted by $\mu_n(o)$. The permutation $\varphi(P) \in S_{n+2}$ will have 1 and $n + 2$ in the same run. The elements $2, 3, \ldots , n + 1$ are inserted one at a time, in a position determined by the corresponding step of P and its weight. The steps of P originating at level k have the following weights:
NE steps (leading to openers of runs): a monomial from $a[k + 2]_{r,s}$,
E steps (leading to singleton runs): a monomial from $a[k + 1]_{r,s}$,
E steps (leading to continuators): a monomial from $[k + 1]_{t,u}$,
SE steps (leading to closers): a monomial from $[k]_{t,u}$.
(These are the same weights for Motzkin paths as in the proof of Theorem 2.1 [Si-St] (with $r = p$, $s = q$, $t = v$, $u = w$), except that the NE steps (the openers) at height k have weight $a[k+2]_{r,s}$, and the horizontal steps for continuators at height k have weight $[k + 1]_{t,u}$.)

Starting with a two-element run $1, n + 2$, we traverse P and if the ith step has weight $a^\alpha r^\beta s^\gamma$ or $t^\delta u^\epsilon$, we insert $i + 1$ in the current (partial) permutation in the leftmost position which ensures the following: for openers and continuators the lsg (resp., rsg) value must be β, δ (resp., γ, ϵ); singletons and closers do not "see" the run containing 1, that is, their lsg (resp., rsg) value *without* counting the run containing 1, must be β, δ (resp., γ, ϵ). The permutations which arise as $\varphi(P)$ satisfy: (i) 1 and $n + 2$ in the same run, and (ii) *in the subword* to the right of the run containing 1, no singleton run is a left-to-right minimum.

Next, we construct a bijection ψ from this subset of S_{n+2} to all permutations of length $n+1$. Suppose the permutation σ of length $n+2$ contains the consecutive runs $1,\ldots,c,n{+}2|d_1,\ldots,d_m$. (Note that $m>1$, otherwise d_1 contradicts condition (ii) for σ.) If $c>d_1$, let $\psi(\sigma)$ be the permutation obtained by replacing the two runs above with the two consecutive runs $1,\ldots,c|d_1,\ldots,d_m$. Note the $\psi(\sigma)$ still satisfies condition (ii). If $c<d_1$, let $\psi(\sigma)$ be the permutation obtained by replacing the two runs above with the run $1,\ldots,c,d_2,\ldots,d_m$, and moving d_1 to the right as a singleton run, in the unique position where it is a left-to-right minimum for the subword to the right of the run containing 1. So d_1 immediately precedes the first run after $1,\ldots,c,d_2,\ldots,d_m$ whose opener is less than d_1. This time, $\psi(\sigma)$ does not satisfy condition (ii). The inverse map ψ^{-1} is easy to describe. If condition (ii) holds, just add $n+2$ to the run with 1. Otherwise find the rightmost singleton which violates (ii), call it d_1, and let c be the the largest element smaller than d_1 occurring in the same run as 1; then insert $n+2$ followed by d_1 immediately after c.

It remains to describe the statistics on permutations to which $\varphi\circ\psi$ maps lsg and rsg. We denote these by lsg^* and rsg^*. If a permutation satisfies condition (ii), then define its lsg^* (rsg^*) to be lsg (rsg), but always include the run with 1 for openers and continuators, and exclude this run for singletons and closers. So for the permutation 278|15|36|4, $lsg^*(4)=2$, $rsg^*(7)=1$. If a permutation does not satisfy (ii), let d_1 be the rightmost singleton run violating (ii). Let $z>d_1$ be the maximum of the run containing 1. This time define lsg^* (rsg^*) as above, with the extra modifications: add 1 for each element in $[d_1,z)$ for the run with 1. Elements in the run with 1 and in $[d_1,z)$ count as left. For example, in the permutation 248|157|69|3, $d_1=3$, $z=7$, $lsg^*(3)=2$, $rsg^*(4)=3$, $lsg^*(5)=2$, $lsg^*(6)=3$.

In view of (6.3) we have the next theorem.

Theorem 7.2. *Let μ_n be given by Theorem 2.1 with $r=p$, $s=q$, $t=v$, $u=w$, $b=1$. Then*

$$\mu_n=\sum_{\sigma\in S_n} r^{\mathrm{lsg^*(sing)+lsg^*(op)}(\sigma)}s^{\mathrm{rsg^*(sing)+rsg^*(op)}(\sigma)}$$
$$t^{\mathrm{lsg^*(cont)+lsg^*(clos)}(\sigma)}u^{\mathrm{rsg^*(cont)+rsg^*(clos)}(\sigma)}a^{\mathrm{run}(\sigma)},$$

for the modified statistics lsg and lrg*.*

8. Equidistributed statistics.

Theorems 2.1 and 2.2 give an interpretation for the moments of the octabasic Laguerre polynomials, in terms of permutation statistics. So we can use facts about the octabasic Laguerre polynomials to give combinatorial theorems about these statistics.

One easy fact is the invariance of the moment under the group G of order 32 in §4.

Theorem 8.1. *The generating functions in Theorems 2.1 and 2.2 are invariant under the group G of order 32 described at the start of §4.*

We can also find analogous theorems for sets other than permutations, for example set partitions. If we take the $r=p=0$ specialization (the orbit (1), 1010, in §4), we find

$$b_n = as^n + [n]_{t,u}$$
$$\lambda_n = aq^{n-1}[n]_{v,w}.$$

Clearly the symmetry group here is a group of order four. We use Proposition 3.3 (2) to find the appropriate statistics for an *RG* function w.

Theorem 8.2. *The generating function for RG functions on* $[n]$*,*

$$\sum_{w \in SP(n)} s^{lrs(sing)(w)} t^{lrg(cont)(w)} u^{lrs(cont)(w)} q^{lrs(op)(w)} v^{lrg(clos)(w)} w^{lrs(clos)(w)} a^{\#blocks},$$

is symmetric under $t \leftrightarrow u$ *and* $v \leftrightarrow w$*.*

For non-crossing set partitions ($r = t = p = v = 0$, orbit 1111), we have

$$b_n = as^n + u^{n-1}, n \geq 1, b_0 = a$$
$$\lambda_n = aq^{n-1}w^{n-1}.$$

Again Proposition 3.3 (2) allows us to take complements.

Theorem 8.3. *The generating function for RG functions of non-crossing set partitions of* $[n]$*,*

$$\sum_{w \in NCSP(n)} s^{lrs(sing)(w)} u^{lrs(cont)(w)} q^{lrs(op)(w)} w^{lrs(clos)(w)} a^{\#blocks},$$

is invariant under $q \leftrightarrow w$*.*

We can use another element of the orbit of 1111 to obtain another equidistribution theorem. If we put $r = u = p = w = 0$, we find the class of set partitions NU (see Proposition 3.9). These are enumerated by the Catalan numbers, just as non-crossing partitions are.

Theorem 8.4. *The 5-tuples*

$$(lrs(sing), lrs(cont), lrs(op), lrs(clos), blocks)$$

and

$$(lrs(sing), lrg(cont), lrs(op), lrg(clos), blocks)$$

are equidistributed on the class of non-crossing set partitions and on the class "NU" of not-under set partitions.

Bijective proofs of Theorems 8.1-8.4 are implicit in the bijection for Theorem 2.1 appearing in [Si-St].

There are similar results, that we do not state here, for orbits whose moments can be interpreted in terms of matchings. Also, Theorems 2.1, 2.2 and 7.2 clearly give equidistribution results. Moreover, Theorem 2.2 provides for alternative versions of Theorems 8.1-8.4.

9. Classical orthogonal polynomials and zero specializations.

We now turn to the classical orthogonal polynomials which arise from our zero specializations. In §4 we gave the combinatorial interpretations for the moments in 54 cases. Several of the cases are related to classical orthogonal polynomials, and we identify them here. We do this assuming the non-zero q's have been put to 1, so that each case will represent a multi-q version of the listed polynomials. We will also discuss some "principal specializations" which lead to known q-analogs of the classical polynomials.

The 27 cases in §4 which have $\lambda_n = 0$ for $n > 1$ have simple explicit formulas, and are not classical orthogonal polynomials. For completeness we state them. If we put $p = q = 0$ (the 0020 case), then clearly

$$p_{n+1}(x) = (x - b_n)p_n(x)$$

for $n > 1$. From this we see that

$$p_n(x) = (x^2 - x(a(r+s) + a + 1) + a^2(r+s)) \prod_{i=3}^{n} (x - a[i+1]_{r,s} - [i]_{t,u}), \quad n \geq 2,$$
$$p_1(x) = x - a.$$

Each of the other 26 cases is a specialization of the above.

This leaves 27 cases, which are listed below. In each case we have included the specialization, that is, which of our 8 bases are set equal to zero, the others being set equal to 1. In each case, we identify the orthogonal polynomials, give the coefficients of the three-term recurrence relation and state the explicit formula for the (monic) polynomials. The Appendix gives the explicit notation for each of the polynomials.

We have used some facts about corecursive polynomials [C2] to give the explicit formulas. We need the following fact: if the coefficient b_0 is perturbed in the recurrence relation (1.1), then the resulting polynomials are perturbed by a polynomial with a related recurrence relation. This occurs in several of the classical cases below. The related recurrence relation in these cases is for the associated orthogonal polynomials.

(0000) Specialization: $\varnothing$, i.e., none of the 8 q's is set equal to zero. For $a = 1$ we obtain Laguerre polynomials. For general a we obtain special Meixner polynomials:

$$b_n = n + a(n+1), \quad \lambda_n = an^2,$$

$$p_n(x) = (-a)^n m_n(x/(1-a), a, 1). \tag{9.0}$$

(1010) Specialization: $r = p = 0$. We have the Charlier polynomials:

$$b_n = n + a, \quad \lambda_n = an,$$

$$p_n(x) = C_n^a(x). \tag{9.1}$$

(1011) Specialization: $r = p = v = 0$. The polynomials are dual Lommel polynomials [C1, (6.11), p.189]:

$$b_n = n + a, \quad \lambda_n = a,$$

$$p_n(x) = (-1)^n K_n(a - x, a). \tag{9.2}$$

(1111) Specialization: $r = t = p = v = 0$. We obtain Chebyshev polynomials:

$$b_0 = a, \quad b_n = a + 1, \; n > 0, \quad \lambda_n = a,$$

$$p_n(x) = U_n(\frac{x - a - 1}{2\sqrt{a}})a^{n/2} + U_{n-1}(\frac{x - a - 1}{2\sqrt{a}})a^{(n-1)/2}. \tag{9.3}$$

(1210) Specialization: $r = t = u = v = 0$. We obtain a sum of associated Hermite polynomials [As-Wm, (4.18)]:

$$b_1 = a + 1, \quad b_n = a, \; n \neq 1, \quad \lambda_n = an,$$

(9.4)

$$p_n(x) = H_n(\frac{x - a}{\sqrt{2a}})(a/2)^{n/2} - H_{n-1}(\frac{x - a}{\sqrt{2a}}, 1)(a/2)^{(n-1)/2} - 2aH_{n-3}(\frac{x - a}{\sqrt{2a}}, 3)(a/2)^{(n-3)/2}.$$

(1211) Specialization: $r = t = u = p = v = 0$. We obtain sums of Chebyshev polynomials:

$$b_1 = a + 1, \quad b_n = a, \; n \neq 1, \quad \lambda_n = a,$$

$$p_n(x) = U_n(\frac{x - a}{2\sqrt{a}})a^{n/2} - (x - a)U_{n-2}(\frac{x - a}{2\sqrt{a}})a^{(n-2)/2}. \tag{9.5}$$

(2010) Specialization: $r = s = p = 0$. We obtain a sum of Charlier and associated Charlier polynomials (see [I-L-V, (1.14) and (4.2)]):

$$b_0 = a, \quad b_n = n, \; n > 0, \quad \lambda_n = an,$$

$$p_n(x) = C_n^a(x + a) - aC_{n-1}^a(x + a, 1). \tag{9.6}$$

(2011) Specialization: $r = s = p = v = 0$. The polynomials are sums of dual Lommel polynomials [C1, (6.11), p.189]:

$$b_0 = a, \quad b_n = n, \; n > 0, \quad \lambda_n = a,$$

$$p_n(x) = (-1)^n K_n(-x, a) + a(-1)^n K_{n-1}(1 - x, a). \tag{9.7}$$

(2111) Specialization: $r = s = t = p = v = 0$. We obtain Chebyshev polynomials:

$$b_0 = a, \quad b_n = 1,\ n > 0, \quad \lambda_n = a,$$

$$p_n(x) = U_n(\frac{x-1}{2\sqrt{a}})a^{n/2} - (a-1)U_{n-1}(\frac{x-1}{2\sqrt{a}})a^{(n-1)/2}. \tag{9.8}$$

(2210) Specialization: $r = s = t = u = p = 0$. We obtain a sum of associated Hermite polynomials:

$$b_0 = a,\ b_1 = 1, \quad b_n = 0,\ n > 1, \quad \lambda_n = an,$$

$$\begin{aligned} p_n(x) =& H_n(x/\sqrt{2a})(a/2)^{n/2} - (1+a)H_{n-1}(x/\sqrt{2a},1)(a/2)^{(n-1)/2} \\ &+ aH_{n-2}(x/\sqrt{2a},2)(a/2)^{(n-2)/2} - 2aH_{n-3}(x/\sqrt{2a},3)(a/2)^{(n-3)/2}. \end{aligned} \tag{9.9}$$

(2211) Specialization: $r = s = t = u = p = v = 0$. We obtain Chebyshev polynomials:

$$b_0 = a, \quad b_1 = 1, \quad b_n = 0,\ n > 1, \quad \lambda_n = a,$$

$$p_n(x) = U_n(\frac{x}{2\sqrt{a}})a^{n/2} - U_{n-1}(\frac{x}{2\sqrt{a}})a^{(n+1)/2} + (a-x)U_{n-2}(\frac{x}{2\sqrt{a}})a^{(n-2)/2}. \tag{9.10}$$

(2110) Specialization: $r = s = t = p = 0$. The polynomials are a sum of Hermite polynomials and associated Hermite polynomials [As-Wm, (4.18)]:

$$b_0 = a, \quad b_n = 1,\ n > 0, \quad \lambda_n = an,$$

$$p_n(x) = H_n(\frac{x-1}{\sqrt{2a}})(a/2)^{n/2} + (1-a)H_{n-1}(\frac{x-1}{\sqrt{2a}},1)(a/2)^{(n-1)/2}. \tag{9.11}$$

(1110) Specialization: $r = t = v = 0$. The polynomials are a sum of Hermite polynomials and associated Hermite polynomials [As-Wm, (4.18)]:

$$b_0 = a, \quad b_n = a+1,\ n > 0, \quad \lambda_n = an,$$

$$p_n(x) = H_n(\frac{x-a-1}{\sqrt{2a}})(a/2)^{n/2} + H_{n-1}(\frac{x-a-1}{\sqrt{2a}},1)(a/2)^{(n-1)/2}. \tag{9.12}$$

(1000) Specialization: $r = 0$. The polynomials are special Meixner polynomials [C1, p. 176]:

$$b_n = n + a, \quad \lambda_n = an^2,$$

$$p_n(x) = \frac{(-c)^n}{(1+c)^n} m_n(((1+c)(x-a)+c)/(1-c), c, 1), \tag{9.13}$$

where $a = c/(1+c)^2$.

(0100) Specialization: $t = 0$. The polynomials are a sum of Meixner polynomials [C1, p. 176] and Pollaczek polynomials:

$$b_0 = a, \quad b_n = a(n+1)+1, \; n > 0 \qquad \lambda_n = an^2,$$

(9.14)
$$p_n(x) = (-1-c)^n m_n(\frac{xc-2c-1}{1-c^2}, c, 1) + P_{n-1}^{1/2}(\frac{x-a/2-1}{-\sqrt{4a-a^2}}, \phi, 1)(4a-a^2)^{(n-1)/2}(-1)^{n-1},$$

where $a = (1+c)^2/c$, $cot(\phi) = \sqrt{\frac{a}{4-a}}$.

(0010) Specialization: $p = 0$. The polynomials are Charlier polynomials:

$$b_n = a(n+1)+n, \quad \lambda_n = an,$$

$$p_n(x) = (1+a)^n C_n^{a/(1+a)^2}(\frac{(1+a)x-a^2}{(1+a)^2}). \tag{9.15}$$

(2000) Specialization: $r = s = 0$. The polynomials are a sum of Meixner polynomials [C1, p. 176] and Pollaczek polynomials:

$$b_0 = a, \quad b_n = n, \; n > 0 \qquad \lambda_n = an^2,$$

(9.16)
$$p_n(x) = (\frac{-c}{1+c})^n m_n(\frac{x+xc+c}{1-c}, c, 1) - aP_{n-1}^{1/2}(\frac{x+1/2}{\sqrt{4a-1}}, \phi, 1)(4a-1)^{(n-1)/2},$$

where $a = c/(1+c)^2$, $cot(\phi) = -1/\sqrt{4a-1}$.

(1100) Specialization: $r = t = 0$. The polynomials are a sum of Meixner polynomials [C1, p. 176] and Pollaczek polynomials

$$b_0 = a, \quad b_n = a+1, \; n > 0 \qquad \lambda_n = an^2,$$

(9.17)
$$p_n(x) = (-a)^{n/2} m_n(\frac{x-a-1-\sqrt{-a}}{2\sqrt{-a}}, -1, 1) + P_{n-1}^{1/2}(\frac{x-a-1}{2\sqrt{a}}, \pi/2, 1)(4a)^{(n-1)/2}.$$

(0200) Specialization: $t = u = 0$. The polynomials are a sum of Meixner polynomials [C1, p. 176] and Pollaczek polynomials:

$$b_1 = 2a+1, \quad b_n = a(n+1), \; n \neq 1 \qquad \lambda_n = an^2,$$

(9.18)
$$p_n(x) = (-1-c)^n m_n(\frac{cx-c-1}{1-c^2}, c, 1) + (a-x)P_{n-2}^{1/2}(\frac{x-a/2}{\sqrt{4a-a^2}}, \phi, 2)(4a-a^2)^{(n-2)/2},$$

where $a = (1+c)^2/c$, $cot(\phi) = \frac{-a}{\sqrt{4a-a^2}}$.

(0110) Specialization: $t = v = 0$. The polynomials are a sum of Charlier polynomials and associated Charlier polynomials:

$$b_0 = a, \quad b_n = a(n+1)+1, \ n > 0, \quad \lambda_n = an,$$

$$p_n(x) = a^n C_n^{1/a}(x/a - 1) + a^{n-1} C_{n-1}^{1/a}(x/a - 1, 1). \tag{9.19}$$

(0011) Specialization: $v = p = 0$. The polynomials are dual Lommel polynomials:

$$b_n = a(n+1) + n, \quad \lambda_n = a,$$

$$p_n(x) = (-1-a)^n K_n\left(\frac{a-x}{1+a}, \frac{a}{(1+a)^2}\right). \tag{9.20}$$

(0111) Specialization: $t = v = p = 0$. The polynomials are sums of dual Lommel polynomials:

$$b_0 = a, \quad b_n = a(n+1)+1, \ n > 0, \quad \lambda_n = a,$$

$$p_n(x) = (-a)^n K_n\left(\frac{1+a-x}{a}, \frac{1}{a}\right) + (-a)^{n-1} K_{n-1}\left(\frac{1+2a-x}{a}, \frac{1}{a}\right). \tag{9.21}$$

(0210) Specialization: $t = u = v = 0$. The polynomials are a sum of Charlier polynomials and associated Charlier polynomials:

$$b_1 = 2a+1, \quad b_n = a(n+1), \ n \neq 1, \quad \lambda_n = an,$$

$$p_n(x) = a^n C_n^{1/a}\left(\frac{x-a+1}{a}\right) + a^{n-2}(a-x) C_{n-2}^{1/a}\left(\frac{x-a+1}{a}, 2\right). \tag{9.22}$$

(1200) Specialization: $r = t = u = 0$. The polynomials are a sum of Meixner polynomials [C1, p. 176] and Pollaczek polynomials:

$$b_1 = a+1, \quad b_n = a, \ n \neq 1, \quad \lambda_n = an^2,$$

(9.23)

$$p_n(x) = (-a)^{n/2} m_n\left(\frac{x-a-\sqrt{-a}}{2\sqrt{-a}}, -1, 1\right) + (a-x) P_{n-2}^{1/2}\left(\frac{x-a}{2\sqrt{a}}, \pi/2, 2\right)(2\sqrt{a})^{n-2}.$$

(2100) Specialization: $r = s = t = 0$. The polynomials are a sum of Meixner polynomials [C1, p. 176] and Pollaczek polynomials:

$$b_0 = a, \quad b_n = 1, \ n \neq 0, \quad \lambda_n = an^2,$$

(9.24)

$$p_n(x) = (-a)^{n/2} m_n\left(\frac{x-1-\sqrt{-a}}{2\sqrt{-a}}, -1, 1\right) + (1-a) P_{n-1}^{1/2}\left(\frac{x-1}{2\sqrt{a}}, \pi/2, 1\right)(2\sqrt{a})^{n-1}.$$

(2200) Specialization: $r = s = t = u = 0$. The polynomials are a sum of Meixner polynomials [C1, p. 176] and Pollaczek polynomials:

$$b_0 = a,\ b_1 = 1,\ \ b_n = 0,\ n > 1,\quad \lambda_n = an^2,$$

$$\begin{aligned} p_n(x) =&(-a)^{n/2} m_n\big(\frac{x-\sqrt{-a}}{2\sqrt{-a}}, -1, 1\big) - a(2\sqrt{a})^{n-1} P_{n-1}^{1/2}\big(\frac{x}{2\sqrt{a}}, \pi/2, 1\big) \\ &-(x-a)(2\sqrt{a})^{n-2} P_{n-2}^{1/2}\big(\frac{x}{2\sqrt{a}}, \pi/2, 2\big). \end{aligned} \tag{9.25}$$

(0211) Specialization: $t = u = v = p = 0$. The polynomials are sums of dual Lommel polynomials:

$$b_1 = 2a + 1,\ \ b_n = a(n+1),\ n \neq 1,\quad \lambda_n = a,$$

$$p_n(x) = (-a)^n K_n\big(\frac{a-x}{a}, \frac{1}{a}\big) + (a-x)(-a)^{n-2} K_{n-2}\big(\frac{3a-x}{a}, \frac{1}{a}\big). \tag{9.26}$$

10. q-analogs of classical orthogonal polynomials.

In §9 we gave several classical orthogonal polynomials which were specializations of the octabasic Laguerre polynomials. So each of these polynomials has its own multi-q version, by "despecializing" the q's which were set to 1. However, it is possible to specialize these remaining q's to obtain known q-analogs of classical orthogonal polynomials. In this section we give these specializations, and state the explicit formulas for the polynomials.

(0000), q-Laguerre [Si-St, (3.6)] ($r = t = b = q^{-2}$, $s = u = a = q^{-1}$, $p = \beta = 1$, $v = q^{-4}$, $w = q^{-3}$, $b_n = q^{-2n}[n]_q + q^{-1-2n}[n+\beta]_q$, $\lambda_n = q^{1-4n}[n]_q[n+\beta-1]_q$)

$$p_n(x) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{j^2-n^2} x^j \prod_{i=0}^{n-j-1} \frac{(1-q^{\beta+j+i})}{(q-1)}.$$

(1011), dual q-Lommel [I, (1.25), (2.10)] ($r = p = v = 0$, $s = 1$, $\{t, u\} = \{1, q^{-1}\}$, $w = q^{-1}$, $b_n = a + [n]_{1/q}$, $\lambda_n = aq^{1-n}$)

$$p_n(x) = q^{-n(n-1)/2}(-1)^n \sum_{j=0}^{n/2} \begin{bmatrix} n-j \\ j \end{bmatrix}_q (-a)^j q^{j^2} \prod_{k=j}^{n-j-1} (q[k]_q - q^k(x-a)).$$

(1010), q-Charlier [deM-St-Wh] ($r = p = 0$, $s = q$, $\{t, u\} = \{v, w\} = \{1, q\}$, $b_n = aq^n + [n]_q$, $\lambda_n = aq^{n-1}[n]_q$)

$$p_n(x) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q (-1)^n a^{n-j} q^{\binom{n-j}{2}} \prod_{k=0}^{j-1} ([k]_q - x).$$

(2110), discrete q-Hermite [Ga-Ra, p. 193] ($r = s = t = p = 0$, $a = u = v = 1$, $w = q$, $b_n = 1$, $\lambda_n = q^{n-1}[n]_q$)

$$p_n(x) = \sum_{j=0}^{n/2} \begin{bmatrix} n \\ 2j \end{bmatrix}_q (x-1)^{n-2j}(-1)^j q^{j(j-1)} \prod_{k=1}^{j} [2k-1]_q.$$

(1111), q-Chebyshev ($r = t = p = v = 0$, $s = u = w = q$, $b_0 = a$, $b_n = aq^n + q^{n-1}$, $n > 0$, $\lambda_n = aq^{2n-2}$)

$$p_n(x) = \sum_{j=0}^{n/2} q^{j(j-1)/2} x^{n-j}(-1)^j \sum_{k=0}^{j} \begin{bmatrix} n-1-k \\ n-1-j \end{bmatrix}_q \begin{bmatrix} n-j+k \\ k \end{bmatrix}_q a^k.$$

(2111), q-Chebyshev (several choices)
($r = s = t = p = v = 0$, $u = 1 = w$, $b_0 = a$, $b_n = 1$, $n > 0$, $\lambda_n = aq^{n-1}$)

$$p_n(x) = \sum_{j=0}^{n/2} (\begin{bmatrix} n-j \\ j \end{bmatrix}_q (x-1)^{n-2j} + (1-a)q^j \begin{bmatrix} n-j-1 \\ j \end{bmatrix}_q (x-1)^{n-2j-1}) q^{j(j-1)}(-a)^j.$$

($r = s = t = p = v = 0$, $w = 1/q$, $b_0 = a$, $b_n = u^{n-1}$, $n > 0$, $\lambda_n = a$, see [Is-Mu] for related results)

$$p_n(x) = \sum_{k=0}^{n} (-1)^k u^{k(k-3)/2} a^{(n-k)/2} \sum_{s=0}^{n-k} \begin{bmatrix} k+s \\ s \end{bmatrix}_u \begin{bmatrix} n-s \\ s \end{bmatrix}_u T_{n-k-2s}(\frac{x}{2\sqrt{a}}) +$$
$$\sum_{k=0}^{n-1} (-1)^k u^{k(k-3)/2} a^{(n-k-1)/2} \sum_{s=0}^{n-k-1} \begin{bmatrix} k+s \\ s \end{bmatrix}_u \begin{bmatrix} n-s-1 \\ s \end{bmatrix}_u T_{n-k-1-2s}(\frac{x}{2\sqrt{a}}).$$

11. Remarks.

Other specializations also are of combinatorial interest. For example, we can obtain up-down permutations, that is, the class $UD(N) \subseteq S_N$ of permutations σ such that $\sigma(1) < \sigma(2) > \sigma(3) < \ldots$. If we take $b_n = 0$ and $\lambda_n = n^2$, then the weighted Motzkin paths enumerated by the moment μ_{2n} have no horizontal steps, hence the corresponding permutations consist exclusively of runs of length 2. Thus, the moments $\mu_{2n} = |UD(2n)|$ are the secant numbers [Go-Ja, p. 169].

Returning briefly to the orbit 2100 (case (24), §4), if we take the specialization $r = s = t = 0$ and the other bases equal to 1, we get the three-term recurrence coefficients $b_n = 1, \lambda_n = n^2$. Comparing the Motzkin paths for this case with those for the previous specialization (this time, singleton runs are permitted and must be separated from the other, length 2, runs), the moments now are $\mu_n = \sum_{j\geq 0} \binom{n}{2j} |UD(2j)|$. In turn, this implies that the moments of the orbit 2100 of §4 have exponential generating function $\sum_{n\geq 0} \mu_n z^n/n! = e^z \sum_{n\geq 0} |UD(2n)| z^n/n! = e^z sec(z)$.

The tangent numbers, $|UD(2n+1)|$, are the moments (up to sign) if $b_n = -1$, $\lambda_n = -n^2$; this is the choice $a = -1$ in (1.1). A set of q-tangent numbers is given in [An-Ge]. We have a natural (r, s)-tangent number if

$$b_n = -[n+1]_{r,s} + [n]_{r,s}, \quad \lambda_n = -[n]_{r,s}^2. \tag{11.1}$$

Let

$$T_{2n+1}(r,s) = (-1)^n \mu_{2n+1}. \tag{11.2}$$

The following theorem is a companion to those in [An-Fo], [An-Ge].

Theorem 11.1. *We have*

$$T_{2n+1}(r,s) = \sum_{w \in UD^{1,2n+2}(2n+2)} r^{lsg'(w)} s^{rsg'(w)},$$

where $UD^{1,2n+2}(2n+2)$ is the set of up-down permutations of $[2n+2]$ in which 1 and $2n+2$ are in the same run, and the statistics lsg' and rsg' are defined by modifying lsg and rsg: closers do not count the run $(1, 2n+2)$. Moreover $(r+s)^n$ divides $T_{2n+1}(r,s)$.

Proof. If we put $\lambda'_{2n+1} = -[n+1]_{r,s}$, $\lambda'_{2n} = [n]_{r,s}$, then the tangent numbers arise from the moments of the odd polynomials. These have a three-term recurrence whose coefficients are a special case of (7.3), namely,

$$b_n(o) = 0, \quad \lambda_n(o) = -[n]_{r,s}[n+1]_{r,s}. \tag{11.3}$$

According to (6.2) and (6.4), $\mu_{2n+1}(e) = -\mu_{2n}(o)$. As in the argument preceding Theorem 7.2, the Motzkin paths enumerated here by the moment $\mu_{2n}(o)$ correspond to permutations in S_{2n+2} via the bijection φ. In this case, the Motzkin paths P have no horizontal steps (since $b_n(o) = 0$), so the permutations produced as $\varphi(P)$ consist exclusively of 2-element runs. In fact, the image of φ in this case is $UD^{1,2n+2}(2n+2)$. The description of the statistics lsg' and rsg' follows directly from the discussion in §7 of the effect of φ on lsg and rsg.

Finally, the Motzkin paths of length $2n$ considered here permit a pairing of every NE step with a SE step between the same two levels. In each such pair, the weight of one the the two steps is a monomial from $[2k]_{r,s}$ for some $k \geq 1$. In turn, $[2k]_{r,s}$ is divisible by $r+s$. Therefore, the sum of the weights of all P which have the same underlying (unweighted) path of length $2n$ is divisible by $(r+s)^n$. □

There is an easy bijection from the permutations in $UD(2n+2)$ which contain the run $1, 2n+2$ to $UD(2n+1)$. This is a modification of the bijection ψ of §7. Since all permutations in $UD^{1,2n+2}(2n+2)$ satisfy the condition (ii) (indeed, vacuously), we delete $2n+2$. But now we must eliminate the two consecutive ascents resulting if $2n+2$ was not in the last position. To this end, we reverse the word to the right of 1. For example, $36|18|25|47 \in UD^{1,8}(8)$ maps to $36|17|45|2 \in UD(7)$. A version of Theorem 11.1 could be given for $UD(2n+1)$.

An (r,s)-analogue of the secant numbers, $E_{2n} = |UD(2n)|$, arises as $E_{2n}(r,s) = \mu_{2n}$ if $b_n = 0$, $\lambda_n = [n]^2_{r,s}$, and we have (see [An-Fo])

$$E_{2n}(r,s) \equiv 1 \mod (r+s)^2.$$

Indeed, the Motzkin paths enumerated by the moments are without horizontal steps and all steps between levels k and $k-1$ have weights from $[k]_{r,s}$. Thus, the unique path which does not exceed level 1 has weight 1, while every other path has at least two steps between levels 1 and 2, and its total contribution to the moment is divisible by $[2]_{r,s}^2$.

Alternative descriptions of the moments for the orbit (3), 1111, of §4 can be given in terms of permutations. Choosing the specialization $r = p = t = v = 0$, it is easy to see from Theorem 2.1 that the moments enumerate the 132-avoiding permutations, i.e., permutations σ such that there is no triple $i < j < k$ with $\sigma(i) < \sigma(k) < \sigma(j)$. Also, for the choice $s = q = u = w = 0$, the moments count the 213-avoiding permutations, i.e., permutations σ such that there is no triple $i < j < k$ with $\sigma(j) < \sigma(i) < \sigma(k)$. The orbit 1111 has size 16 here, so there are 14 other subsets of permutations which are enumerated by the Catalan numbers. Analogous statements can be made for the other orbits.

We could define an octabasic version of the Laguerre polynomials $L_n^\alpha(x)$, by putting

$$(11.4)\quad b_n = (\alpha + r^{n-1}s + \cdots + s^n) + [n]_{t,u}, \quad \lambda_n = [n]_{p,q}(\alpha + v^{n-2}w + \cdots + w^{n-1}).$$

Appendix.

We list the explicit formulas for each of the classical polynomials in §7. We use the notation $(a)_k$ for the rising factorial. We also give the recurrence relation coefficients for the monic forms.

Charlier, $b_n = n + a, \lambda_n = an$,[C1, p. 170]

$$C_n^a(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{x}{k} k!(-a)^{n-k}.$$

Meixner, $b_n = ((1+c)n + c\beta)/(1-c), \lambda_n = cn(n+\beta-1)/(1-c)^2$, [C1, p. 176, (3.5)]

$$m_n(x, c, \beta) = n!(-1)^n \sum_{k=0}^{n} \binom{-x-\beta}{n-k}\binom{x}{k}(1/c)^k.$$

Chebyshev, $b_n = 0, \lambda_n = 1$,[C1, p. 143]

$$U_n(x) = \sum_{m=0}^{n/2} \binom{n-m}{m}(-1)^m (2x)^{n-2m}.$$

Hermite, $b_n = 0, \lambda_n = n/2$,[C1, p. 146]

$$H_n(x) = n! \sum_{k=0}^{n/2} \frac{(-1)^k (2x)^{n-2k}}{k!(n-2k)!}.$$

Associated Hermite, $b_n = 0, \lambda_n = (n+c)/2$,[As-Wm, (4.18)]

$$H_n(x, c) = \sum_{k=0}^{n/2} \frac{(-2)^k (c)_k (n-k)!}{k!(n-2k)!} H_{n-2k}(x).$$

Associated Charlier, $b_n = n + a + c, \lambda_n = a(n + c)$, [I-L-V]

$$C_n^a(x, c) = (-1)^n (c+1)_n \sum_{0 \le k+p \le n} \frac{(x+1)_k}{k!} \frac{(-x)_{n-k-p}}{(n-k-p)!} \frac{ca^p}{(c+k)_{p+1}}.$$

Dual Lommel polynomials, $b_n = -n, \lambda_n = \beta$, [Ma]

$$K_n(x, \beta) = \sum_{k=0}^{n/2} \binom{n-k}{k} (x+k)_{n-2k} (-\beta)^k.$$

Meixner-Pollaczek,[C1, p. 186, (5.13)], $(c = 0, \alpha = cot(\phi))$, $b_n = -\alpha(n+\mu)$, $\lambda_n = n(n + 2\mu - 1)(1 + \alpha^2)/4$,

$$P_n^\mu(x, \phi) = \frac{(2\mu)_n}{n!} e^{in\phi} \sum_{k=0}^{n} \binom{n}{k} \frac{(\mu + ix)_k}{(2\mu)_n} (e^{-2i\phi} - 1)^k.$$

Associated Meixner-Pollaczek (Pollaczek), [C1, p. 186, (5.13)] $b_n = -\alpha(n + \mu + c)$, $\lambda_n = (n + c)(n + 2\mu + c - 1)(1 + \alpha^2)/4$,

$$P_n^\mu(x, \phi, c) = \frac{(c+1)_n}{(2sin(\phi))^n} \sum_{k=0}^{n} \frac{c}{c+k} P_k^{1-\mu}(-x, \phi) P_{n-k}^\mu(x, \phi).$$

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