AN ALGORITHMIC INVOLUTION FOR $P(N)$

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Abstract. An involution ϕ is given on the set of all integer partitions of n. The cardinality of the fixed point set of ϕ is $< n^2$ and has the same parity as $p(n)$. Moreover, $\phi(\lambda)$ can found in polynomial time.

1. Introduction.

Let $p(n)$ be the number of integer partitions of n [1]. It is well-known that $p(n)$ grows exponentially in n, and that $p(n)$ can be computed in from Euler's recurrence formula. Euler's recurrence requires about $\sqrt{n}$ previous $p(i)$ to find $p(n)$. Thus the parity of $p(n)$ can be found in polynomial time. Wilf [6] asked if there was a constructive approach for the parity question: are there sets P and M such that partitions of n with parts sizes from P and multiplicities from M grow polynomially, and have the same parity as $p(n)$. We do not answer Wilf's question in this paper, rather we answer a combinatorial form of his question: find an involution ϕ on the set of all partitions of n whose fixed point set has cardinality which is polynomial in n. The fixed points of ϕ clearly have the same parity as $p(n)$ and would be a constructive solution. In this paper we provide such an involution.

The "best involution" ϕ would have the following two properties. λ would be a fixed point of ϕ if, and only if, some easily verified condition held, e.g. in $O(1)$ time. One may also ask for $O(1)$ time to find the image $\phi(\lambda)$. The definition of ϕ in this paper allows a fixed point to be found in $O(n)$ time, and the image $\phi(\lambda)$ also in $O(n)$ time once n^2 computations have initially been done.

The involution is given in §2, and depends upon a certain listing algorithm for partitions. This listing idea applies in other situations, and two examples are given in §3.

Perhaps what makes this problem unwieldy is the lack of sets of partitions of n which are polynomial in n. In fact, the only non-trivial naturally occurring such sets known to the authors are t-cores (see [2]) and those in this paper.

2. The involution.

Before we set notation, we give the basic idea of the involution ϕ. Consider an upper triangular matrix whose entry positions will be called *cells*. Each partition λ of n will correspond to a partition which lies in one of these cells. A given cell may be empty or may contain several partitions. There will be a unique path from a

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partition in a cell, along other partitions in other cells, which terminates at the cell [1,1]. A swapping procedure on these paths, similar to the swapping of tails in the Gessel-Viennot lattice path involution, will be the involution. The cardinality of the fixed point set will come from the precise definition of the cells and the paths.

Let $[m, k]$ denote the cell in the m th column and k th row for $1 \leq k \leq m < \infty$. We shall place in cell $[m, k]$ all partitions of m whose largest part is k which do not contain 1's as parts. We additionally place the partition 1 in the cell [1,1]. See Figure 2. Thus the cell [8,4] contains the partitions 44 and 422. Note that [1,1] is the only cell which contains a partition with a part of size 1.

If $f(m, k)$ is the number of partitions in cell [m,k], then

$$f(m, k) = f(m-1, k-1) + f(m-k, k), \tag{2.1}$$

where

$$f(n, 1) = \begin{cases} 1 & \text{for n = 1,} \\ 0 & \text{otherwise.} \end{cases}$$

(2.1) is interpreted as removing the largest part if it is repeated, or removing one from the largest part if is is not repeated. We call the cells $[m-1, k-1]$ and $[m-k, k]$ the two cells *previous* to $[m, k]$.

Given a partition λ of n, let λ^* be the partition obtained from λ by removing all parts of size 1. Then λ^* lies in some cell $[m, k]$, $1 \leq k \leq m \leq n$, except for the partition $11 \cdots 1$, which lies in cell [1,1]. So

$$p(n) = \sum_{m=1}^{n} \sum_{k=1}^{m} f(m, k).$$

Thus the parity of $p(n)$ is determined by the parity of $f(m, k)$, for $1 \leq k \leq m \leq n$. Note that there are $\binom{n+1}{2}$ such numbers $f(m, k)$.

Given a partition in cell $[m, k]$, there is a unique path along others cells back to the cell [1,1]. The path is determined by the recurrence (2.1): remove the largest part if it repeats, otherwise delete one from the largest part. So the path for 66332 is

$$66332 \to 6332 \to 5332 \to 4332 \to 3332 \to 332 \to 32 \to 22 \to 2 \to 1.$$

The path can be encoded by a word with letters L (for moving left, if the largest part is deleted), and D (for moving diagonally northwest) if 1 is subtracted from the largest part. The word for the above example is $LDDDLLDLD$.

Proposition 1. *Every partition λ in a cell corresponds to a unique path $Path(\lambda)$ of partitions back to 1 in cell [1,1], or equivalently some word in L and D.*

We need one more ingredient to define the involution ϕ. Suppose that the parity of $f(m, k)$ has been computed from (2.1) for $1 \leq k \leq m \leq n$. We define the *label* for the cell $[m, k]$ as the parity of $f(m, k)$. The two cells previous to any cell labeled with a 1 must have labels 0 and 1. Thus we have the next Proposition.

Proposition 2. *To each cell C labeled 1 their corresponds a unique path of cells, $Path(C)$, each labeled 1, back to cell [1,1]. This path uniquely defines a partition inside cell C.*

We now define $\phi(\lambda)$. Let w be the word of λ^*. Assume that $f(m, k) \mod 2$ has been computed for $1 \leq k \leq m \leq n$. Consider the labels of the cells in the path $Path(\lambda^*)$ from Proposition 1. If all of the labels are 1, then λ is fixed by ϕ. Otherwise find the last cell C in $Path(\lambda^*)$ whose label is 0. The two cells previous to C, C_1 and C_2, must have equal labels, and since one of them is on $Path(\lambda^*)$, the label must be 1. From Proposition 2, there are unique paths from C_1 and C_2 back to [1,1]. Since the labels on $Path(\lambda^*)$ are 1 past C, we can assume that the path from C_1 to [1,1] is the tail of $Path(\lambda^*)$. We replace this tail by the path from C_2 to [1,1], and also replace the edge $C \to C_1$ by $C \to C_2$. This defines $\phi(\lambda^*)$. We then readjoin the 1's to define $\phi(\lambda)$.

For an example, we need the labels of the cells $[m, k]$ given in Figure 1. Let $\lambda = 43222111$ so that $\lambda^* = 43222$ lies in cell [13,4]. Then

$$Path(\lambda^*) = 43222 \to 33222 \to 3222 \to 2222 \to 222 \to 22 \to 2 \to 1 = DLDLLLD.$$

The label sequence is 10011111, so that the last 0 label occurs at cell [9,3], for 3222. The cells previous to [9,3] are [8,2] and [6,3], whose unique paths back to [1,1] give the partitions 2222= $LLLD$ and 33= LDD. This gives $\phi(3222) = 333$. So we replace the tail $LLLD$ of $DLDLLLD$ by LDD, and the third letter D ($[9,3] \to [8,2]$) by L ($[9,3] \to [6,3]$) to define $\phi(43222) = DLLLDD = 4333$. Adding the 1's gives $\phi(\lambda) = 4333111$.

It is clear that the involution ϕ has exactly one fixed point for each cell labeled 1. Thus the number of fixed points as ϕ acts on the set of all partitions of n is at most $\binom{n+1}{2} \leq n^2$.

Proposition 3. *If the involution ϕ is restricted to partitions of m whose largest part is k and which contain no 1's, then the fixed point set of ϕ has cardinality 0 or 1.*

3. Remarks.

The technique for constructing the involution ϕ applies in great generality. All that is required is a recursive formula such as (2.1) with two terms on the right side, and a combinatorial interpretation of that recurrence formula so that Proposition 1 holds. Some simple examples include binomial coefficients and Stirling numbers of both kinds. For example, the Stirling numbers of the second kind satisfy

$$S(n,k) = S(n-1,k-1) + k \quad S(n-1,k). \tag{3.1}$$

There is an easy involution on the set of blocks of a set partition which reduces the coefficient $k \mod 2$. This allows paths to be defined which have no edges from $[n-1,k]$ to $[n,k]$ if k is even. The parity of $S(n,k)$ is then determined by cell $[n,k]$.

What makes the involution ϕ remarkable is that a single "primitive" pairing (e.g. $\phi(3222) = 333$) induces infinitely many pairings in an infinite tree. We see that $\phi(3222) = 333$ implies $\phi(3^j 222) = 3^{j+2}$ (moving horizontally across Figure 2), and $\phi(m222) = m33$, (moving diagonally down Figure 2). Moreover, each of these new pairings each induces infinitely many new pairings in the same manner. In this way it is possible to have a polynomial number of primitive pairings induce an exponential number of pairings. The remaining fixed point set is polynomial.

The technique in §2 applies to other sets of partitions. The parity of the number of partitions of n into distinct parts is known from the Euler pentagonal theorem [1]. Moreover Franklin's involution, which proves it, has a fixed point set of size zero or one. However Franklin's involution naturally appears if we list partitions with distinct parts as in Figure 2. Since the part 1 can appear at most once, all such partitions of n occur in columns n or $n-1$. Clearly a partition in cell $[n-1,k]$ has natural pairing with a partition in cell $[n,k+1]$, by increasing the largest part by one. This pairing is a special case of the Franklin involution. The remaining unpaired partitions all lie in column n, and have an initial run of length at least 2. The Franklin involution completes the job.

Bijections may also be found using the listing technique. For example, one may take Euler's "odds"="distinct" theorem. List the partitions with odd parts as in Figure 2. The conjugate of a partition with distinct parts must contain all parts sizes from 1 to its largest part. We list these partitions as in Figure 2, modified so that one part of size 1 is always included. In this way the additional 1's from the odd parts correspond exactly to the additional ones from the conjugates of the partitions with distinct parts. A careful consideration of these two figures leads to Sylvester's "fishhook" bijection. It would be natural to try the technique on other sets of partitions, such as the Rogers-Ramanujan identities.

Unfortunately the technique in §2 applies only to congruences mod 2. For general primes more techniques are available see [3,4,5].

1	2	3	4	5	6	7	8	9	n/k
1	0	0	0	0	0	0	0	0	1
	1	0	1	0	1	0	1	0	2
		1	0	1	1	1	1	0	3
			1	0	1	1	0	1	4
				1	0	1	1	0	5
					1	0	1	1	6
						1	0	1	7
							1	0	8
								1	9

Figure 1: f(n,k) mod 2

1	2	3	4	5	6	7	8	9	n/k
1									1
	2		22		222		2222		2
		3		32	33	322	332	3222,333	3
			4		42	43	44,422	432	4
				5		52	53	54,522	5
					6		62	63	6
						7		72	7
							8		8
								9	9

Figure 2: Partitions counted by f(n,k)

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