ON q-INTEGRALS OVER ORDER POLYTOPES

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Abstract. A combinatorial study of multiple q-integrals is developed. This includes a q-volume of a convex polytope, which depends upon the order of q-integration. A multiple q-integral over an order polytope of a poset is interpreted as a generating function of linear extensions of the poset. Specific modifications of posets are shown to give predictable changes in q-integrals over their respective order polytopes. This method is used to combinatorially evaluate some generalized q-beta integrals. One such application is a combinatorial interpretation of a q-Selberg integral. New generating functions for generalized Gelfand-Tsetlin patterns and reverse plane partitions are established. A q-analogue to a well known result in Ehrhart theory is generalized using q-volumes and q-Ehrhart polynomials.

Contents

1. Introduction

The main object in this paper is the q-integral

$$\int_0^1 f(x) d_q x = (1-q)\sum_{i=0}^{n} f(q^i)q^i,$$

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which was introduced by Thomae [23] and Jackson [12]. The q-integral is a q-analogue of the Riemann integral. Fermat used it to evaluate $\int_0^1 x^n dx$. See [2, §10.1] for more details of the history of q-integrals. Many important integrals have q-analogues in terms of q-integrals, such as q-beta integrals and q-Selberg integrals. In this paper we develop combinatorial methods to study q-integrals.

The original motivation of this paper was to generalize Stanley's combinatorial interpretation of the Selberg integral [19]

$$\begin{aligned} S_n(\alpha,\beta,\gamma) &= \int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^{\alpha-1}(1-x_i)^{\beta-1} \prod_{1\le i<j\le n} |x_i-x_j|^{2\gamma} dx_1\cdots dx_n \\ &= \prod_{j=1}^n \frac{\Gamma(\alpha+(j-1)\gamma)\Gamma(\beta+(j-1)\gamma)\Gamma(1+j\gamma)}{\Gamma(\alpha+\beta+(n+j-2)\gamma)\Gamma(1+\gamma)}, \end{aligned}$$

where n is a positive integer and α,β,γ are complex numbers such that $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$, and $\mathrm{Re}(\gamma) > -\min\{1/n, \mathrm{Re}(\alpha)/(n-1), \mathrm{Re}(\beta)/(n-1)\}$. Stanley [22, Exercise 1.10 (b)] found a combinatorial interpretation of the above integral when $\alpha-1,\beta-1$ and 2γ are nonnegative integers in terms of permutations. His idea is to interprete the integral as the probability that a random permutation satisfies certain properties. This idea uses the fact that a real number $x\in(0,1)$ can be understood as the probability that a random number selected from $(0,1)$ lies on an interval of length x is equal to x.

In order to find a combinatorial interpretation of a q-analogue of this integral, see (12), we take a different approach. We interpret q-integrals as generating functions in q. This is not surprising, as the q-integral itself is a sum. Here is a brief summary of our approach to this problem. We will define a q-volume of a polytope by a certain multiple q-integral. The polytopes of interest are order polytopes of posets. We shall see that simple operations on posets correspond to insertions of polynomials in the integrands of the multiple q-integral. Using these simple operations, we can define a poset whose order polytope has a q-volume given by a q-Selberg integral. We show that the q-volume of an order polytope is a generating function for linear extensions of the poset. This gives a combinatorial interpretation of a q-Selberg integral, see Corollary 7.7.

The purpose of this paper is not limited to answering the motivational question on the Selberg integral. We have examples and applications of the combinatorial methods developed in this paper.

The key property of q-integrals is the failure of Fubini's theorem, but in a controlled way, see Corollary 4.9. We show in Theorem 4.4 that q-volume of the order polytope of a poset P is equal to a generating function for (P,ω)-partitions, where the labeling ω of the poset P corresponds to the order of integration. Equivalently, using a well known fact in (P,ω)-partition theory due to Stanley [20], this is equal to a generating function for the linear extensions of P.

The remainder of this paper is organized as follows.

In Section 2 we give definitions that are used throughout the paper.

In Section 3 we prove basic properties of the q-integrals. We investigate how Fubini's theorem fails and when it holds. We give an expansion formula for the q-integral over a polytope determined by certain inequalities.

In Section 4 we study q-integrals over order polytopes of posets. We show that the q-volume of the order polytope of a poset is the *maj*-generating function for the linear extensions of the poset, up to a constant factor.

In Section 5 we consider simple operations on P such as adding a new chain. We show how the q-integral changes over the order polytope of P when these operations are performed.

In Section 6, using the results in the previous sections, we consider several q-integrals: the q-beta integral, a q-analogue of Dirichlet's integral, a generalized q-beta integral due to Andrews and Askey [1].

In Section 7 we give a combinatorial interpretation of Askey's q-Selberg integral in terms of a generating function of the linear extensions of a poset.

In Section 8 we study reverse plane partitions using Selberg-type q-integrals which involve Schur functions. We show that these q-integrals are essentially generating functions of reverse plane partitions with certain weights. By using known evaluation formulas for these q-integrals we obtain a formula for the generating function for the reverse plane partitions with fixed shape (both shifted and normal) and fixed diagonal entries. This can be restated as a generating function for generalized Gelfand-Tsetlin patterns. Taking the sum of these generating functions yields the well known trace-generating function formulas for reverse plane partitions of fixed (shifted or normal) shape. We also show that Askey's q-Selberg integral is equivalent to a new generating function for reverse plane partitions of a square shape. This allows us to obtain a new product formula for the generating function for reverse plane partitions of a square shape with a certain weight.

In Section 9 we study q-Ehrhart polynomials and q-Ehrhart series of order polytopes using q-integrals. We show that the q-Ehrhart function of an order polytope is a polynomial in a particular sense whose leading coefficient is the q-volume of the order polytope of the dual poset.

2. Definitions

In this section we give the necessary definitions with examples for q-integration, multiple q-integration, and order polytopes of posets.

Throughout this paper we assume $0 < q < 1$. We will use the following notation for q-series:

$$[n]_q = \frac{1-q^n}{1-q}, \qquad [n]_q! = [1]_q[2]_q\cdots[n]_q, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!},$$

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \qquad (a_1,a_2,\dots,a_k;q)_n = (a_1;q)_n\cdots(a_k;q)_n.$$

We also use the notation $[n] := \{1,2,\dots,n\}$. We denote by S_n the set of permutations on $[n]$.

Definition 2.1. Let $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$. An integer $i \in [n-1]$ is called a *descent* of π if $\pi_i > \pi_{i+1}$. Let $\mathrm{Des}(\pi)$ be the set of descents of π. We define $\mathrm{des}(\pi)$ and $\mathrm{maj}(\pi)$ to be the number of descents of π and the sum of descents of π, respectively. We denote by $\mathrm{inv}(\pi)$ the number of pairs (i,j) such that $1 \le i < j \le n$ and $\pi_i > \pi_j$.

First, recall [2, §10.1] the q-integral of a function f over (a,b).

Definition 2.2. For $0 < q < 1$, the *q-integral from a to b* is defined by

$$\int_a^b f(x)d_qx = (1-q)\sum_{i=0}^{\infty}\left(f(bq^i)bq^i - f(aq^i)aq^i\right). \tag{1}$$

In the limit as $q \to 1$, the q-integral becomes the usual integral. It is easy to see that

$$\int_a^b x^n d_qx = \frac{b^{n+1}-a^{n+1}}{[n+1]_q}.$$

We extend the definition of a q-integral to a multiple q-integral over a convex polytope. Here the ordering π of the variables is important because the iterated q-integral is not, in general, independent of the ordering.

Definition 2.3. Let $\pi = \pi_1\cdots\pi_n \in S_n$. For a function $f(x)$ of n-variables $x = (x_1,\dots,x_n)$ and a convex polytope $D \in \mathbb{R}^n$, *the q-integral of $f(x)$ over D with respect to order π of integration* is defined by

$$\int_D f(x_1,\dots,x_n)d_qx_{\pi_1}\cdots d_qx_{\pi_n} = \int_{\min(D_n)}^{\max(D_n)}\cdots\int_{\min(D_1)}^{\max(D_1)} f(x_1,\dots,x_n)d_qx_{\pi_1}\cdots d_qx_{\pi_n},$$

where D_i is the set of real numbers depending on the values of $x_{\pi_{i+1}},\dots,x_{\pi_n}$ given by

$$D_i = D_i(x_{\pi_{i+1}},\dots,x_{\pi_n}) = \{y_{\pi_i} : (y_1,\dots,y_n) \in D, \quad y_{\pi_j} = x_{\pi_j} \text{ for } j > i\}.$$

If a convex polytope D is determined by a family Q of inequalities of the sorted variables $x = (x_1,\dots,x_n)$, then we will also write

$$\int_Q f(x)d_qx = \int_D f(x)d_qx, \qquad \text{where } d_qx = d_qx_1\cdots d_qx_n.$$

Example 2.4. We have

$$\int_{\substack{a\le x\le b\\ c\le y\le d}} f(x,y)d_qxd_qy = \int_{[a,b]\times[c,d]} f(x,y)d_qxd_qy = \int_c^d\int_a^b f(x,y)d_qxd_qy,$$

and, for $x=(x_1,\dots,x_6)$,

$$\begin{aligned}\int_{a\le x_3\le x_1\le x_5\le x_2\le x_4\le x_6\le b} f(x)d_qx &= \int_{\{(x_1,\dots,x_6):a\le x_3\le x_1\le x_5\le x_2\le x_4\le x_6\le b\}} f(x)d_qx\\ &= \int_a^b\int_a^{x_6}\int_{x_5}^{x_6}\int_a^{x_5}\int_{x_5}^{x_4}\int_{x_3}^{x_5} f(x_1,\dots,x_6)d_qx_1\cdots d_qx_6.\end{aligned}$$

Note that we have

$$\lim_{q\to1}\int_D f(x_1,\dots,x_n)d_qx_{\pi_1}\cdots d_qx_{\pi_n} = \int_D f(x_1,\dots,x_n)dx_{\pi_1}\cdots dx_{\pi_n},$$

which is independent of the ordering π if Fubini's theorem holds. Unlike usual integrals, we do not have always Fubini's theorem for q-integrals of well behaved functions. For example,

$$\int_{0\le x_1\le x_2\le 1} d_qx_1d_qx_2 = \frac{1}{1+q}, \qquad \int_{0\le x_1\le x_2\le 1} d_qx_2d_qx_1 = \frac{q}{1+q}.$$

We next define the q-volume of a convex polytope. Since the q-integral depends on the order of integration, we need to specify that ordering.

Definition 2.5. Suppose that D is a convex polytope in $\mathbb{R}^n$ and $w=w_1\cdots w_n$ is a permutation of the coordinates $x_1,x_2,\dots,x_n$ of $\mathbb{R}^n$. Then the *q-volume of D with respect to w* is

$$V_q(D,w) = \int_D d_qw_1\cdots d_qw_n.$$

If $w=x_1x_2\cdots x_n$, then we will omit w and simply write $V_q(D)=V_q(D,w)$, that is,

$$V_q(D) = \int_D d_qx_1\cdots d_qx_n.$$

Example 2.6. If $D_1=\{(x_1,x_2,y)\in[0,1]^3 : x_1\le y\le x_2\}$, then

$$V_q(D_1, yx_1x_2) = \int_{0\le x_1\le y\le x_2\le 1} d_qyd_qx_1d_qx_2,$$

and if $D_2=\{(x_1,x_2,x_3)\in[0,1]^3 : x_3\le x_1\le x_2\}$, then

$$V_q(D_2) = \int_{0\le x_3\le x_1\le x_2\le 1} d_qx_1d_qx_2d_qx_3.$$

In most of this paper we will integrate over order polytopes of partially ordered sets (posets).

Definition 2.7. If P is a poset with n elements, a *labeling* of P is a bijection $\omega : P\to[n]$. A pair (P,ω) of a poset P and its labeling ω is called a *labeled poset*. If $\omega(x)\le\omega(y)$ for any $x\le_P y$, we say that ω is a *natural labeling* of P, or P is *naturally labeled*.

We need some rudiments of P-partition theory, which appear in [22, Chapter 3].

Definition 2.8. Let (P,ω) be a labeled poset. A *(P,ω)-partition* is a function $\sigma : P\to\{0,1,2,\dots\}$ such that

- $\sigma(x)\ge\sigma(y)$ if $x\le_P y$,
- $\sigma(x)>\sigma(y)$ if $x\le_P y$ and $\omega(x)>\omega(y)$.

For a (P,ω)-partition σ, the *size of* σ is defined by

$$|\sigma| = \sum_{x\in P}\sigma(x).$$

Definition 2.9. A *linear extension of* P is an arrangement $(t_1,t_2,\dots,t_n)$ of the elements in P such that if $t_i<_P t_j$ then $i<j$.

Definition 2.10. The *Jordan-Hölder set* $\mathcal{L}(P,\omega)$ of P is the set of permutations of the form $\omega(t_1)\omega(t_2)\cdots\omega(t_n)$ for some linear extension $(t_1,t_2,\dots,t_n)$ of P.

It is well known [22, Theorem 3.15.7] that

$$\sum_{\sigma} q^{|\sigma|} = \frac{\sum_{\pi\in\mathcal{L}(P,\omega)} q^{\mathrm{maj}(\pi)}}{(q;q)_n}, \tag{2}$$

where the sum is over all (P,ω)-partitions σ.

We next define a polytope obtained from a poset in a natural way. For simplicity, we will use the same letter x_i for the elements x_i of a poset P, the coordinates of $\mathbb{R}^n$, and also the integration variables.

Definition 2.11. Let P be a poset on $\{x_1,\dots,x_n\}$. For an n-dimensional box

$$I = \{(x_1,\dots,x_n) : a_i \le x_i \le b_i\},$$

the truncated order polytope of P inside I is defined by

$$\mathcal{O}_I(P) = \{(x_1,\dots,x_n)\in I : x_i \le x_j \text{ if } x_i \le_P x_j\}.$$

The order polytope of P is defined by

$$\mathcal{O}(P) = \mathcal{O}_{[0,1]^n}(P).$$

An important special case of order polytopes is a simplex, which is an order polytope of a chain. Let us first define chains and anti-chains.

Definition 2.12. Let P be a poset. Two elements x and y are called *comparable* if $x\le_P y$ or $y\le_P x$, and *incomparable* otherwise. A *chain* is a poset in which any two elements are comparable. An *anti-chain* is a poset in which any two distinct elements are incomparable.

Definition 2.13. For $\pi=\pi_1\cdots\pi_n\in S_n$, we denote by P_π the chain on $\{x_1,\dots,x_n\}$ with relations $x_{\pi_1}<x_{\pi_2}<\cdots<x_{\pi_n}$. For real numbers $a<b$, we call

$$\mathcal{O}_{[a,b]^n}(P_\pi) = \{(x_1,\dots,x_n)\in\mathbb{R}^n : a\le x_{\pi_1}\le\cdots\le x_{\pi_n}\le b\}$$

the *truncated simplex*.

Note that $\mathcal{O}_{[0,1]^n}(P_\pi)=\mathcal{O}(P_\pi)$ is the standard simplex which corresponds to a permutation π and has volume $1/n!$.

We end this section with definitions for partitions and Schur functions.

Definition 2.14. A *partition* is a weakly decreasing sequence $\lambda=(\lambda_1,\dots,\lambda_n)$ of non-negative integers. Each λ_i is called a *part* of λ. The *length* $\ell(\lambda)$ of λ is the number of nonzero parts. We identify a partition λ with its *Young diagram*

$$\lambda = \{(i,j) : 1\le i\le \ell(\lambda),\quad 1\le j\le\lambda_i\}.$$

The *transpose* λ' of a partition λ is defined by

$$\lambda' = \{(j,i) : 1\le i\le \ell(\lambda),\quad 1\le j\le\lambda_i\}.$$

If λ has m_i parts equal to i for $i\ge 1$, we also write λ as $(1^{m_1},2^{m_2},\dots)$. For two partitions $\lambda=(\lambda_1,\dots,\lambda_n)$ and $\mu=(\mu_1,\dots,\mu_n)$ we define

$$\lambda+\mu = (\lambda_1+\mu_1,\lambda_2+\mu_2,\dots,\lambda_n+\mu_n).$$

We also define

$$\delta_n = (n-1,n-2,\dots,1,0).$$

Definition 2.15. Let Par_n denote the set of all partitions with length at most n. For a partition $\lambda\in\mathrm{Par}_n$, we denote

$$b(\lambda) = \sum_{i=1}^n (i-1)\lambda_i,$$

and

$$q^\lambda = (q^{\lambda_1},\dots,q^{\lambda_n}).$$

Definition 2.16. For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, the *alternant* $a_\lambda(x_1, \dots, x_n)$ is defined by

$$a_\lambda(x_1, \dots, x_n) = \det(x_j^{\lambda_i+n-i})_{i,j=1}^n.$$

Definition 2.17. For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, the *Schur function* $s_\lambda(x_1, \dots, x_n)$ is defined by

$$s_\lambda(x_1, \dots, x_n) = \frac{a_{\lambda+\delta_n}(x_1, \dots, x_n)}{a_{\delta_n}(x_1, \dots, x_n)}.$$

Remark 2.18. Note that denominator of the Schur function is the Vandermonde determinant

$$a_{\delta_n}(x_1, \dots, x_n) = \prod_{1\le i<j\le n} (x_i - x_j) = \Delta(x).$$

We shall also use a version of the Vandermonde determinant which is positive on $x_1 \le x_2 \le \cdots \le x_n$,

$$\overline{\Delta}(x) = \prod_{1\le i<j\le n} (x_j - x_i) = (-1)^{\binom{n}{2}}\Delta(x).$$

3. Properties of q-integrals

In this section we prove several basic properties of the q-integrals. We give explicit examples when Fubini's theorem of interchanging the order of integration is allowed (see Proposition 3.2, Corollary 3.6), and one example how it fails (see Proposition 3.3). Finally we give a general technical expansion of a q-integral over a special polytope in Proposition 3.4. It is applied to an arbitrary interchange of the order of integration in Corollary 3.7.

Lemma 3.1. *If a and b are integers such that $a \le b$, then*

$$\int_{q^b}^{q^a} f(x) d_qx = (1-q)\sum_{i=a}^{b-1} f(q^i)q^i.$$

Proof. This follows easily using the definition (1). ☐

When the range for x and y are independent, then we can change the order of integration for these variables.

Proposition 3.2. *We have*

$$\int_c^d \int_a^b f(x,y) d_qx d_qy = \int_a^b \int_c^d f(x,y) d_qy d_qx.$$

Proof. This follows immediately from the definition (1) of the q-integral. ☐

An example of the failure of Fubini's theorem is the following two variable computation on triangles. Note that if $q \to 1$, the difference is 0.

Proposition 3.3. *For $a \le b$, we have*

$$\int_{a\le x\le y\le b} f(x,y) d_qx d_qy - \int_{a\le x\le y\le b} f(x,y) d_qy d_qx = (1-q)\int_{a\le x\le b} x f(x,x) d_qx.$$

Proof. The first integral on the left side is

$$\begin{aligned} L_1 &= (1-q)\int_{a\le y\le b} \sum_{i=0}^{\infty} \big(yf(yq^i,y) - af(aq^i,y)\big) q^i d_qy \\ &= (1-q)^2 \sum_{i,j=0}^{\infty} \Bigg(\Big(b^2q^{i+2j} f(bq^{i+j}, bq^j) - abq^{i+j} f(aq^i, bq^j) \Big) \\ &\qquad - \Big(a^2q^{i+2j} f(aq^{i+j}, aq^j) - a^2q^{i+j} f(aq^i, bq^j) \Big) \Bigg). \end{aligned}$$

Similarly the second integral on the left side is

$$L_2 = (1-q)\int_{a\le x\le b}\sum_{j=0}^{\infty}\big(bf(x,bq^j)-xf(x,xq^j)\big)q^j d_qx$$
$$= (1-q)^2\sum_{i,j=0}^{\infty}\Bigg(\Big(b^2q^{i+j}f(bq^i,bq^j)-b^2q^{2i+j}f(bq^i,bq^{i+j})\Big)$$
$$-\Big(abq^{i+j}f(aq^i,bq^j)-a^2q^{2i+j}f(aq^i,bq^{i+j})\Big)\Bigg).$$

So the difference is

$$L_1-L_2=(1-q)^2\sum_{i,j=0}^{\infty}\Bigg(b^2\left(q^{i+2j}f(bq^{i+j},bq^j)-q^{i+j}f(bq^i,bq^j)+q^{2i+j}f(bq^i,bq^{i+j})\right)$$
$$-a^2\left(q^{i+2j}f(aq^{i+j},aq^j)-q^{i+j}f(aq^i,aq^j)+q^{2i+j}f(aq^i,aq^{i+j})\right)\Bigg).$$

Let's check when a term $q^{s+t}f(bq^s,bq^t)$ occurs in the first three terms, for non-negative integers s and t. The first term allows $s\ge t$, the third term $s\le t$, while the second term is all s,t. So the remaining terms are

$$(1-q)^2\sum_{s=0}^{\infty}\left(b^2q^{2s}f(bq^s,bq^s)-a^2q^{2s}f(aq^s,aq^s)\right)=(1-q)\int_a^b xf(x,x)d_qx.$$

$\square$

Note that Proposition 3.3 implies that if $f(x,y)$ vanishes on the boundary $x=y$, then we can exchange the order of integration

$$\int_{a\le x\le y\le b}f(x,y)d_qxd_qy=\int_{a\le x\le y\le b}f(x,y)d_qyd_qx.$$

We show that this is also true for certain polytopes, see Corollary 3.5.

Proposition 3.4 gives a general expansion to evaluate q-integrals over special polytopes.

Proposition 3.4. *Suppose that S is a set of pairs (i,j) with $1\le i\ne j\le n$. For each element $(i,j)\in S$, let $t_{i,j}$ be an integer. For fixed integers $r_i\ge s_i\ge 0$, let*

$$Q=\{q^{r_i}\le x_i\le q^{s_i}:1\le i\le n\}\cap\{q^{t_{i,j}}x_i\le x_j:(i,j)\in S\}.$$

Then for $\pi=\pi_1\dots\pi_n\in S_n$ we have

$$\int_Q f(x_1,\dots,x_n)d_qx_{\pi_1}\cdots d_qx_{\pi_n}=(1-q)^n\sum_{k_1,\dots,k_n}f(q^{k_1},\dots,q^{k_n})q^{k_1+\dots+k_n},$$

where the sum is over all integers $k_1,\dots,k_n$ satisfying

- $s_i\le k_i<r_i$ *for* $1\le i\le n$,
- $t_{i,j}+k_i\ge k_j$ *if* $(i,j)\in S$ *and* $\pi^{-1}(i)<\pi^{-1}(j)$,
- $t_{i,j}+k_i>k_j$ *if* $(i,j)\in S$ *and* $\pi^{-1}(i)>\pi^{-1}(j)$.

Proof. Let D be the set of points in $\mathbb{R}^n$ satisfying the inequalities in Q. By definition, the left hand side is

$$\int_{\min(D_n)}^{\max(D_n)}\cdots\int_{\min(D_1)}^{\max(D_1)}f(x_1,\dots,x_n)d_qx_{\pi_1}\cdots d_qx_{\pi_n}, \tag{3}$$

where

$$D_i=D_i(x_{\pi_{i+1}},\dots,x_{\pi_n})=\{y_{\pi_i}:(y_1,\dots,y_n)\in D, y_{\pi_j}=x_{\pi_j}\text{ for }j>i\}.$$

Note that D_i is the set of all real numbers y_{π_i} such that

- $q^{r_{\pi_i}}\le y_{\pi_i}\le q^{s_{\pi_i}}$,
- $q^{t_{\pi_i,\pi_j}}y_{\pi_i}\le x_{\pi_j}$ for all $j>i$ with $(\pi_i,\pi_j)\in S$,

- $q^{t_{\pi_j,\pi_i}} x_{\pi_j} \le y_{\pi_i}$ for all $j > i$ with $(\pi_j, \pi_i) \in S$.

Thus

$$\max(D_i) = \min\{q^{s_{\pi_i}}, q^{-t_{\pi_i,\pi_j}} x_{\pi_j} : j > i, (\pi_i, \pi_j) \in S\},$$
$$\min(D_i) = \max\{q^{r_{\pi_i}}, q^{t_{\pi_j,\pi_i}} x_{\pi_j} : j > i, (\pi_i, \pi_j) \in S\}.$$

In other words, if $x_{\pi_j} = q^{k_{\pi_j}}$ for $j = i+1, \dots, n$, then $\max(D_i) = q^{a_i}$ and $\min(D_i) = q^{b_i}$, where

$$a_i = a_i(k_{\pi_{i+1}}, \dots, k_{\pi_n}) = \max(\{s_{\pi_i}\} \cup \{k_{\pi_j} - t_{\pi_i,\pi_j} : j > i, (\pi_i, \pi_j) \in S\}),$$
$$b_i = b_i(k_{\pi_{i+1}}, \dots, k_{\pi_n}) = \min(\{r_{\pi_i}\} \cup \{k_{\pi_j} + t_{\pi_j,\pi_i} : j > i, (\pi_j, \pi_i) \in S\}).$$

Thus, by Lemma 3.1, (3) is equal to

$$(1-q)^n \sum_{k_{\pi_n}=a_n}^{b_n-1} \cdots \sum_{k_{\pi_1}=a_1}^{b_1-1} f(q^{k_1}, \dots, q^{k_n}) q^{k_1+\cdots+k_n}.$$

Note that once $k_{\pi_{i+1}}, \dots, k_{\pi_n}$ are determined, we have $a_i \le k_{\pi_i} < b_i$ if and only if

- $s_{\pi_i} \le k_{\pi_i} < r_{\pi_i}$,
- $k_{\pi_j} - t_{\pi_i,\pi_j} \le k_{\pi_i}$ for all $j > i$ with $(\pi_i, \pi_j) \in S$,
- $k_{\pi_i} < k_{\pi_j} + t_{\pi_j,\pi_i}$ for all $j > i$ with $(\pi_j, \pi_i) \in S$.

One can easily check that the above sum is equivalent to the right hand side of the equation in this proposition. □

An immediate corollary of Proposition 3.4 is that if $f(x_1, \dots, x_n)$ vanishes on the boundary $q^{t_{i,j}} x_i = x_j$ for all $(i,j) \in S$, then we can change the order of integration.

Corollary 3.5. *We follow the same notation in Proposition 3.4. Suppose that the function f satisfies that $f(x_1, \dots, x_n) = 0$ if $q^{t_{i,j}} x_i = x_j$ for any pair $(i,j) \in S$. Then for any permutations $\pi, \sigma \in S_n$, we have*

$$\int_Q f(x_1, \dots, x_n) d_q x_{\pi_1} \cdots d_q x_{\pi_n} = \int_Q f(x_1, \dots, x_n) d_q x_{\sigma_1} \cdots d_q x_{\sigma_n}.$$

Corollary 3.6 gives a sufficient condition for changing the domain of a q-integral from one standard simplex to another. We will use this corollary later in this paper.

Corollary 3.6. *Suppose that $f(x_1, \dots, x_n)$ is symmetric in $x_1, \dots, x_n$ and $f(x_1, \dots, x_n) = 0$ if $x_i = x_j$ for any $i \ne j$. Then for $\pi \in S_n$ and $0 < q < 1$, we have*

$$\int_{0 \le x_{\pi_1} \le \cdots \le x_{\pi_n} \le 1} f(x_1, \dots, x_n) d_q x_1 \cdots d_q x_n = \int_{0 \le x_1 \le \cdots \le x_n \le 1} f(x_1, \dots, x_n) d_q x_1 \cdots d_q x_n.$$

Proof. Let $\sigma = \pi^{-1}$. By renaming the variables $x_i \mapsto x_{\sigma_i}$, the left hand side becomes

$$\int_{0 \le x_1 \le \cdots \le x_n \le 1} f(x_{\sigma_1}, \dots, x_{\sigma_n}) d_q x_{\sigma_1} \cdots d_q x_{\sigma_n}.$$

Since f is symmetric, this is equal to

$$\int_{0 \le x_1 \le \cdots \le x_n \le 1} f(x_1, \dots, x_n) d_q x_{\sigma_1} \cdots d_q x_{\sigma_n}.$$

We finish the proof by applying Corollary 3.5. □

Corollary 3.7 allows us to change the order of integration by modifying the polytope.

Corollary 3.7. *Let Q be a family of inequalities of the form $q^{t_{i,j}} x_i \le x_j$ for $1 \le i \ne j \le n$ and the inequalities $q^{r_i} \le x_i \le q^{s_i}$ for $1 \le i \le n$. Then for $\pi, \sigma \in S_n$ we have*

$$\int_Q f(x_1, \dots, x_n) d_q x_{\pi_1} \cdots d_q x_{\pi_n} = \int_{Q'} f(x_1, \dots, x_n) d_q x_{\sigma_1} \cdots d_q x_{\sigma_n},$$

where Q' is the family of inequalities obtained from Q as follows.

- *If Q contains the inequality $q^{t_{i,j}}x_i \le x_j$ such that $\pi^{-1}(i) > \pi^{-1}(j)$ and $\sigma^{-1}(i) < \sigma^{-1}(j)$, then replace this inequality by $q^{t_{i,j}-1}x_i \le x_j$.*
- *If Q contains the inequality $q^{t_{i,j}}x_i \le x_j$ such that $\pi^{-1}(i) < \pi^{-1}(j)$ and $\sigma^{-1}(i) > \sigma^{-1}(j)$, then replace this inequality by $q^{t_{i,j}+1}x_i \le x_j$.*
- *The remaining inequalities of Q are unchanged.*

Proof. This is proved by expanding both sides using Proposition 3.4. □

4. q-integrals over order polytopes

In this section we consider q-integrals over order polytopes of posets. The main result (Theorem 4.4) is that the q-volume of an order polytope of a poset may be written, up to a factor, as the maj-generating function of the linear extensions of that poset. Thus these q-integrals may be evaluated using permutation enumeration.

Recall the truncated order polytope $\mathcal{O}_I(P)$ in Definition 2.11. We first use Proposition 3.4 to evaluate an arbitrary integral over $\mathcal{O}_I(P)$ as a sum over restricted (P,ω)-partitions σ.

Theorem 4.1. *Let P be a poset on $\{x_1,\dots,x_n\}$ and $\omega_n : P \to [n]$ be the labeling of P given by $\omega_n(x_i) = i$ for $1 \le i \le n$. For integers $r_1, r_2, \dots, r_n, s_1, s_2, \dots, s_n$ with $r_i \ge s_i \ge 0$, let*

$$I = \{(x_1,\dots,x_n) : q^{r_i} \le x_i \le q^{s_i}\}.$$

Then

$$\int_{\mathcal{O}_I(P)} f(x_1,\dots,x_n) d_qx_1 \cdots d_qx_n = (1-q)^n \sum_{\sigma} f(q^{\sigma(x_1)},\dots,q^{\sigma(x_n)}) q^{|\sigma|},$$

where the sum is over all (P,ω_n)-partitions σ satisfying $s_i \le \sigma(x_i) < r_i$ for $1 \le i \le n$.

Proof. This is obtained immediately from Proposition 3.4 by taking

$$\pi = 12\cdots n, \qquad S = \{(i,j) : x_i \le_P x_j\}, \qquad t_{i,j} = 0.$$

□

Note that in Theorem 4.1 the labeling of the poset is closely related to the order of integration.

The next corollary expresses a q-integral over a truncated order polytope as a sum of q-integrals of truncated simplices.

Corollary 4.2. *Let P be a poset on $\{x_1,\dots,x_n\}$ and $\omega_n : P \to [n]$ the labeling of P given by $\omega_n(x_i) = i$ for $1 \le i \le n$. For integers $r_1, r_2, \dots, r_n, s_1, s_2, \dots, s_n$ with $r_i \ge s_i \ge 0$, let*

$$I = \{(x_1,\dots,x_n) : q^{r_i} \le x_i \le q^{s_i}\}.$$

Then

$$\int_{\mathcal{O}_I(P)} f(x_1,\dots,x_n) d_qx_1 \cdots d_qx_n = \sum_{\pi \in \mathcal{L}(P,\omega_n)} \int_{\mathcal{O}_I(P_\pi)} f(x_1,\dots,x_n) d_qx_1 \cdots d_qx_n,$$

P_π is the chain $x_{\pi_1} \le \cdots \le x_{\pi_n}$.

Proof. This can be proved by the standard arguments in the (P,ω)-partition theory, see [22, Lemma 3.15.3]. □

We shall need the following lemma later.

Lemma 4.3. *Let $f(x_1,\dots,x_n)$ be a function such that $f(x_1,\dots,x_n) = 0$ if $x_i = x_j$ for any $i \ne j$. Then*

$$\sum_{\mu \in \mathrm{Par}_n} q^{|\mu+\delta_n|} f(q^{\mu+\delta_n}) = \frac{1}{(1-q)^n} \int_{0 \le x_1 \le \cdots \le x_n \le 1} f(x_1,\dots,x_n) d_qx_1 \cdots d_qx_n.$$

Proof. By Theorem 4.1 and the assumption on the function $f(x_1,\dots,x_n)$, the right side is equal to

$$\sum_{i_1 \ge i_2 \ge \cdots \ge i_n \ge 0} f(q^{i_1},\dots,q^{i_n}) q^{i_1+\cdots+i_n} = \sum_{i_1 > i_2 > \cdots > i_n \ge 0} f(q^{i_1},\dots,q^{i_n}) q^{i_1+\cdots+i_n}.$$

This is equal to the left side. □

By taking $I = [0,1]^n$ and $f(x_1,\dots,x_n) = 1$ in Theorem 4.1, we obtain that the q-volume of the order polytope $\mathcal{O}(P)$ is the generating function for (P,ω)-partitions. The equivalence of the two equations in Theorem 4.4 follows from the well known fact (2) on P-partition theory.

Theorem 4.4. *[q-volume of order polytope] Let P be a poset on $\{x_1,\dots,x_n\}$ with labeling ω_n given by $\omega_n(x_i) = i$. Then*

$$V_q(\mathcal{O}(P)) = \int_{\mathcal{O}(P)} d_qx_1 \cdots d_qx_n = (1-q)^n \sum_{\sigma} q^{|\sigma|},$$

where the sum is over all (P,ω_n)-partitions σ. Equivalently,

$$V_q(\mathcal{O}(P)) = \int_{\mathcal{O}(P)} d_qx_1 \cdots d_qx_n = \frac{1}{[n]_q!} \sum_{\pi\in\mathcal{L}(P,\omega_n)} q^{\mathrm{maj}(\pi)}.$$

As a special case of Theorem 4.4, let us consider the anti-chain P on $\{1,2,\dots,n\}$. Then $\mathcal{O}(P)$ is the n-dimensional unit cube whose q-volume is 1 and $\mathcal{L}(P,\omega) = S_n$. If we apply Theorem 4.4 to P, we obtain

$$1 = \frac{1}{[n]_q!} \sum_{\pi\in S_n} q^{\mathrm{maj}(\pi)}.$$

This is a well known result for the maj-generating function for permutations, see [22].

Let's consider another special case of order polytopes, which are truncated simplices. The following lemma will be used to evaluate the q-volume of a truncated simplex.

Lemma 4.5. *Let $\pi \in S_n$ and $r > s \ge 0$. Then*

$$\sum_{\substack{r>i_1\ge\cdots\ge i_n\ge s\\ i_j>i_{j+1} \text{ if } j\in\mathrm{Des}(\pi)}} q^{i_1+\cdots+i_n} = q^{sn+\mathrm{maj}(\pi)} \frac{(q^{r-s-\mathrm{des}(\pi)};q)_n}{(q;q)_n}.$$

Proof. Let A be the set of partitions $\lambda = (\lambda_1,\dots,\lambda_n)$ such that $r > \lambda_1 \ge \cdots \ge \lambda_n \ge s$ and $\lambda_j > \lambda_{j+1}$ if $j \in \mathrm{Des}(\pi)$. Observe that if $\lambda \in A$, then considering the Young diagram of the transpose λ' of λ we have $(n^s) \subseteq \lambda' \subseteq (n^{r-1})$ and $j \in \lambda'$ for $j \in \mathrm{Des}(\pi)$. Thus the left hand side is equal to

$$\sum_{\lambda\in A} q^{|\lambda|} = q^{sn} q^{\mathrm{maj}(\pi)} \begin{bmatrix} r-s-1-\mathrm{des}(\pi)+n \\ n \end{bmatrix}_q = q^{sn} q^{\mathrm{maj}(\pi)} \frac{(q^{r-s-\mathrm{des}(\pi)};q)_n}{(q;q)_n},$$

which finishes the proof. □

We now have a formula for the q-volume of a truncated simplex. This will be used later to evaluate the q-beta integral and give a combinatorial interpretation for the q-Selberg integral.

Corollary 4.6. *[q-volume of a truncated simplex] For $\pi \in S_n$ and real numbers $a < b$, the q-volume of the truncated simplex $\mathcal{O}_{[a,b]^n}(P_\pi)$ is*

$$V_q(\mathcal{O}_{[a,b]^n}(P_\pi)) = \int_{a\le x_{\pi_1}\le\cdots\le x_{\pi_n}\le b} d_qx_1 \cdots d_qx_n = \frac{b^n q^{\mathrm{maj}(\pi)}}{[n]_q!} (aq^{-\mathrm{des}(\pi)}/b;q)_n.$$

Proof. Since both sides are polynomials in a and b, it is sufficient to show the following for integers $r > s \ge 0$:

$$\int_{q^r\le x_{\pi_1}\le\cdots\le x_{\pi_n}\le q^s} d_qx_1 \cdots d_qx_n = \frac{q^{sn+\mathrm{maj}(\pi)}}{[n]_q!} (q^{r-s-\mathrm{des}(\pi)};q)_n.$$

This follows from Theorem 4.1 and Lemma 4.5. □

By considering the q-volume of the box $[a,1]^n$, we obtain an identity for a generating function for permutations with maj and des statistics.

Corollary 4.7. *For a non-negative integer n, we have*

$$\sum_{\pi\in S_n} q^{\mathrm{maj}(\pi)} (aq^{-\mathrm{des}(\pi)};q)_n = (1-a)^n [n]_q!.$$

Proof. Let $I=[a,1]^n$ and P the anti-chain on $\{x_1,\dots,x_n\}$. Then $\mathcal{O}_I(P)=I$ and $V_q(\mathcal{O}_I(P))=(1-a)^n$. On the other hand, by Corollary 4.2 and Corollary 4.6, we have

$$(1-a)^n=V_q(\mathcal{O}_I(P))=\sum_{\pi\in S_n}V_q(\mathcal{O}_{[a,1]^n}(P_\pi))=\sum_{\pi\in S_n}\frac{q^{\operatorname{maj}(\pi)}}{[n]_q!}(aq^{-\operatorname{des}(\pi)};q)_n.$$

By multiplying both sides by $[n]_q!$, we have the stated result. $\square$

Theorem 4.8 is another generating function for permutations with maj and des statistics due to MacMahon. This result is often called Carlitz's formula [6], see [10, p. 6].

Theorem 4.8. *We have*

$$\sum_{\pi\in S_n}t^{\operatorname{des}(\pi)}q^{\operatorname{maj}(\pi)}=(t;q)_{n+1}\sum_{i\ge 0}[i+1]_q^n t^i.$$

We show that Corollary 4.6 and Theorem 4.8 are equivalent. By expanding $(aq^{-\operatorname{des}(\pi)};q)_n$ using the q-binomial theorem and comparing the coefficients of a^k in both sides, one can restate Corollary 4.6 as follows: for $0\le k\le n$,

$$\sum_{\pi\in S_n}(q^{-k})^{\operatorname{des}(\pi)}q^{\operatorname{maj}(\pi)}=q^{-\binom{k}{2}}\binom{n}{k}(q;q)_k(q;q)_{n-k}. \tag{4}$$

On the other hand, by expanding the numerator of $[i+1]_q^n=(1-q^{i+1})^n/(1-q)^n$, using the binomial theorem and the geometric series, one can check that Theorem 4.8 can be restated as

$$\sum_{\pi\in S_n}t^{\operatorname{des}(\pi)}q^{\operatorname{maj}(\pi)}=\frac{1}{(1-q)^n}\sum_{i=0}^{n}\binom{n}{i}(-q)^i(t;q)_i(q^{i+1}t;q)_{n-i}. \tag{5}$$

If we substitute $t=q^{-k}$ in (5) for $0\le k\le n$ then we obtain (4). In order to obtain (5) from (4) one can argue as follows. By (4), we know that (5) is true when $t=q^{-k}$ for $0\le k\le n$. Since both sides of (5) are polynomials in t of degree at most n and the equation has $n+1$ different solutions, both sides are the same as polynomials in t.

When $a=0$ and $b=1$ in Corollary 4.6 we obtain the following corollary. It explicitly demonstrates the failure of Fubini's theorem by finding the differences in the q-volumes of the standard $n!$ simplices which lie inside an n-dimensional cube.

Corollary 4.9. *[q-volume of a simplex] For $\pi\in S_n$, the q-volume of the simplex $\mathcal{O}(P_\pi)$ is*

$$V_q(\mathcal{O}(P_\pi))=\int_{0\le x_{\pi_1}\le\cdots\le x_{\pi_n}\le 1}d_qx_1\cdots d_qx_n=\frac{q^{\operatorname{maj}(\pi)}}{[n]_q!}.$$

5. Operations on posets

Suppose that a poset P is modified to obtain another poset P'. Is $V_q(\mathcal{O}(P'))$ related to the q-integral defining $V_q(\mathcal{O}(P))$? In this section we answer this question for the following types of modifications:

(1) attaching a chain below an element (Lemma 5.1),
(2) attaching a chain above an element (Lemma 5.2),
(3) attaching a chain between two elements (Lemma 5.3),
(4) inserting a chain which interlaces another chain (Lemmas 5.5 and 5.6).

First we insert a chain below a fixed element. See the left figure in Figure 1.

Lemma 5.1 (Scaredy Cat Lemma). *Let P be a poset on $\{x_1,\dots,x_n\}$ and $t\in[n]$. Let Q be the poset on $\{x_1,\dots,x_n,y_1,\dots,y_m\}$ with relations $x_i\le_Q x_j$ if and only if $x_i\le_P x_j$, and $y_1\le_Q\cdots\le_Q y_m\le_Q x_t$. In other words, Q is obtained from P by attaching a chain $y_1\le\cdots\le y_m$ below x_t. Then we have*

$$\int_{\mathcal{O}(P)}x_t^m f(x_1,\dots,x_n)d_qx_1\cdots d_qx_n=[m]_q!\int_{\mathcal{O}(Q)}f(x_1,\dots,x_n)d_qy_1\cdots d_qy_md_qx_1\cdots d_qx_n.$$

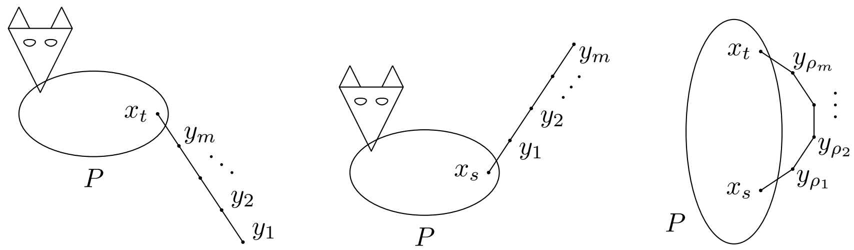

FIGURE 1. Illustrations of the scaredy cat lemma (left), the happy cat lemma (middle), and the attaching chain lemma (right).

Proof. By Corollary 4.6, we have

$$\int_{0\le y_1\le\cdots\le y_m\le x_t} d_qy_1\cdots d_qy_m = \frac{x_t^m}{[m]_q!}.$$

Thus, the left hand side of the equation can be written as

$$[m]_q!\int_{\mathcal{O}(P)}\left(\int_{0\le y_1\le\cdots\le y_m\le x_t} d_qy_1\cdots d_qy_m\right) f(x_1,\dots,x_n)d_qx_1\cdots d_qx_n.$$

This is equal to the right hand side of the equation. □

Next we insert a chain above a fixed element. See the middle figure in Figure 1.

Lemma 5.2 (Happy Cat Lemma). *Let P be a poset on $\{x_1,\dots,x_n\}$ and $s\in[n]$. Let Q be the poset on $\{x_1,\dots,x_n,y_1,\dots,y_m\}$ with relations $x_i\le_Q x_j$ if and only if $x_i\le_P x_j$, and $\le_Q x_s\le y_1\le_Q\cdots\le_Q y_m$. In other words, Q is obtained from P by attaching a chain $y_1\le\cdots\le y_m$ above x_s. Then we have*

$$\int_{\mathcal{O}(P)}(qx_s;q)_m f(x_1,\dots,x_n)d_qx_1\cdots d_qx_n = [m]_q!\int_{\mathcal{O}(Q)} f(x_1,\dots,x_n)d_qx_1\cdots d_qx_nd_qy_1\cdots d_qy_m.$$

Proof. By Corollary 4.6, we have

$$\int_{qx_s\le y_1\le\cdots\le y_m\le 1} d_qy_1\cdots d_qy_m = \frac{(qx_s;q)_m}{[m]_q!}.$$

Thus, the left hand side of the equation can be written as

$$\begin{aligned}&[m]_q!\int_{\mathcal{O}(P)}\left(\int_{qx_s\le y_1\le\cdots\le y_m\le 1} d_qy_1\cdots d_qy_m\right) f(x_1,\dots,x_n)d_qx_1\cdots d_qx_n\\ =&[m]_q!\int_D f(x_1,\dots,x_n)d_qy_1\cdots d_qy_md_qx_1\cdots d_qx_n,\end{aligned}$$

where D is the set of inequalities given by

$$D=\{0\le x_i\le x_j\le 1: x_i\le_P x_j\}\cup\{0\le y_1\le\cdots\le y_m\le 1\}\cup\{qx_s\le y_i: i\in[m]\}.$$

By Corollary 3.7, we have

$$\int_D f(x_1,\dots,x_n)d_qy_1\cdots d_qy_md_qx_1\cdots d_qx_n = \int_{D'} f(x_1,\dots,x_n)d_qx_1\cdots d_qx_nd_qy_1\cdots d_qy_m,$$

where

$$D'=\{0\le x_i\le x_j\le 1: x_i\le_P x_j\}\cup\{0\le y_1\le\cdots\le y_m\le 1\}\cup\{x_s\le y_i: i\in[m]\}.$$

Since D' and $\mathcal{O}(Q)$ represent the same domain, we get the lemma. □

Now we state but do not prove the attaching chain lemma, since the proof is similar to the previous proofs. See the right figure in Figure 1.

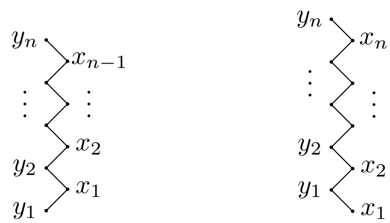

FIGURE 2. Illustrations of $x \prec y$ for $x = (x_1, \dots, x_{n-1})$, $y = (y_1, \dots, y_n)$ on the left and $x \prec y$ for $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ on the right.

Lemma 5.3 (Attaching chain lemma)**.** *Let $\rho \in S_m$. Let P be a poset on $\{x_1, \dots, x_n\}$ with $x_s \le_P x_t$. Define Q to be the poset on $\{x_1, \dots, x_n, y_1, \dots, y_m\}$ with relations $x_i \le_Q x_j$ if and only if $x_i \le_P x_j$, and $x_s \le_Q y_{\rho_1} \le_Q \cdots \le_Q y_{\rho_m} \le_Q x_t$. Then, we have*

$$\int_{\mathcal{O}(P)} q^{\operatorname{maj}(\rho)} x_t^m (q^{-\operatorname{des}(\rho)} x_s/x_t; q)_m f(x_1, \dots, x_n) d_qx_1 \cdots d_qx_n \\ = [m]_q! \int_{\mathcal{O}(Q)} f(x_1, \dots, x_n) d_qy_1 \cdots d_qy_m d_qx_1 \cdots d_qx_n, \tag{6}$$

$$\int_{\mathcal{O}(P)} q^{\operatorname{maj}(\rho)} x_t^m (q^{1-\operatorname{des}(\rho)} x_s/x_t; q)_m f(x_1, \dots, x_n) d_qx_1 \cdots d_qx_n \\ = [m]_q! \int_{\mathcal{O}(Q)} f(x_1, \dots, x_n) d_qx_1 \cdots d_qx_s d_qy_1 \cdots d_qy_m d_qx_{s+1} \cdots d_qx_n. \tag{7}$$

We note that (7) holds if the order of integration on the right hand side is obtained from $d_qx_1 \cdots d_qx_n$ by inserting $d_qy_1 \cdots d_qy_m$ anywhere between x_s and x_t.

There is another way of attaching a chain. We need some definitions.

Definition 5.4. For two sets of variables $x = (x_1, x_2, \dots, x_{n-1})$ and $y = (y_1, y_2, \dots, y_n)$ with y having one more variable than x, we denote by $x \prec y$ the set of inequalities

$$\{y_1 \le x_1 \le y_2 \le x_2 \le \cdots \le y_{n-1} \le x_{n-1} \le y_n\}.$$

Moreover, if $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ have the same number of variables, we also denote $x \prec y$ the set of inequalities

$$\{x_1 \le y_1 \le x_2 \le y_2 \le \cdots \le x_n \le y_n\}.$$

See Figure 2.

Lemma 5.5 (Interlacing chain lemma 1)**.** *Let P be a poset on $\{x_1, \dots, x_N\}$. Suppose that $y = (y_1 <_P \cdots <_P y_n)$ is a chain in P. Let Q be the poset obtained from P by adding a new chain $z = (z_1 < \cdots < z_{n-1})$ which interlaces with y as follows:*

$$y_1 <_Q z_1 <_Q y_2 <_Q z_2 <_Q \cdots <_Q y_{n-1} <_Q z_{n-1} <_Q y_n.$$

Then, for a partition λ with $\ell(\lambda) < n$, we have

$$\int_{\mathcal{O}(P)} s_\lambda(y) \overline{\Delta}(y) f(x) d_qx_1 \cdots d_qx_N \\ = \prod_{i=1}^{n-1} [\lambda_i + n - i]_q \int_{\mathcal{O}(Q)} s_\lambda(z) \overline{\Delta}(z) f(x) d_qz_1 \cdots d_qz_{n-1} d_qx_1 \cdots d_qx_N.$$

Proof. This follows from Lemma 5.7 below. $\square$

Lemma 5.6 (Interlacing chain lemma 2)**.** *Let P be a poset on $\{x_1,\dots,x_N\}$. Suppose that $y=(y_1<_P\cdots<_P y_n)$ is a chain in P. Let Q be the poset obtained from P by adding a new chain $z=(z_1<\cdots<z_n)$ which interlaces with y as follows:*

$$z_1<_Q y_1<_Q z_2<_Q y_2<_Q\cdots<_Q z_n<_Q y_n.$$

Then, for a partition λ with $\ell(\lambda)=n$, we have

$$\int_{\mathcal{O}(P)} s_\lambda(y)\overline{\Delta}(y)f(x)d_qx_1\cdots d_qx_N$$
$$=\prod_{i=1}^n[\lambda_i+n-i]_q\int_{\mathcal{O}(Q)} s_{\lambda-(1^n)}(z)\overline{\Delta}(z)f(x)d_qz_1\cdots d_qz_nd_qx_1\cdots d_qx_N.$$

Proof. This follows from Lemma 5.8 below. □

The following lemma is a special case of [18, Theorem 1], which has Macdonald polynomials instead of Schur functions.

Lemma 5.7. *Let $z=(z_1,z_2,\dots,z_{n-1})$ and $y=(y_1,y_2,\dots,y_n)$. If $\ell(\lambda)<n$, we have*

$$s_\lambda(y)\overline{\Delta}(y)=\prod_{i=1}^{n-1}[\lambda_i+n-i]_q\int_{z\prec y} s_\lambda(z)\overline{\Delta}(z)d_qz_1\cdots d_qz_{n-1}.$$

Proof. We note that the integrand on the right side is a determinant. Applying the q-integrals gives another determinant, which is the left side. Specifically,

$$s_\lambda(z_1,\dots,z_{n-1})\overline{\Delta}(z)=(-1)^{\binom{n-1}{2}}\det\left(z_i^{\lambda_j+n-1-j}\right)_{1\le i,j\le n-1}.$$

Evaluating the q-integral on z_i from y_i to y_{i+1} evaluates the right side of Lemma 5.7 as

$$(-1)^{\binom{n-1}{2}}\det\left(y_{i+1}^{\lambda_j+n-j}-y_i^{\lambda_j+n-j}\right)_{1\le i,j\le n-1}.$$

For the left side,

$$s_\lambda(y_1,\dots,y_n)\overline{\Delta}(y)=(-1)^{\binom{n}{2}}\det\left(y_i^{\lambda_j+n-j}\right)_{1\le i,j\le n}$$

has an n^{th} column of all 1's because $\lambda_n=0$. By successively subtracting the i^{th} row from the $i+1^{th}$ row for $i=n-1$ to $i=1$ we obtain a final column of $(1,0,\dots,0)^T$. Expanding the determinant along this column gives

$$\begin{aligned}s_\lambda(y_1,\dots,y_n)\overline{\Delta}(y)=&(-1)^{\binom{n}{2}}(-1)^{n-1}\det\left(y_{i+1}^{\lambda_j+n-j}-y_i^{\lambda_j+n-j}\right)_{1\le i,j\le n-1}\\ =&(-1)^{\binom{n-1}{2}}\det\left(y_{i+1}^{\lambda_j+n-j}-y_i^{\lambda_j+n-j}\right)_{1\le i,j\le n-1}\end{aligned}$$

which is the right side. □

The version of Lemma 5.7 when $\lambda_n>0$ is next.

Lemma 5.8. *Let $z=(z_1,z_2,\dots,z_n)$ and $y=(y_1,y_2,\dots,y_n)$. If $\ell(\lambda)=n$, we have*

$$s_\lambda(y)\overline{\Delta}(y)=\prod_{i=1}^{n}[\lambda_i+n-i]_q\int_{z\prec y} s_{\lambda-(1^n)}(z)\overline{\Delta}(z)d_qz_1\cdots d_qz_n.$$

Proof. Let $u=(u_1,u_2,\dots,u_{n+1})=(0,y_1,y_2,\dots,y_n)$. By Lemma 5.7, we have

$$s_{\lambda-(1^n)}(u_1,\dots,u_{n+1})\overline{\Delta}(u)=\prod_{i=1}^{n}[(\lambda_i-1)+(n+1)-i]_q\int_{z\prec u} s_{\lambda-(1^n)}(z_1,\dots,z_n)\overline{\Delta}(z)d_qz_1\cdots d_qz_n.$$

Since

$$s_{\lambda-(1^n)}(u_1,\dots,u_{n+1})=s_{\lambda-(1^n)}(y_1,\dots,y_n)=s_\lambda(y_1,\dots,y_n)/y_1\cdots y_n$$

and

$$\overline{\Delta}(u)=y_1\cdots y_n\overline{\Delta}(y)$$

we are done. □

6. Examples of q-integrals

In this section we use the constructions in Section 4 to evaluate the q-integrals. This includes the q-beta integral, a q-analogue of Dirichlet's integral, and a general q-beta integral due to Andrews and Askey [1]. We will then find a connection with linear extensions of forest posets.

6.1. The q-beta integral.

The following is the well known integral called the q-beta integral. We now prove this using our methods. The idea is to add two chains to a point: one chain below it and the other chain above it, naturally labeled. The resulting poset is again a chain whose q-volume is easily computed.

Corollary 6.1. *We have*

$$\int_0^1 x^n (xq;q)_m d_q x = \frac{[n]_q![m]_q!}{[n+m+1]_q!}.$$

Proof. By (5.2) and (5.1) we have

$$\int_0^1 x^n (xq;q)_m d_q x = [n]_q![m]_q! \int_{0\le y_1\le\cdots\le y_n\le x\le z_1\le\cdots\le z_m\le 1} d_q y_1\cdots d_q y_n d_q x d_q z_1\cdots d_q z_m.$$

By Corollary 4.9, we get the q-beta integral formula. □

6.2. A q-analog of Dirichlet integral.

We now consider the simplex

$$\Omega_n = \{(x_1,\dots,x_n)\in[0,1]^n : x_1+\cdots+x_n\le 1\}.$$

Dirichlet's integral is the following, see [2, Theorem 1.8.6]:

$$\int_{\Omega_n} x_1^{\alpha_1-1}\cdots x_n^{\alpha_n-1}(1-x_1-\cdots-x_n)^{a_{n+1}-1}dx_1\cdots dx_n = \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_{n+1})}{\Gamma(\alpha_1+\cdots+\alpha_{n+1})}. \tag{8}$$

By introducing new variables $y_i = x_1+\cdots+x_i$ and integers $k_i=\alpha_i-1$, we get an equivalent version of (8)

$$\int_{0\le y_1\le y_2\le\cdots\le y_n\le 1} y_1^{k_1}(y_2-y_1)^{k_2}\cdots(y_n-y_{n-1})^{k_n}(1-y_n)^{k_{n+1}}dy_1\cdots dy_n = \frac{k_1!\cdots k_{n+1}!}{(n+k_1+\cdots+k_{n+1})!}. \tag{9}$$

A q-analogue of (9) is given by the next corollary. It generalizes Corollary 6.1 which is the case $n=1$.

Corollary 6.2. *For nonnegative integers $k_1,\dots,k_{n+1}$, we have*

$$\int_{0\le y_1\le y_2\le\cdots\le y_n\le 1} y_1^{k_1}(qy_n;q)_{k_{n+1}}\prod_{i=2}^n y_i^{k_i}(qy_{i-1}/y_i;q)_{k_i} d_q y_1\cdots d_q y_n = \frac{[k_1]_q!\cdots[k_{n+1}]_q!}{[n+k_1+\cdots+k_{n+1}]_q!}.$$

Proof. We generalize the proof of Corollary 6.1. We start with a chain $y_1\le y_2\le\cdots\le y_n$. Attach a chain with k_i elements between y_{i-1} and y_i, for $2\le i\le n$, and also a chain with k_1 elements below y_1, and a chain with k_{n+1} elements above y_n. We obtain a chain with $n+k_1+\cdots+k_{n+1}$ elements which is naturally labeled.

By applying the scaredy cat lemma once, the second part (7) in the attaching chain lemma with ρ the identity permutation $n-1$ times, and the happy cat lemma once, we obtain the left side multiplied by some q-factorials. The right side results from Corollary 4.9 with π being the identity. □

6.3. **The general q-beta integral of Andrews and Askey.** Andrews and Askey [1] generalized the q-beta integral as follows: for $|q|<1$,

$$\int_a^b \frac{(qx/a;q)_\infty(qx/b;q)_\infty}{(Ax/a;q)_\infty(Bx/b;q)_\infty} d_qx = \frac{(1-q)(q;q)_\infty(AB;q)_\infty ab(a/b;q)_\infty(b/a;q)_\infty}{(A;q)_\infty(B;q)_\infty(a-b)(Ba/b;q)_\infty(Ab/a;q)_\infty}. \tag{10}$$

In this subsection, by computing the q-volume of a truncated simplex in two different ways, we will show the following proposition which is equivalent to the special case of (10) with substitution

$$(a,b,A,B)\mapsto(aq^{r-k_1}, bq^{k_2+1}, q^{r+1}, q^{s+1}).$$

Proposition 6.3. *Let n,r,s,k_1,k_2 be nonnegative integers such that $n=r+s+1$, $k_1\le r$, $k_2\le s$, and $k=k_1+k_2+1$ if $s\ge 1$ and $k=k_1$ if $s=0$. Then*

$$\int_a^b x^r(aq^{-k_1}/x;q)_r(xq^{-k_2}/b;q)_s d_qx = \frac{[r]_q![s]_q!}{[n]_q!} b^{r+1} q^{(k-k_1)(r+1)}(aq^{-k}/b;q)_n.$$

Proof. Let $\pi\in S_n$. By Corollary 4.6 the q-volume of the truncated simplex $\mathcal{O}_{[a,b]^n}(P_\pi)$ is

$$V_q(\mathcal{O}_{[a,b]^n}(P_\pi)) = \int_{a\le x_{\pi_1}\le\cdots\le x_{\pi_n}\le b} d_qx_1\cdots d_qx_n = \frac{b^n q^{\mathrm{maj}(\pi)}}{[n]_q!}(aq^{-\operatorname{des}(\pi)}/b;q)_n.$$

Now we compute this q-volume in a different way by decomposing the chain $x_{\pi_1}\le\cdots\le x_{\pi_n}$ into two chains.

First we decompose π into $\pi=\sigma n\tau$ using the largest integer n. Suppose that σ and τ have r and s letters respectively and $\operatorname{des}(\sigma)=k_1$, $\operatorname{des}(\tau)=k_2$. Then $n=r+s+1$ and $k=k_1+k_2+1$ if $k_2\ge 1$ and $k=k_1$ if $k_2=0$. The q-volume of $\mathcal{O}_{[a,b]^n}(P_\pi)$ can be written as

$$\int_a^b \left(\int_{a\le x_{\sigma_1}\le\cdots\le x_{\sigma_r}\le x_n} d_qy_1\cdots d_qy_r \int_{x_n\le x_{\tau_1}\le\cdots\le x_{\tau_s}\le b} d_qz_1\cdots d_qz_s \right) d_qx_n,$$

where $y_1,\dots,y_r$ and $z_1,\dots,z_s$ are obtained by rearranging $x_{\sigma_1},\dots,x_{\sigma_r}$ and $x_{\tau_1},\dots,x_{\tau_s}$ respectively so that subscripts are increasing. By applying Corollary 4.6 to the two inside integrals, the above is equal to

$$\int_a^b \frac{x^r q^{\mathrm{maj}(\sigma)}}{[r]_q!}(aq^{-k_1}/x;q)_r \frac{b^s q^{\mathrm{maj}(\tau)}}{[s]_q!}(xq^{-k_2}/b;q)_s d_qx. \tag{11}$$

Note that $\mathrm{maj}(\pi)=\mathrm{maj}(\sigma)+\mathrm{maj}(\tau)+(r+1)(k_2+1)$ if $s\ge1$ and $\mathrm{maj}(\pi)=\mathrm{maj}(\sigma)$ if $s=0$. In either case we can write $\mathrm{maj}(\pi)=\mathrm{maj}(\sigma)+\mathrm{maj}(\tau)+(r+1)(k-k_1)$. This completes the proof. □

We now consider the case $s\ge1$ in Proposition 6.3 so that $k=k_1+k_2+1$. One can rewrite the integral in Proposition 6.3 as

$$\begin{aligned}(-1)^r a^r q^{\binom{r}{2}-k_1r} &\int_a^b \frac{(xq^{1-r+k_1}/a;q)_\infty(xq^{-k_2}/b;q)_\infty}{(xq^{k_1+1}/a;q)_\infty(xq^{s-k_2}/b;q)_\infty} d_qx \\ &= (-1)^r a^r q^{\binom{r}{2}-k_1r}\int_{aq^{r-k_1}}^{bq^{k_2+1}} \frac{(xq^{1-r+k_1}/a;q)_\infty(xq^{-k_2}/b;q)_\infty}{(xq^{k_1+1}/a;q)_\infty(xq^{s-k_2}/b;q)_\infty} d_qx,\end{aligned}$$

where the equality follows from the fact that the integrand is 0 if $x=bq^j$ for $0\le j\le k_2$ and $x=aq^j$ for $0\le j\le r-k_1-1$. Thus Proposition 6.3 is equivalent to

$$\int_{aq^{r-k_1}}^{bq^{k_2+1}} \frac{(xq^{1-r+k_1}/a;q)_\infty(xq^{-k_2}/b;q)_\infty}{(xq^{k_1+1}/a;q)_\infty(xq^{s-k_2}/b;q)_\infty} d_qx = \frac{(-1)^r b^{r+1}[r]_q![s]_q! q^{kr+k-k_1-\binom{r}{2}}(aq^{-k}/b;q)_n}{a^r[n]_q!}.$$

This is the $(a,b,A,B)\mapsto(aq^{r-k_1}, bq^{k_2+1}, q^{r+1}, q^{s+1})$ case of (10).

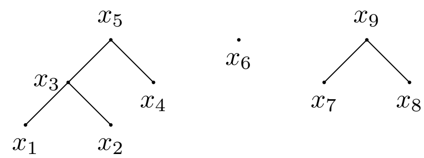

FIGURE 3. A forest poset.

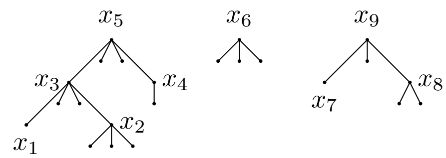

FIGURE 4. The forest poset F_a, where F is the forest in Figure 3 and $a = (a_1, \dots, a_9)$.

6.4. **q-integrals of monomials over the order polytope of a forest poset.** In this subsection we consider special posets called forests and evaluate the q-integral of a monomial over the order polytope coming from these posets. These forests have an order polytope whose q-volume has a hook formula, see Corollary 6.9.

Definition 6.4. A poset is called a *forest* if every element is covered by at most one element. For a forest poset F and its element x, the *hook length* $h_F(x)$ of x is defined to be the number of elements $y \in F$ such that $y \le_F x$. An element of a forest is called *isolated* if it has no relation with other elements. A *leaf* is a non-isolated element whose hook length is 1.

Example 6.5. Let F be the forest in Figure 3. Then x_6 is an isolated element and x_1, x_2, x_4, x_7, x_8 are leaves. These elements have hook length 1. The hook lengths of other elements are $h_F(x_3) = 3$, $h_F(x_5) = 5$, $h_F(x_9) = 3$.

Definition 6.6. For a forest F and a sequence $a = (a_1, \dots, a_n)$ of nonnegative integers, let F_a be the poset obtained from F by attaching a_i leaves to x_i for each $i \in [n]$, see Figure 4.

Example 6.7. Let F be the poset in Figure 3 and

$$a = (a_1, \dots, a_9) = (0, 3, 2, 1, 2, 3, 0, 2, 1).$$

The forest F_a is shown in Figure 4, where the short edges are newly added from F in Figure 3. We have

$$h_{F_a}(x_1) = 1, \quad h_{F_a}(x_2) = 4, \quad h_{F_a}(x_3) = 8, \quad h_{F_a}(x_4) = 2, \quad h_{F_a}(x_5) = 13,$$

$$h_{F_a}(x_6) = 4, \quad h_{F_a}(x_7) = 1, \quad h_{F_a}(x_8) = 3, \quad h_{F_a}(x_9) = 6.$$

The next corollary allows us to evaluate any multiple q-integral of a monomial over the order polytope of a naturally labeled forest.

Theorem 6.8. *Let F be a forest on $\{x_1, \dots, x_n\}$ with labeling ω_n given by $\omega_n(x_i) = i$. Suppose that ω_n is a natural labeling. Let $a = (a_1, \dots, a_n)$ be a sequence of nonnegative integers. Then, we have*

$$\int_{\mathcal{O}(F)} x_1^{a_1} \cdots x_n^{a_n} d_q x_1 \cdots d_q x_n = \prod_{v \in F_a} \frac{1}{[h_{F_a}(v)]_q}.$$

Proof. We prove this using induction on n. If $n = 1$, then both sides are equal to $1/[a+1]_q$. Suppose that $n > 1$ and the theorem is true for $n - 1$. Since ω is a natural labeling, x_1 is an isolated element or a leaf in F.

Case 1: x_1 is an isolated element in F. Then the range for x_1 in the q-integral is $0 \le x_1 \le 1$. Let $F' = F - \{x_1\}$ and $a' = (a_2, \dots, a_n)$. By the induction hypothesis, we have

$$\begin{aligned}\int_{\mathcal{O}(F)} x_1^{a_1} \cdots x_n^{a_n} d_q x_1 \cdots d_q x_n &= \int_0^1 x_1^{a_1} d_q x_1 \int_{\mathcal{O}(F')} x_2^{a_2} \cdots x_n^{a_n} d_q x_2 \cdots d_q x_n \\ &= \frac{1}{[a_1+1]_q} \prod_{v \in F'_{a'}} \frac{1}{[h_{F'_{a'}}(v)]_q} = \prod_{v \in F_a} \frac{1}{[h_{F_a}(v)]_q}.\end{aligned}$$

Case 2: x_1 is a leaf in F. Let x_k be the unique element covering x_1. Then the range for x_1 in the q-integral is $0 \le x_1 \le x_k$. Let $F' = F - \{x_1\}$ and $a' = (a'_2, \dots, a'_n)$, where $a'_i = a_i$ if $i \ne k$ and $a'_k = a_k + a_1 + 1$. By the induction hypothesis, we have

$$\begin{aligned}\int_{\mathcal{O}(F)} x_1^{a_1} \cdots x_n^{a_n} d_q x_1 \cdots d_q x_n &= \int_{\mathcal{O}(F')} \left(\int_0^{x_k} x_1^{a_1} d_q x_1 \right) x_2^{a_2} \cdots x_n^{a_n} d_q x_2 \cdots d_q x_n \\ &= \int_{\mathcal{O}(F')} \left(\frac{x_k^{a_1+1}}{[a_1+1]_q} \right) x_2^{a_2} \cdots x_n^{a_n} d_q x_2 \cdots d_q x_n \\ &= \frac{1}{[a_1+1]_q} \int_{\mathcal{O}(F')} x_2^{a'_2} \cdots x_n^{a'_n} d_q x_2 \cdots d_q x_n = \frac{1}{[a_1+1]_q} \prod_{v \in F'_{a'}} \frac{1}{[h_{F'_{a'}}(v)]_q}.\end{aligned}$$

It is easy to check that

$$\frac{1}{[a_1+1]_q} \prod_{v \in F'_{a'}} \frac{1}{[h_{F'_{a'}}(v)]_q} = \prod_{v \in F_a} \frac{1}{[h_{F_a}(v)]_q},$$

which completes the proof. $\square$

If $a = (0, \dots, 0)$ in Theorem 6.8 we obtain the q-volume of the order polytope of a forest. Using Theorem 4.4 we also obtain the maj-generating function for the linear extensions of a forest, which was first proved by Björner and Wach [5].

Corollary 6.9. *Let F be a forest on $\{x_1, \dots, x_n\}$ with labeling ω_n given by $\omega_n(x_i) = i$. Suppose that ω_n is naturally labeled. Then, we have*

$$V_q(\mathcal{O}(F)) = \frac{1}{\prod_{v \in F} [h_F(v)]_q},$$

$$\sum_{\pi \in \mathcal{L}(F, \omega_n)} q^{\mathrm{maj}(\pi)} = \frac{[n]_q!}{\prod_{v \in F} [h_F(v)]_q}.$$

7. q-Selberg integrals

In this section we will find a combinatorial interpretation for a q-Selberg integral. We will use the first three lemmas of Section 4 to "insert" linear factors in q-integrals by building a "Selberg poset" whose q-volume is the q-Selberg integral, up to a constant. As a corollary of the q-Selberg integral evaluation, we obtain the factorization of the *maj*-generating function for the linear extensions, see Corollary 7.7.

There are many generalizations of the Selberg integral, see [8]. We consider three q-Selberg integrals which are closely related.

Definition 7.1. Let α β be complex numbers with $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, and m a non-negative integer. The *Askey-Selberg integral* $\mathrm{AS}_n(\alpha, \beta, m)$ is defined by

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^{\alpha-1} (qx_i; q)_{\beta-1} \prod_{1 \le i < j \le n} x_j^{2m} \left(q^{1-m} x_i / x_j; q \right)_{2m} d_q x_1 \cdots d_q x_n.$$

The *Kadell-Selberg integral* $\mathrm{KS}_n(\alpha,\beta,m)$ is defined by

$$\int_0^1\cdots\int_0^1\prod_{i=1}^n x_i^{\alpha-1}(qx_i;q)_{\beta-1}\prod_{1\le i<j\le n} x_j^{2m-1}\left(q^{1-m}x_i/x_j;q\right)_{2m-1}\overline{\Delta}(x)d_qx_1\cdots d_qx_n.$$

The *Askey-Kadell-Selberg integral* $\mathrm{AKS}_n(\alpha,\beta,m)$ is defined by

$$\int_{0\le x_1\le\cdots\le x_n\le 1}\prod_{i=1}^n x_i^{\alpha-1}(qx_i;q)_{\beta-1}\prod_{1\le i<j\le n} x_j^{2m-1}\left(q^{1-m}x_i/x_j;q\right)_{2m-1}\overline{\Delta}(x)d_qx_1\cdots d_qx_n.$$

It is easy to check that the integrand in $\mathrm{KS}_n(\alpha,\beta,m)$ is symmetric in the variables $x_1,x_2,\dots,x_n$. Recall [2, p. 493] that the q-gamma function is defined by

$$\Gamma_q(x)=\frac{(q;q)_{x-1}}{(1-q)^{x-1}}.$$

Askey [3] conjectured that

$$\mathrm{AS}_n(\alpha,\beta,m)=q^{\alpha m\binom{n}{2}+2m^2\binom{n}{3}}\prod_{j=1}^n\frac{\Gamma_q(\alpha+(j-1)m)\Gamma_q(\beta+(j-1)m)\Gamma_q(1+jm)}{\Gamma_q(\alpha+\beta+(n+j-2)m)\Gamma_q(1+m)},\tag{12}$$

which has been proved independently by Habsieger [11] and Kadell [13]. Kadell [14, Eq. (4.11)] showed that

$$\mathrm{AS}_n(\alpha,\beta,m)=\frac{[n]_{q^m}!}{n!}\,\mathrm{KS}_n(\alpha,\beta,m).\tag{13}$$

More generally Kadell showed Lemma 7.2 which implies (13).

Lemma 7.2. *For an anti-symmetric function $f(x)$, we have*

$$\int_{[a,b]^n}f(x)\prod_{1\le i<j\le n}(x_i-px_j)d_qx=\frac{[n]_p!}{n!}\int_{[a,b]^n}f(x)\overline{\Delta}(x)d_qx.$$

Since the integrand of $\mathrm{KS}_n(\alpha,\beta,m)$ is symmetric under any permutation of $x_1,\dots,x_n$ and zero whenever $x_i=x_j$, by Corollary 3.6 we have

$$\mathrm{KS}_n(\alpha,\beta,m)=n!\,\mathrm{AKS}_n(\alpha,\beta,m).\tag{14}$$

By (12), (13) and (14), we have

$$\mathrm{AKS}_n(\alpha,\beta,m)=q^{\alpha m\binom{n}{2}+2m^2\binom{n}{3}}\prod_{j=1}^n\frac{\Gamma_q(\alpha+(j-1)m)\Gamma_q(\beta+(j-1)m)\Gamma_q(jm)}{\Gamma_q(\alpha+\beta+(n+j-2)m)\Gamma_q(m)}.\tag{15}$$

We will give a combinatorial interpretation of $\mathrm{AKS}_n(\alpha,\beta,m)$ when $\alpha-1=r$, $\beta-1=s$ and m are non-negative integers. This requires defining a poset, the Selberg poset, whose order polytope has q-volume given by the q-Selberg integral. Basically we start with a chain C of n elements. Insert two independent chains with m elements between any two elements of C. For each element of $c\in C$, insert a chain with s elements above c and a chain with r elements below c.

Definition 7.3. We define the *Selberg poset* $P(n,r,s,m)$ to be the poset in which the elements are $x_i,y_i^{(a)},z_i^{(b)},w_{i,j}^{(k)}$ for $i,j\in[n],a\in[r],b\in[s],k\in[m]$ with $i\ne j$, and the covering relations are as follows:

- $x_i<w_{i,j}^{(1)}<\cdots<w_{i,j}^{(m)}<x_j$ for $1\le i<j\le n$,
- $x_i<w_{j,i}^{(m)}<\cdots<w_{j,i}^{(1)}<x_j$ for $1\le i<j\le n$,
- $y_i^{(1)}<\cdots<y_i^{(r)}<x_i<z_i^{(1)}<\cdots<z_i^{(s)}$ for $1\le i\le n$.

We define W to be the following permutation of the elements of $P(n,r,s,m)$:

$$W = \left(\prod_{1\le i<j\le n} w_{i,j}^{(1)}\cdots w_{i,j}^{(m)}\right)\left(\prod_{1\le i<j\le n} w_{j,i}^{(1)}\cdots w_{j,i}^{(m)}\right) \left(\prod_{i=1}^{n} y_i^{(1)}\cdots y_i^{(r)}\right)\left(\prod_{i=1}^{n} x_i\right)\left(\prod_{i=1}^{n} z_i^{(1)}\cdots z_i^{(s)}\right),$$

where $\prod_{1\le i<j\le n} w_{i,j}^{(1)}\cdots w_{i,j}^{(m)}$ means the concatenation of the word $w_{i,j} = w_{i,j}^{(1)}\cdots w_{i,j}^{(m)}$ for $1 \le i < j \le n$ so that $w_{i,j}$ appears before $w_{i',j'}$ if $i < i'$ or ($i = i'$ and $j < j'$). The other products are defined in the same way, for instance, $\prod_{i=1}^{n} x_i$ means the word $x_1\cdots x_n$. Finally, we define ω to be the labeling of $P(n,r,s,m)$ such that for $u \in P(n,r,s,m)$, $\omega(u)$ is the position of u in W.

Example 7.4. Let P be the Selberg poset $P(n,r,s,m)$ in Figure 5. Then

$$\begin{aligned}W = {} & w_{1,2}^{(1)}w_{1,2}^{(2)}w_{1,2}^{(3)}w_{1,3}^{(1)}w_{1,3}^{(2)}w_{1,3}^{(3)}w_{1,4}^{(1)}w_{1,4}^{(2)}w_{1,4}^{(3)}w_{2,3}^{(1)}w_{2,3}^{(2)}w_{2,3}^{(3)}w_{2,4}^{(1)}w_{2,4}^{(2)}w_{2,4}^{(3)}w_{3,4}^{(1)}w_{3,4}^{(2)}w_{3,4}^{(3)}\\ & w_{2,1}^{(1)}w_{2,1}^{(2)}w_{2,1}^{(3)}w_{3,1}^{(1)}w_{3,1}^{(2)}w_{3,1}^{(3)}w_{4,1}^{(1)}w_{4,1}^{(2)}w_{4,1}^{(3)}w_{3,2}^{(1)}w_{3,2}^{(2)}w_{3,2}^{(3)}w_{4,2}^{(1)}w_{4,2}^{(2)}w_{4,2}^{(3)}w_{4,3}^{(1)}w_{4,3}^{(2)}w_{4,3}^{(3)}\\ & y_1^{(1)}y_1^{(2)}y_2^{(1)}y_2^{(2)}y_3^{(1)}y_3^{(2)}y_4^{(1)}y_4^{(2)}x_1x_2x_3x_4z_1^{(1)}z_2^{(1)}z_3^{(1)}z_4^{(1)}.\end{aligned}$$

The labeling ω of $P(n,r,s,m)$ is shown in Figure 5.

The following theorem implies that the Askey-Kadell-Selberg integral is a q-volume of the order polytope $\mathcal{O}(P(n,r,s,m))$ up to a certain factor.

Theorem 7.5. *We have*

$$\int_{0\le x_1\le\cdots\le x_n\le 1}\prod_{i=1}^{n} x_i^r(qx_i;q)_s \prod_{1\le i<j\le n} x_j^{2m-1}\left(q^{1-m}x_i/x_j;q\right)_{2m-1}\overline{\Delta}(x)d_qx_1\cdots d_qx_n$$
$$= q^{-\binom{m}{2}\binom{n}{2}}([r]_q!)^n([s]_q!)^n([m]_q!)^{2\binom{n}{2}}\cdot V_q(\mathcal{O}(P(n,r,s,m)),W).$$

Proof. Observe that the left side is equal to

$$\int_{0\le x_1\le\cdots\le x_n\le 1}\prod_{i=1}^{n} x_i^r(qx_i;q)_s \prod_{1\le i<j\le n} x_j^{m}\left(q^{1-m}x_i/x_j;q\right)_{m} x_j^m\left(x_i/x_j;q\right)_m d_qx_1\cdots d_qx_n.$$

Then we get the right side by applying the happy cat lemma to each factor $(qx_i;q)_s$, the scaredy cat lemma to each factor x_i^r, the first part (6) of attaching chain lemma with $\rho = m(m-1)\cdots 1$ to each factor $x_j^m\left(q^{1-m}x_i/x_j;q\right)_m$ and the first part (6) of attaching chain lemma with $\rho = 12\cdots m$ to each factor $x_j^m\left(x_i/x_j;q\right)_m$. ☐

The following corollary gives a combinatorial interpretation for the Askey-Kadell-Selberg integral in terms of linear extensions.

Corollary 7.6. *We have*

$$\int_{0\le x_1\le\cdots\le x_n\le 1}\prod_{i=1}^{n} x_i^r(qx_i;q)_s \prod_{1\le i<j\le n} x_j^{2m-1}\left(q^{1-m}x_i/x_j;q\right)_{2m-1}\overline{\Delta}(x)d_qx_1\cdots d_qx_n$$
$$= \frac{q^{-\binom{m}{2}\binom{n}{2}}([r]_q!)^n([s]_q!)^n([m]_q!)^{2\binom{n}{2}}}{[N]_q!}\sum_{\pi\in\mathcal{L}(P(n,r,s,m),\omega)} q^{\mathrm{maj}(\pi)},$$

where

$$N = n(r+s+1) + 2m\binom{n}{2},$$

and, ω is the labeling of the Selberg poset $P(n,r,s,m)$ defined in Definition 7.3.

Proof. This is an immediate consequence of Theorem 7.5 and Theorem 4.4. ☐

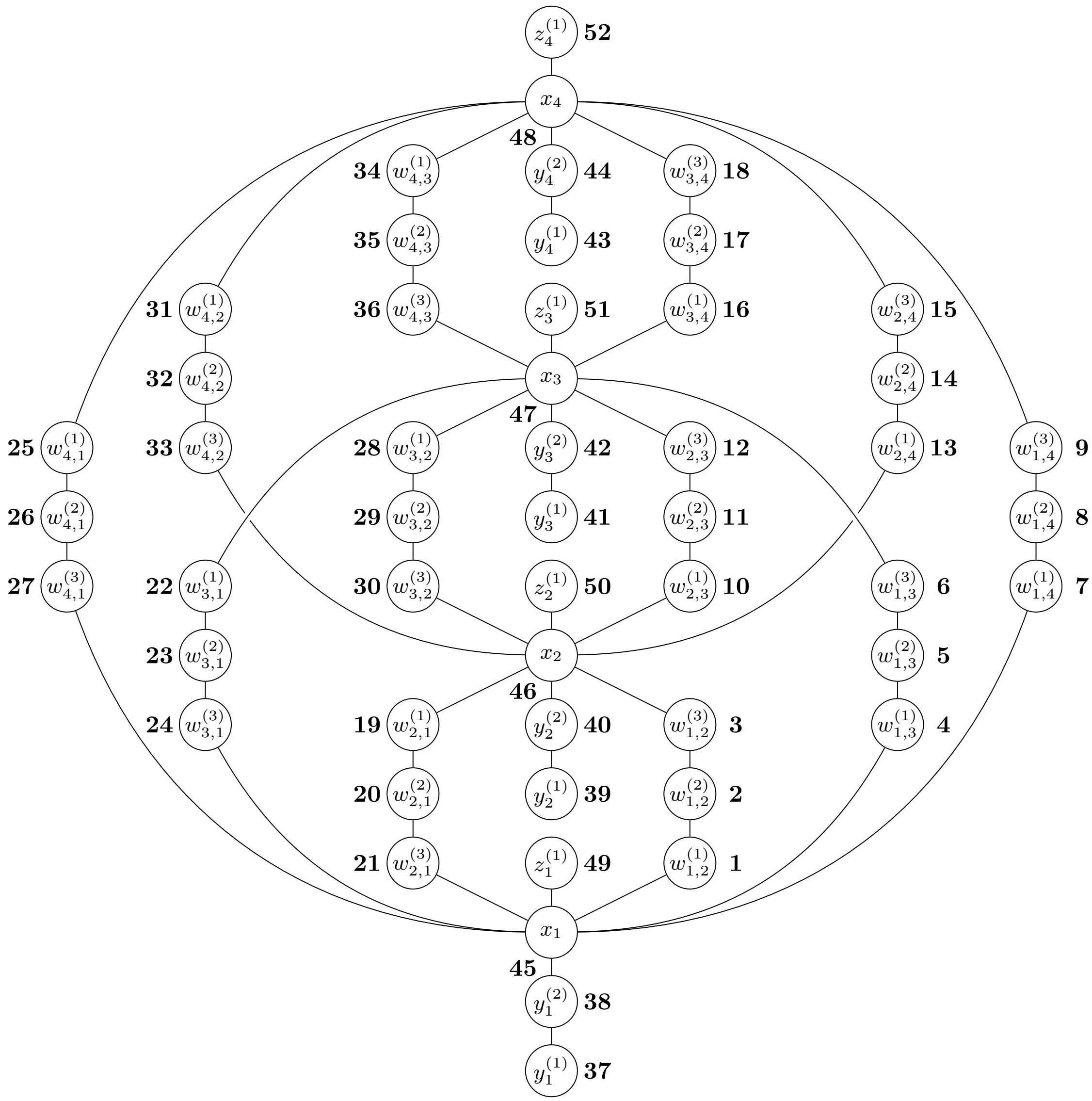

FIGURE 5. The Selberg poset $P(n,r,s,m)$ for $n=4, r=2, s=1, m=3$ with labeling.

Using the q-Selberg integral formula, we obtain an explicit product formula for the maj-generating function for linear extensions of the Selberg poset.

Corollary 7.7. *We have*

$$\sum_{\pi\in\mathcal{L}(P(n,r,s,m),\omega)} q^{\mathrm{maj}(\pi)}$$
$$= \frac{q^{\binom{m}{2}\binom{n}{2}+(r+1)m\binom{n}{2}+2m^2\binom{n}{3}}[N]_q!}{([r]_q!)^n([s]_q!)^n([m]_q!)^{2\binom{n}{2}}} \prod_{j=1}^{n} \frac{[r+(j-1)m]_q![s+(j-1)m]_q![jm-1]_q!}{[r+s+1+(n+j-2)m]_q![m-1]_q!},$$

where N and ω are the same as in Corollary 7.6.

Proof. This is an immediate consequence of Corollary 7.6 and (15). $\square$

We note that Kim and Oh [16] used Stanley's combinatorial interpretation to study the Selberg integral. They found a connection with certain combinatorial objects called Young books, which

0	0	1	1
0	2	2	
4			

0	0	0	1	1	1	2	3	3
	0	0	1	2	2	2	4	
		1	1	6	6			
			8	8				
				8				

FIGURE 6. The left diagrams is an element in $\mathrm{RPP}((4,3,1))$ and the right diagram is an element in $\mathrm{RPP}((\delta_{5+1}+(4,3,1))^*)$.

generalizes standard Young tableaux of certain shapes. It would be interesting to see if their results can be generalized using ours. We also note that Kim and Okada [17] considered a different q-Selberg integral. By evaluating the q-integral, they showed that it is a generating function for Young books with certain weight.

8. Reverse plane partitions

In this section we consider q-Selberg type integrals with Schur functions in the integrand. We then relate the resulting integral evaluations to generating functions for reverse plane partitions and generalized Gelfand-Tsetlin patterns. In particular we

(1) define a Schur poset whose truncated order polytope has q-volume which is basically a Schur function $s_\lambda(x_1,\dots,x_n)$ (Lemma 8.6),
(2) use q-integral evaluations to give generating functions for shifted and non-shifted reverse plane partitions with a given shape and diagonal (Theorem 8.7 and Corollary 8.11),
(3) show that a q-integral of Warnaar may be interpreted as the trace generating function for reverse plane partitions of given shape (Theorems 8.12 and 8.13), and give an analogous Warnaar-type q-integral for the shifted version due to Gansner (Theorem 8.16),
(4) define a generalized Gelfand-Tsetlin pattern, and give a new generating function for these objects (Theorem 8.10),
(5) relate the Askey-Kadell-Selberg integral to a new weighted generating function for reverse plane partitions of square shape (Theorems 8.19 and 8.20).

Before we get to the specific subsections, we need some notation and definitions.

For a partition λ with $\ell(\lambda)\le n$, let $(\delta_{n+1}+\lambda)^*$ be the shifted Young diagram obtained from λ by attaching the shifted staircase $\delta_{n+1}=(n,n-1,\dots,1)$.

Definition 8.1. Let λ be a Young diagram or shifted Young diagram. A *reverse plane partition* of shape λ is a filling of λ with non-negative integers which are weakly increasing along rows and columns. We denote by $\mathrm{RPP}(\lambda)$ the set of reverse plane partitions of shape λ, see Figure 6.

Definition 8.2. Let λ be a partition or a shifted partition. The *k-diagonal* of λ is the set of cells in λ which are in row i and column j with $j-i=k$. For $T\in\mathrm{RPP}(\lambda)$, the *reverse diagonal* of T is the sequence $\mathrm{rdiag}(T)=(\mu_1,\dots,\mu_n)$ obtained by reading the entries of T in the 0-diagonal of λ from southeast to northwest. Also the *trace* is defined by $\mathrm{tr}(T)=\mu_1+\cdots+\mu_n$, and the sum of entries of T is denoted $|T|$.

Example 8.3. Let T_1 be the left diagram and T_2 the right diagram in Figure 6. Then $\mathrm{tr}(T_1)=2$, $\mathrm{tr}(T_2)=17$, $|T_1|=10$, $|T_2|=60$, $\mathrm{rdiag}(T_1)=(2,0)$ and $\mathrm{rdiag}(T_2)=(8,8,1,0,0)$. Note that the reverse diagonal of a reverse plane partition is always a partition.

Definition 8.4. Let λ be a partition with $\ell(\lambda)\le n$. For $0\le k\le\lambda_1+n-1$, let a_k be the size of the k-diagonal of $(\delta_{n+1}+\lambda)^*$. We label the cells in the k-diagonal $(\delta_{n+1}+\lambda)^*$ by $x_1^{(k)},\dots,x_{a_k}^{(k)}$ from southeast to northwest. The *0-Schur poset* $P^0_{\mathrm{Schur}}(n,\lambda)$ is the poset on

$$\{x_i^{(k)}:0\le k\le\lambda_1+n-1,\quad 1\le i\le a_k\}$$

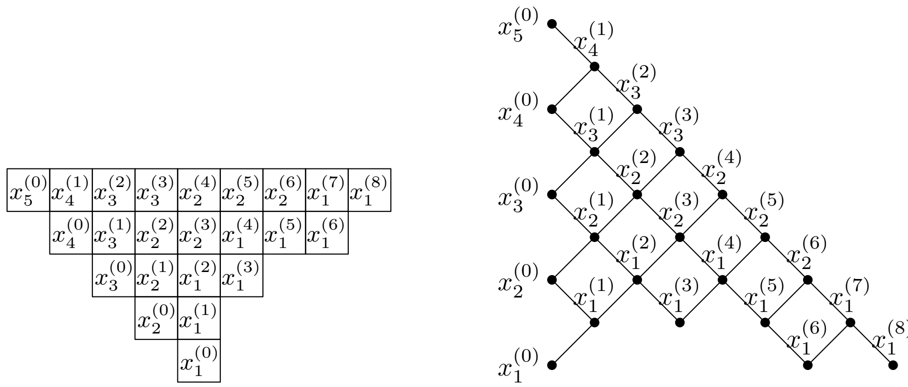

FIGURE 7. The left diagram shows the labels of the cells of $(\delta_{n+1}+\lambda)^*$, and the right diagram shows the Hasse diagram of the 0-Schur poset $P^0_{\mathrm{Schur}}(n,\lambda)$ for $n=5$ and $\lambda=(4,3,1)$.

with relations $x^{(k_1)}_{i_1} \le x^{(k_2)}_{i_2}$ if the cell $x^{(k_2)}_{i_2}$ is weakly northwest of the cell $x^{(k_1)}_{i_1}$ in $(\delta_{n+1}+\lambda)^*$. See Figure 7. Then we define the *Schur poset* $P_{\mathrm{Schur}}(n,\lambda)$ to be the poset obtained from the *0-Schur poset* $P^0_{\mathrm{Schur}}(n,\lambda)$ by removing the elements $x^{(0)}_1,\dots,x^{(0)}_n$.

Note that we have

$$\begin{aligned} P_{\mathrm{Schur}}(n,\lambda) &\cong P^0_{\mathrm{Schur}}(n-1,\lambda) \qquad \text{if } \ell(\lambda)\le n-1,\\ P_{\mathrm{Schur}}(n,\lambda) &\cong P^0_{\mathrm{Schur}}(n,\lambda-(1^n)) \qquad \text{if } \ell(\lambda)= n. \end{aligned}$$

The next proposition shows that the Schur function $s_\lambda(x)$ can be written as a q-volume of a truncated order polytope of a Schur poset up to a constant factor.

Proposition 8.5. *Let λ be a partition with $\ell(n)\le n$ and a_k the size of the k-diagonal of $(\delta_{n+1}+\lambda)^*$. Let $x=(x_1,\dots,x_n)$ be a sequence of fixed real numbers $x_1<x_2<\cdots<x_n$ and*

$$I=\begin{cases} \{(x^{(k)}_i)\in\mathbb{R}^{\binom{n}{2}+|\lambda|} : x_j\le x^{(1)}_j\le x_{j+1}, 1\le j\le n-1\}, & \text{if } \ell(\lambda)\le n-1,\\ \{(x^{(k)}_i)\in\mathbb{R}^{\binom{n}{2}+|\lambda|} : x_{j-1}\le x^{(1)}_j\le x_j, 1\le j\le n, x_0=0\}, & \text{if } \ell(\lambda)= n,\end{cases}$$

where $(x^{(k)}_i)$ means a point in $\mathbb{R}^{\binom{n}{2}+|\lambda|}$ with indices $1\le k\le\lambda_1+n-1$ and $1\le i\le a_k$. Then

$$s_\lambda(x)\overline{\Delta}(x)=\prod_{j=1}^n[\lambda_j+n-j]_q!\int_{\mathcal{O}_I(P_{\mathrm{Schur}}(n,\lambda))} d_qx^{(\lambda_1+n-1)}d_qx^{(\lambda_1+n-2)}\cdots d_qx^{(1)},$$

where $d_qx^{(k)}=d_qx^{(k)}_1\cdots d_qx^{(k)}_{a_k}$.

We will prove Lemma 8.6 which is equivalent to Proposition 8.5. The idea is to introduce new sets of variables which are interlacing with the previous sets of variables.

Lemma 8.6. *Let λ be a partition with $\ell(n)\le n$ and a_k the size of the k-diagonal of $(\delta_{n+1}+\lambda)^*$. For $0\le k\le\lambda_1+n-1$, we consider sequences of variables $x^{(k)}=(x^{(k)}_1,\dots,x^{(k)}_{a_k})$ with $x^{(0)}_i=x_i$. Then for the set*

$$Q=\{x^{(\lambda_1+n-1)}\prec\cdots\prec x^{(0)}\}$$

of inequalities, we have

$$s_\lambda(x)\overline{\Delta}(x)=\prod_{j=1}^n[\lambda_j+n-j]_q!\int_Q d_qx^{(\lambda_1+n-1)}d_qx^{(\lambda_1+n-2)}\cdots d_qx^{(1)}, \tag{16}$$

where $d_qx^{(k)}=d_qx^{(k)}_1\cdots d_qx^{(k)}_{a_k}$.

Proof. We use induction on n. If $n=1$, the left hand side of (16) is $s_\lambda(x_1)=x_1^{\lambda_1}$. By Corollary 4.9, the right hand side of (16) is

$$[\lambda_1]_q! \int_{0\le x_1^{(\lambda_1)}\le\cdots\le x_1^{(1)}\le x_1} d_q x_1^{(\lambda_1)}\cdots d_q x_1^{(1)} = x_1^{\lambda_1}.$$

Now suppose that (16) is true for $n-1$ and consider n.

CASE 1: $\ell(\lambda)\le n-1$. By Lemma 5.7, we have

$$s_\lambda(x)\overline{\Delta}(x) = \prod_{j=1}^{n-1}[\lambda_j+n-j]_q \int_{w\prec x} s_\lambda(w)\overline{\Delta}(w) d_q w_1\cdots d_q w_{n-1}. \tag{17}$$

Since $\ell(\lambda)\le n-1$ and $w=(w_1,\dots,w_{n-1})$ has $n-1$ variables, by the induction hypothesis, we have

$$s_\lambda(w)\overline{\Delta}(w) = \prod_{j=1}^{n-1}[\lambda_j+n-j-1]_q! \int_{Q'} d_q w^{(\lambda_1+n-2)} d_q w^{(\lambda_1+n-3)}\cdots d_q w^{(1)}, \tag{18}$$

where $w^{(k)}=(w_1^{(k)},\dots,w_{a'_k}^{(k)})$ is a sequence of variables for $0\le k\le \lambda_1+n-2$ with $w_i^{(0)}=w_i$, a'_k is the size of the k-diagonal of $(\delta_n+\lambda)^*$, and

$$Q'=\{w^{(\lambda_1+n-2)}\prec\cdots\prec w^{(0)}\}.$$

Note that $a'_k=a_{k+1}$. By substituting $w_i^{(k)}=x_i^{(k+1)}$, we have

$$\begin{aligned}\int_{w\prec x}\left(\int_{Q'} d_q w^{(\lambda_1+n-2)} d_q w^{(\lambda_1+n-3)}\cdots d_q w^{(1)}\right) d_q w_1\cdots d_q w_{n-1}\\ = \int_Q d_q x^{(\lambda_1+n-1)} d_q x^{(\lambda_1+n-2)}\cdots d_q x^{(1)}.\end{aligned} \tag{19}$$

By (17), (18), and (19), we have

$$s_\lambda(x)\overline{\Delta}(x) = \prod_{j=1}^{n-1}[\lambda_j+n-j]_q! \int_Q d_q x^{(\lambda_1+n-1)} d_q x^{(\lambda_1+n-2)}\cdots d_q x^{(1)}.$$

Since $\lambda_n=0$, we have

$$\prod_{j=1}^{n-1}[\lambda_j+n-j]_q! = \prod_{j=1}^{n}[\lambda_j+n-j]_q!,$$

which implies that (16) is also true for n.

CASE 2: $\ell(\lambda)=n$. Let $\lambda_n=m$. Note that for $0\le k\le m$, we have $x^{(k)}=(x_1^{(k)},\dots,x_n^{(k)})$. By Lemma 5.8, for $0\le k\le m-1$, we have

$$s_{\lambda-(k^n)}(x^{(k)})\overline{\Delta}(x^{(k)}) = \prod_{j=1}^{n}[\lambda_j-k+n-j]_q \int_{x^{(k+1)}\prec x^{(k)}} s_{\lambda-((k+1)^n)}(x^{(k+1)})\overline{\Delta}(x^{(k+1)}) d_q x^{(k+1)}. \tag{20}$$

Applying (20) for $k=0,1,\dots,m-1$, iteratively, we get

$$s_\lambda(x)\overline{\Delta}(x) = \prod_{k=0}^{m-1}\prod_{j=1}^{n}[\lambda_j-k+n-j]_q \int_{x^{(m)}\prec\cdots\prec x^{(0)}} s_{\lambda-(m^n)}(x^{(m)})\overline{\Delta}(x^{(m)}) d_q x^{(m)}\cdots d_q x^{(1)}. \tag{21}$$

Since $\lambda-(m^n)$ has length at most $n-1$, by *CASE 1*, we get

$$s_{\lambda-(m^n)}(x^{(m)})\overline{\Delta}(x^{(m)}) = \prod_{j=1}^{n}[\lambda_j-m+n-j]_q! \int_{Q'} d_q x^{(\lambda_1+n-1)} d_q x^{(\lambda_1+n-2)}\cdots d_q x^{(m+1)}, \tag{22}$$

where $Q' = \{x^{(\lambda_1+n-1)} \prec x^{(\lambda_1+n-2)} \prec \cdots \prec x^{(m)}\}$. By (21) and (22), we have that $s_\lambda(x)\overline{\Delta}(x)$ is equal to

$$\left(\prod_{k=0}^{m-1}\prod_{j=1}^{n}[\lambda_j-k+n-j]_q\right)\prod_{j=1}^{n}[\lambda_j-m+n-j]_q!\int_Q d_qx^{(\lambda_1+n-1)}d_qx^{(\lambda_1+n-2)}\cdots d_qx^{(1)},$$

which is equal to the right hand side of (16). Thus (16) is also true for n. By induction (16) is true for all nonnegative integers n. □

8.1. **Shifted reverse plane partitions with fixed diagonal entries.** The main result in this subsection, Theorem 8.7, reinterprets the q-integral evaluation in Lemma 8.6 as a generating function for shifted reverse plane partitions.

Theorem 8.7. *For $\lambda, \mu \in \mathrm{Par}_n$ we have*

$$\sum_{\substack{T\in \mathrm{RPP}((\delta_{n+1}+\lambda)^*)\\ \mathrm{rdiag}(T)=\mu}} q^{|T|} = \frac{q^{-b(\delta_{n+1}+\lambda)}}{\prod_{j=1}^n (q;q)_{\lambda_j+n-j}} q^{|\mu+\delta_n|}s_\lambda(q^{\mu+\delta_n})\overline{\Delta}(q^{\mu+\delta_n}).$$

Proof. Let

$$x=(x_1,x_2,\dots,x_n)=(q^{\mu_1+n-1},q^{\mu_2+n-2},\dots,q^{\mu_n})=q^{\mu+\delta_n}.$$

Following the notations in Proposition 8.5, we have

$$s_\lambda(x)\overline{\Delta}(x)=\prod_{j=1}^n[\lambda_j+n-j]_q!\int_{\mathcal{O}_I(P_{\mathrm{Schur}}(n,\lambda))} d_qx^{(\lambda_1+n-1)}d_qx^{(\lambda_1+n-2)}\cdots d_qx^{(1)}.$$

Let ω be the labeling of $P_{\mathrm{Schur}}(n,\lambda)$ defined by $\omega(x_i^{(k)})=j$ if $d_qx_i^{(k)}$ is in the jth position of $d_qx^{(\lambda_1+n-1)}d_qx^{(\lambda_1+n-2)}\cdots d_qx^{(1)}$. Then by Theorem 4.1, we have

$$\int_{\mathcal{O}_I(P_{\mathrm{Schur}}(n,\lambda))} d_qx^{(\lambda_1+n-1)}d_qx^{(\lambda_1+n-2)}\cdots d_qx^{(1)}=(1-q)^{\binom{n}{2}+|\lambda|}\sum_\sigma q^{|\sigma|},$$

where the sum is over all $(P_{\mathrm{Schur}}(n,\lambda),\omega)$-partitions σ such that

$$\begin{aligned}
&\mu_j+n-j>\sigma(x_j^{(1)})\ge\mu_{j+1}+n-j-1, \quad 1\le j\le n-1 \quad \text{if } \ell(\lambda)\le n-1,\\
&\mu_{j-1}+n-j+1>\sigma(x_j^{(1)})\ge\mu_j+n-j, \quad 1\le j\le n \quad \text{if } \ell(\lambda)=n,
\end{aligned}$$

where $\mu_0=\infty$. There is an obvious way to regard such σ as a reverse plane partition $U\in\mathrm{RPP}((\delta_{n+1}+\lambda)^*)$ with $|\sigma|=|U|-|\mu+\delta_n|$ such that the $\mathrm{rdiag}(U)=q^{\mu+\delta_n}$ and in each column the entries are strictly increasing from top to bottom. Thus, by Theorem 4.1, we have

$$\int_{\mathcal{O}_I(P_{\mathrm{Schur}}(n,\lambda))} d_qx^{(\lambda_1+n-1)}d_qx^{(\lambda_1+n-2)}\cdots d_qx^{(1)}=(1-q)^{|\lambda|+\binom{n}{2}}\sum_U q^{|U|-\sum_{i=1}^n(\mu_i+n-i)},$$

where the sum is over all column-strict reverse plane partitions $U\in\mathrm{RPP}((\delta_{n+1}+\lambda)^*)$ with $\mathrm{rdiag}(U)=q^{\mu+\delta_n}$. For such U, let T be the reverse plane partition obtained from it by decreasing the entries in row i by $i-1$ for $i=1,2,\dots,n$. Then $T\in\mathrm{RPP}((\delta_{n+1}+\lambda)^*)$, $\mathrm{rdiag}(T)=\mu$, and $|U|=|T|+b(\delta_{n+1}+\lambda)$. Summarizing these, we obtain

$$s_\lambda(x)\overline{\Delta}(x)=\prod_{j=1}^n[\lambda_j+n-j]_q!(1-q)^{|\lambda|+\binom{n}{2}}\sum_{\substack{T\in \mathrm{RPP}((\delta_{n+1}+\lambda)^*)\\ \mathrm{rdiag}(T)=\mu}} q^{|T|+b(\delta_{n+1}+\lambda)-\sum_{i=1}^n(\mu_i+n-i)},$$

which is equivalent to the theorem. □

8.2. **Generalized Gelfand-Tsetlin patterns.** In this subsection we restate Theorem 8.7 in terms of generalized Gelfand-Tsetlin patterns, see Theorem 8.10.

Definition 8.8. For $\lambda,\mu\in \mathrm{Par}_n$, an (n,λ,μ)-*Gelfand-Tsetlin pattern* is a triangular array

$$\{(x_{i,j}) : 1\le j\le n, \quad 1-\lambda_j\le i\le n+1-j\}$$

of nonnegative integers such that $x_{i,j+1}\ge x_{i,j}\ge x_{i+1,j}$ and $x_{i,n+1-i}=\mu_i$. We assume that the inequality holds if any index lies outside the stated domain. We denote by $\mathrm{GT}_n(\lambda,\mu)$ the set of (n,λ,μ)-Gelfand-Tsetlin patterns.

Example 8.9. Let $n=4$, $\lambda=(3,1)$ and $\mu=(3,2,2,1)$. Then the coordinates of an (n,λ,μ)-Gelfand-Tsetlin pattern are arranged as follows.

$$\begin{array}{ccccccc}
 & & & x_{1,4} & & & \\
 & & & x_{1,3} & x_{2,3} & & \\
 & & x_{0,2} & x_{1,2} & x_{2,2} & x_{3,2} & \\
x_{-2,1} & x_{-1,1} & x_{0,1} & x_{1,1} & x_{2,1} & x_{3,1} & x_{4,1}
\end{array}$$

An example of an (n,λ,μ)-Gelfand-Tsetlin is shown below.

$$\begin{array}{ccccccc}
 & & & 3 & & & \\
 & & & 3 & 2 & & \\
 & & 4 & 3 & 2 & 2 & \\
7 & 5 & 3 & 3 & 2 & 1 & 1
\end{array}$$

Note that if $\lambda=\varnothing$, then $\mathrm{GT}_n(\varnothing,\mu)$ is basically the set of Gelfand-Tsetlin patterns, see [21, (7.37), p. 313]. There is a well known bijection [21, p. 314] between Gelfand-Tsetlin patterns and column strict tableaux which gives

$$\sum_{T\in \mathrm{GT}_n(\varnothing,\mu)} q^{|T|} = q^{|\mu|} s_\mu(q^{\delta_n}). \tag{23}$$

Note also that, if rotated by 180°, the (λ,μ)-Gelfand-Tsetlin patterns are the same as the reverse plane partitions of shape $(\delta_{n+1}+\lambda)^*$ with reverse diagonal entries given by μ. Thus Theorem 8.7 can be restated as Theorem 8.10, which generalizes (23) to arbitrary λ.

Theorem 8.10. *For $\lambda,\mu\in\mathrm{Par}_n$, we have*

$$\sum_{T\in \mathrm{GT}_n(\lambda,\mu)} q^{|T|} = q^{|\mu|-b(\lambda)} s_\mu(q^{\delta_n}) s_\lambda(q^{\mu+\delta_n}) \prod_{j=1}^n \frac{(q;q)_{n-j}}{(q;q)_{\lambda_j+n-j}}.$$

Equivalently,

$$\sum_{T\in \mathrm{GT}_n(\lambda,\mu)} q^{|T|} = \frac{q^{|\mu|-b(\delta_n+\lambda)}}{\prod_{j=1}^n (q;q)_{\lambda_j+n-j}} s_\lambda(q^{\mu+\delta_n})\overline{\Delta}(q^{\mu+\delta_n}).$$

Proof. By Theorem 8.7, we have

$$\sum_{T\in \mathrm{GT}_n(\lambda,\mu)} q^{|T|} = \sum_{\substack{T\in \mathrm{RPP}((\delta_{n+1}+\lambda)^*)\\ \mathrm{rdiag}(T)=\mu}} q^{|T|} = \frac{q^{-b(\delta_{n+1}+\lambda)}}{\prod_{j=1}^n (q;q)_{\lambda_j+n-j}} q^{|\mu+\delta_n|} s_\lambda(q^{\mu+\delta_n})\overline{\Delta}(q^{\mu+\delta_n}).$$

Then we obtain the theorem using the fact [21, (7.105)]

$$s_\mu(q^{\delta_n}) = \prod_{1\le i<j\le n} \frac{q^{\mu_j+n-j}-q^{\mu_i+n-i}}{q^{i-1}-q^{j-1}} = \frac{\overline{\Delta}(q^{\mu+\delta_n})}{q^{b(\delta_n)}\prod_{j=1}^{n-1}(q;q)_j}.$$

□

8.3. **The trace-generating function for reverse plane partitions.** In this subsection we give two results on reverse plane partitions. In Corollary 8.11 we give a generating function for reverse plane partitions with a fixed shape and diagonal. Then we reinterpret an integral of Warnaar as the trace generating function for reverse plane partitions of a given shape, Theorem 8.13 and Corollary 8.14.

The first result gives the generating function for reverse plane partitions with a fixed shape and diagonal by decomposing via the Durfee square. Here, the *Durfee square* of a partition λ is the largest square containing the cell in row 1 and column 1 in λ.

Corollary 8.11. *Let $\lambda = (\lambda_1, \dots, \lambda_n)$, $\mu = (\mu_1, \dots, \mu_n)$ and ν the partition obtained from (n^n) by attaching λ to the right and μ' at the bottom. Then for a partition $\rho = (\rho_1, \dots, \rho_n)$, we have*

$$\sum_{\substack{T\in \mathrm{RPP}(\nu)\\ \mathrm{rdiag}(T)=\rho}} q^{|T|} = \frac{q^{\binom{n}{2}-b(\delta_{n+1}+\lambda)-b(\delta_{n+1}+\mu)}q^{|\rho+\delta_n|}}{\prod_{j=1}^n (q;q)_{\lambda_j+n-j}(q;q)_{\mu_j+n-j}} s_\lambda(q^{\rho+\delta_n}) s_\mu(q^{\rho+\delta_n}) \overline{\Delta}(q^{\rho+\delta_n})^2. \tag{24}$$

Proof. Since

$$\sum_{\substack{T\in \mathrm{RPP}(\nu)\\ \mathrm{rdiag}(T)=\rho}} q^{|T|} = q^{-|\rho|} \sum_{\substack{T\in \mathrm{RPP}((\delta_{n+1}+\lambda)^*)\\ \mathrm{rdiag}(T)=\rho}} q^{|T|} \sum_{\substack{T\in \mathrm{RPP}((\delta_{n+1}+\mu)^*)\\ \mathrm{rdiag}(T)=\rho}} q^{|T|},$$

we get the desired formula by Theorem 8.10. □

The following q-integral evaluation is the special case $k = 1$ of Warnaar's integral [24, Theorem 1.4], which has Macdonald polynomials instead of Schur functions.

Theorem 8.12. [24, Special case of Theorem 1.4] *Let $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_n)$ be partitions,* $\mathrm{Re}(\alpha) > -\lambda_n - \mu_n$ *and* $0 < q < 1$. *Then*

$$\int_{0\le x_1\le \dots \le x_n \le 1} s_\lambda(x) s_\mu(x) \prod_{i=1}^n x_i^{\alpha-1} \overline{\Delta}(x)^2 d_q x = (1-q)^n q^{\alpha\binom{n}{2}+2\binom{n}{3}}$$
$$\times s_\lambda(1,q,\dots,q^{n-1}) s_\mu(1,q,\dots,q^{n-1}) \prod_{i=1}^{n-1} (q;q)_i^2 \prod_{i,j=1}^n \frac{1}{1-q^{\alpha+2n-i-j+\lambda_i+\mu_j}}.$$

Proof. If $k = 1$ in [24, Theorem 1.4], we have

$$\int_{[0,1]^n} s_\lambda(x) s_\mu(x) \prod_{i=1}^n x_i^{\alpha-1} \prod_{1\le i<j\le n} x_j^2 \left(\frac{x_i}{x_j}\right)_2 d_q x$$
$$= q^{\alpha\binom{n}{2}+2\binom{n}{3}} (1-q)^{n^2} s_\lambda(1,q,\dots,q^{n-1}) s_\mu(1,q,\dots,q^{n-1})$$
$$\times \prod_{i=1}^n \Gamma_q(i)\Gamma_q(i+1) \prod_{i,j=1}^n \frac{1}{1-q^{\alpha+2n-i-j+\lambda_i+\mu_j}}.$$

By the same argument as in the proof of (15), one can show that these two integral evaluations are equivalent. □

We now show that Warnaar's q-integral Theorem 8.12 can be interpreted as a trace-generating function of a reverse plane partition of given shape. First we show that the trace generating function is a q-integral.

Theorem 8.13. *Let $\lambda=(\lambda_1,\dots,\lambda_n)$, $\mu=(\mu_1,\dots,\mu_n)$ and ν the partition obtained from (n^n) by attaching λ to the right and μ' at the bottom. Then*

$$\sum_{T\in \mathrm{RPP}(\nu)} q^{|T|}(q^a)^{\mathrm{tr}(T)} = \frac{q^{(1-a)\binom{n}{2}-b(\delta_{n+1}+\lambda)-b(\delta_{n+1}+\mu)}}{\prod_{j=1}^n (q;q)_{\lambda_j+n-j}(q;q)_{\mu_j+n-j}} \times \frac{1}{(1-q)^n}\int_{0\le x_1\le\cdots\le x_n\le 1} s_\lambda(x)s_\mu(x)\prod_{i=1}^n x_i^a \overline{\Delta}(x)^2 d_qx.$$

Proof. Let $\rho=(\rho_1,\dots,\rho_n)$ be a fixed partition. For $T\in\mathrm{RPP}(\nu)$ with $\mathrm{rdiag}(T)=\rho$, we have

$$q^{a\cdot \mathrm{tr}(T)} = q^{a|\rho|} = q^{-a\binom{n}{2}}\left(q^{|\rho+\delta_n|}\right)^a. \tag{25}$$

Thus, by Corollary 8.11,

$$\sum_{\substack{T\in \mathrm{RPP}(\nu)\\ \mathrm{rdiag}(T)=\rho}} q^{|T|}(q^a)^{\mathrm{tr}(T)} = \frac{q^{(1-a)\binom{n}{2}-b(\delta_{n+1}+\lambda)-b(\delta_{n+1}+\mu)}\left(q^{|\rho+\delta_n|}\right)^{a+1}}{\prod_{j=1}^n (q;q)_{\lambda_j+n-j}(q;q)_{\mu_j+n-j}} \times s_\lambda(q^{\rho+\delta_n})s_\mu(q^{\rho+\delta_n})\overline{\Delta}(q^{\rho+\delta_n})^2. \tag{26}$$

By Lemma 4.3, if we take the sum of both sides of (26) over all $\rho\in\mathrm{Par}_n$, we get the theorem. $\square$

As a corollary of Theorems 8.12 and 8.13, one can prove the trace-generating function for reverse plane partitions, which is a special case of [9, Theorem 5.1]. One uses the principally specialized Schur functions as products, [21, Lemma 7.21.1] and [21, Lemma 7.21.2]. We do not give the details.

Corollary 8.14. *Let ν be a partition. Then*

$$\sum_{T\in\mathrm{RPP}(\nu)} (q^a)^{\mathrm{tr}(T)}q^{|T|} = \prod_{u\in\nu}\frac{1}{1-(q^a)^{\chi(u)}q^{h(u)}},$$

where $\chi(u)=1$ if u is in the Durfee square of ν and $\chi(u)=0$ otherwise.

8.4. **The trace-generating function for shifted reverse plane partitions.** In this subsection we find a "shifted"-counterpart to Warnaar's q-integral, which gives a trace generating function for shifted reverse plane partitions.

We need a special case of Gansner's result on the trace-generating function for shifted reverse plane partitions. We note that Gansner [9] considered a more general weight which involves the entries in the k-diagonal for each k.

Theorem 8.15. [9, A special case of Theorem 7.1] *For $\lambda\in\mathrm{Par}_n$, we have*

$$\sum_{T\in\mathrm{RPP}((\lambda+\delta_{n+1})^*)} x^{\mathrm{tr}(T)}q^{|T|} = \prod_{u\in(\lambda+\delta_{n+1})^*}\frac{1}{1-x^{\chi(u)}q^{h(u)}},$$

where $\chi(u)=1$ if u is in column j for $1\le j\le n$, and $\chi(u)=0$ otherwise, and $h(u)$ is the length of the shifted hook at u, see [9] for the definition. Equivalently, this can be restated as

$$\sum_{T\in\mathrm{RPP}((\lambda+\delta_{n+1})^*)} x^{\mathrm{tr}(T)}q^{|T|} = q^{-b(\lambda)}s_\lambda(1,q,\dots,q^{n-1})\prod_{j=1}^n\frac{(q;q)_{j-1}}{(q;q)_{\lambda_j+n-j}} \times \prod_{1\le i\le j\le n}\frac{1}{1-xq^{1+2n-i-j+\lambda_i+\lambda_{j+1}}}.$$

We do not prove the equivalence of the second statement. It follows from routine facts on Schur functions.

Theorem 8.15 gives the generating function for shifted reverse plane partitions, but we want to find the corresponding q-integral. Here is the "shifted" version of Warnaar's q-integral which corresponds to Gansner's theorem.

0	1	3
0	3	3
2	3	3

FIGURE 8. A reverse plane partition of square shape (3^3).

Theorem 8.16. *If* $\lambda \in \mathrm{Par}_n$,

$$\int_{0\le x_1\le\cdots\le x_n\le 1} s_\lambda(x)\prod_{i=1}^n x_i^{\alpha-1}\overline{\Delta}(x)d_qx = (1-q)^n q^{\alpha\binom{n}{2}+\binom{n}{3}} s_\lambda(1,q,\dots,q^{n-1}) \times \prod_{i=1}^{n-1}(q;q)_i \prod_{1\le i\le j\le n}\frac{1}{1-q^{\alpha+2n-i-j+\lambda_i+\lambda_{j+1}}}.$$

Proof. Let $a=\alpha-1$. By Theorem 8.10, we have

$$\sum_{\substack{T\in\mathrm{RPP}((\delta_{n+1}+\lambda)^*)\\ \mathrm{rdiag}(T)=\mu}} (q^a)^{\mathrm{tr}(T)}q^{|T|} = q^{-a\binom{n}{2}}q^{a|\mu+\delta_n|}\frac{q^{-b(\delta_{n+1}+\lambda)}}{\prod_{j=1}^n (q;q)_{\lambda_j+n-j}} q^{|\mu+\delta_n|}s_\lambda(q^{\mu+\delta_n})\overline{\Delta}(q^{\mu+\delta_n}).$$

By summing both sides over all $\mu\in\mathrm{Par}_n$ and using Lemma 4.3, we obtain

$$\begin{aligned}\sum_{T\in\mathrm{RPP}((\delta_{n+1}+\lambda)^*)} (q^a)^{\mathrm{tr}(T)}q^{|T|} &= \frac{q^{-a\binom{n}{2}-b(\delta_{n+1}+\lambda)}}{\prod_{j=1}^n (q;q)_{\lambda_j+n-j}}\sum_{\mu\in\mathrm{Par}_n} q^{|\mu+\delta_n|}q^{a|\mu+\delta_n|}s_\lambda(q^{\mu+\delta_n})\overline{\Delta}(q^{\mu+\delta_n})\\ &= \frac{q^{-(a+1)\binom{n}{2}-\binom{n}{3}-b(\lambda)}}{\prod_{j=1}^n (q;q)_{\lambda_j+n-j}}\frac{1}{(1-q)^n}\int_{0\le x_1\le\cdots\le x_n\le 1} s_\lambda(x)\prod_{i=1}^n x_i^a\overline{\Delta}(x)d_qx.\end{aligned}$$

We finish the proof by comparing this with Theorem 8.15. ☐

8.5. **Reverse plane partitions and the q-Selberg integral.** In this subsection we show that the Askey-Kadell-Selberg integral is equivalent to a generating function for reverse plane partitions of a square shape with certain weight. The weight has 3 parameters and generalizes the sum of the entries of a reverse plane partition.

Definition 8.17. For a reverse plane partition $T\in\mathrm{RPP}(n^n)$ with $\mathrm{rdiag}(T)=(\mu_1,\dots,\mu_n)$, let $v_i=\mu_i+n-i$ for $i=1,2,\dots,n$. For integers $a,b\ge 0$ and $m\ge 1$, we define

$$\mathrm{wt}_{a,b,m}(T) = q^{|T|+a\cdot\mathrm{tr}(T)}(q^{v_1+1},\dots,q^{v_n+1};q)_b\prod_{1\le i<j\le n}\frac{(q^{v_j})^{2m-1}(q^{1-m+v_i-v_j};q)_{2m-1}}{q^{v_j}-q^{v_i}}.$$

Example 8.18. Let T be the reverse plane partition in Figure 8. Then $|T|=18$, $\mathrm{tr}(T)=6$, $\mathrm{rdiag}(T)=(3,3,0)$ and $(v_1,v_2,v_3)=(3,3,0)+(2,1,0)=(5,4,0)$. Thus

$$\begin{aligned}\mathrm{wt}_{a,b,m}(T) &= q^{18+6a}(q^6,q^5,q^1;q)_b\cdot\frac{(q^4)^{2m-1}(q^{1-m+5-4};q)_{2m-1}}{q^4-q^5}\cdot\frac{(q^0)^{2m-1}(q^{1-m+5-0};q)_{2m-1}}{q^0-q^5}\\ &\quad\times\frac{(q^0)^{2m-1}(q^{1-m+4-0};q)_{2m-1}}{q^0-q^4}\\ &= q^{18+6a}(q^6,q^5,q^1;q)_b\cdot\frac{q^{8m-8}(q^{2-m},q^{6-m},q^{5-m};q)_{2m-1}}{(1-q)(1-q^5)(1-q^4)}.\end{aligned}$$

Note that

$$\begin{aligned}\mathrm{wt}_{a,b,1}(T) &= q^{|T|+a\cdot\mathrm{tr}(T)}(q^{v_1+1},\dots,q^{v_n+1};q)_b,\\ \mathrm{wt}_{a,0,1}(T) &= q^{|T|+a\cdot\mathrm{tr}(T)},\\ \mathrm{wt}_{0,0,1}(T) &= q^{|T|}.\end{aligned}$$

The following theorem shows that the Askey-Kadell-Selberg integral is a generating function for reverse plane partitions of a square shape, up to simple constants.

Theorem 8.19. *The weighted generating function for reverse plane partitions of a square shape is given by the Askey-Kadell-Selberg integral*

$$\sum_{T\in \mathrm{RPP}(n^n)} \mathrm{wt}_{a,b,m}(T) = \frac{q^{(-1-a)\binom{n}{2}-2\binom{n}{3}}}{(1-q)^n\prod_{j=1}^{n-1}(q;q)_j^2}\cdot \mathrm{AKS}_n(a+1,b+1,m).$$

Proof. Let us fix $\mu=(\mu_1,\dots,\mu_n)\in \mathrm{Par}_n$ and $v=(v_1,\dots,v_n)$ with $v_i=\mu_i+n-i$. Then

$$\sum_{\substack{T\in \mathrm{RPP}(n^n)\\ \mathrm{rdiag}(T)=\mu}} \mathrm{wt}_{a,b,m}(T) = \sum_{\substack{T\in \mathrm{RPP}(n^n)\\ \mathrm{rdiag}(T)=\mu}} q^{|T|+a\cdot \mathrm{tr}(T)} C(q^{\mu+\delta_n}),$$

where

$$C(x)=\prod_{i=1}^n (qx_i;q)_b \prod_{1\le i<j\le n} x_j^{2m-1}(q^{1-m}x_i/x_j;q)_{2m-1}\overline{\Delta}(x)^{-1}.$$

By the special case $\lambda=\mu=(0^n)$ of Corollary 8.11 and (25), we have

$$\sum_{\substack{T\in \mathrm{RPP}(n^n)\\ \mathrm{rdiag}(T)=\mu}} q^{|T|+a\cdot \mathrm{tr}(T)} = \frac{q^{(1-a)\binom{n}{2}-2b(\delta_{n+1})}(q^{|\mu+\delta_n|})^{a+1}}{\prod_{j=1}^{n-1}(q;q)_j^2}\overline{\Delta}(q^{\mu+\delta_n})^2.$$

Combining these two equations we get

$$\sum_{\substack{T\in \mathrm{RPP}(n^n)\\ \mathrm{rdiag}(T)=\mu}} \mathrm{wt}_{a,b,m}(T) = \frac{q^{(1-a)\binom{n}{2}-2b(\delta_{n+1})}(q^{|\mu+\delta_n|})^{a+1}}{\prod_{j=1}^{n-1}(q;q)_j^2}\overline{\Delta}(q^{\mu+\delta_n})^2 C(q^{\mu+\delta_n}).$$

By summing over all $\mu\in \mathrm{Par}_n$ and using Lemma 4.3, we get

$$\sum_{T\in \mathrm{RPP}(n^n)} \mathrm{wt}_{a,b,m}(T) = \frac{q^{(1-a)\binom{n}{2}-2b(\delta_{n+1})}}{\prod_{j=1}^{n-1}(q;q)_j^2}\frac{1}{(1-q)^n}\mathrm{AKS}_n(a+1,b+1,m).$$

Then we finish the proof using the fact $b(\delta_{n+1})=\binom{n}{2}+\binom{n}{3}$. □

Using Theorem 8.19 and the evaluation (15), we obtain a product formula for the weighted generating function.

Theorem 8.20. *The weighted generating function for reverse plane partitions of square shape has the explicit product formula*

$$\sum_{T\in \mathrm{RPP}(n^n)} \mathrm{wt}_{a,b,m}(T) = \frac{q^{(1-a+m+am)\binom{n}{2}-2\binom{n+1}{3}+2m^2\binom{n}{3}}}{(1-q)^{n^2}} \times \prod_{j=1}^n \frac{[a+(j-1)m]_q![b+(j-1)m]_q![jm-1]_q!}{[a+b+(n+j-2)m+1]_q![m-1]_q!([j-1]_q!)^2}.$$

We conclude with some special cases of Theorem 8.20.

If $m=1$ in Theorem 8.20, we have

Corollary 8.21. *A weighted generating function for reverse plane partitions of square shape is*

$$\sum_{T\in \mathrm{RPP}(n^n)} q^{|T|+a\cdot \mathrm{tr}(T)}(q^{v_1+1},\dots,q^{v_n+1};q)_b = \frac{1}{(1-q)^{n^2}}\prod_{j=1}^n \frac{[a+j-1]_q![b+j-1]_q!}{[a+b+n+j-1]_q![j-1]_q!}.$$

where the reverse diagonal entries of T are $(\mu_1,\dots,\mu_n)$, and $v_i=\mu_i+n-i$, $1\le i\le n$.

If $b = 0$ in Corollary 8.21, we have the trace generating function for square shapes, see [21]. If also $a = 0$, we have the generating function for reverse plane partitions of square shape.

Kamioka [15] has another weighted generating function for reverse plane partitions of a square shape.

9. q-Ehrhart polynomials

In this section, using q-integrals, we study q-Ehrhart polynomials and q-Ehrhart series of order polytopes with some faces removed. We refer the reader to [7] for the more details in q-Ehrhart polynomials of lattice polytopes. This section was initiated by discussions with Josuat-Vergès.

Throughout this section we assume that P is a poset on $\{x_1, \dots, x_n\}$ and ω_n is the bijection from $\{x_1, \dots, x_n\}$ to $[n]$ given by $\omega_n(x_i) = i$ for $1 \le i \le n$.

We denote by $\overline{P}$ the dual of P, i.e., the poset obtained by reversing the orders in P.

Definition 9.1. For a bounded set X of points in $\mathbb{R}^n$ and a positive integer m, we define the *q-Ehrhart function*

$$E_q(X, m) = \sum_{(x_1,\dots,x_n)\in mX\cap\mathbb{Z}^n} q^{x_1+\cdots+x_n}.$$

Definition 9.2. Let P be a poset on $\{x_1, \dots, x_n\}$. We define $\Delta(P)$ to be the set of points $(x_1, \dots, x_n) \in [0, 1]^n$ such that

- $x_i \ge x_j$ if $x_i \le_P x_j$,
- $x_i > x_j$ if $x_i \le_P x_j$ and $i > j$.

Note that $\Delta(P)$ is obtained from the order polytope $\mathcal{O}(\overline{P})$ by removing some faces. In general $\Delta(P)$ is not a polytope. However, if (P, ω_n) is naturally labeled, then $\Delta(P) = \mathcal{O}(\overline{P})$. Therefore, by adding the assumption that (P, ω_n) is naturally labeled, every result in this section has a corollary stated in terms of the order polytope $\mathcal{O}(\overline{P})$. For instance, see Corollary 9.4.

Now we show some properties of the q-Ehrhart function of $\Delta(P)$: it is represented as a q-integral of a truncated order polytope of P and it is a polynomial in $[m]_q$ whose leading coefficient is the q-volume of the order polytope of P.

Theorem 9.3. *Let P be a poset on $\{x_1, \dots, x_n\}$. Then, for an integer $m \ge 0$, we have*

$$E_q(\Delta(P), m) = \frac{1}{(1-q)^n} V_q(\mathcal{O}_{[q^{m+1},1]^n}(P)),$$

and, equivalently,

$$E_q(\Delta(P), m) = \sum_{\pi\in\mathcal{L}(P,\omega_n)} q^{\mathrm{maj}(\pi)} \begin{bmatrix} m+n-\mathrm{des}(\pi) \\ n \end{bmatrix}_q.$$

Moreover, the q-Ehrhart function $E_q(\Delta(P), m)$ is a polynomial in $[m]_q$ whose coefficients are rational functions in q and whose leading coefficient is the q-volume $V_q(\mathcal{O}(P))$.

Proof. By definition, we have

$$E_q(\Delta(P), m) = \sum_{\sigma} q^{|\sigma|},$$

where the sum is over all (P, ω_n)-partitions σ with $\max(\sigma) \le m$. By Theorem 4.1, this is equal to $\frac{1}{(1-q)^n} V_q(\mathcal{O}_{[q^{m+1},1]^n}(P))$.

By Corollary 4.2, we have

$$V_q(\mathcal{O}_{[q^{m+1},1]^n}(P)) = \sum_{\pi\in\mathcal{L}(P,\omega_n)} V_q(\mathcal{O}_{[q^{m+1},1]^n}(P_\pi)).$$

By Corollary 4.6, we have

$$V_q(\mathcal{O}_{[q^{m+1},1]^n}(P_\pi)) = (1-q)^n q^{\mathrm{maj}(\pi)} \begin{bmatrix} m+n-\mathrm{des}(\pi) \\ n \end{bmatrix}_q,$$

which is a polynomial in $[m]_q$ whose coefficients are rational functions of q. Since $E_q(\Delta(P),m)$ is a sum of these q-volumes divided by $(1-q)^n$, it is also a polynomial in $[m]_q$. The leading coefficient of $E_q(\Delta(P),m)$ as a polynomial in $[m]_q$ is

$$\lim_{m\to\infty}\frac{E_q(\Delta(P),m)}{[m]_q^n}=\lim_{m\to\infty}\frac{1}{(1-q^m)^n}V_q(\mathcal{O}_{[q^{m+1},1]^n}(P))=V_q(\mathcal{O}(P)).$$

□

If we add the assumption that (P,ω_n) is naturally labeled in Theorem 9.3 then we obtain a result on the order polytope $\mathcal{O}(\overline{P})$.

Corollary 9.4. *Let P be a poset on $\{x_1,\dots,x_n\}$. Suppose that (P,ω_n) is naturally labeled. Then, for an integer $m\ge 0$, we have*

$$E_q(\mathcal{O}(\overline{P}),m)=\frac{1}{(1-q)^n}V_q(\mathcal{O}_{[q^{m+1},1]^n}(P)).$$

Moreover, the q-Ehrhart function $E_q(\mathcal{O}(\overline{P}),m)$ is a polynomial in $[m]_q$ whose coefficients are rational functions in q and whose leading coefficient is the q-volume $V_q(\mathcal{O}(P))$.

We note that $E_q(X,m)$ is not always a polynomial in $[m]_q$. For example, if $n=1$ and $X=\{1/k\}$ then $E_q(X,m)=q^{m/k}$ if m is divisible by k and $E_q(X,m)=0$ otherwise.

Remark 9.5. In [7], Chapoton defines the q-volume of a poset P on $\{x_1,\dots,x_n\}$ to be the leading coefficient of the q-Ehrhart polynomial $E_q(\mathcal{O}(P),m)$ times $[n]_q!$. If $(\overline{P},\omega_n)$ is naturally labeled, we have $\Delta(\overline{P})=\mathcal{O}(P)$. By Corollary 9.4, the leading coefficient of the q-Ehrhart polynomial $E_q(\mathcal{O}(P),m)$ is the q-volume $V_q(\mathcal{O}(\overline{P}))$. Thus, Chapoton's q-volume of P is equal to $[n]_q!V_q(\mathcal{O}(\overline{P}))$.

Next we consider the q-Ehrhart series.

Definition 9.6. For a set X of points in $\mathbb{R}^n$, we define the *q-Ehrhart series* of X by

$$E_q^*(X,t)=\sum_{m\ge 0}E_q(X,m)t^m.$$

We now show that the q-Ehrhart series of $\Delta(P)$ is a generating function for the linear extensions of P.

Corollary 9.7. *For a poset P on $\{x_1,\dots,x_n\}$, we have*

$$E_q^*(\Delta(P),t)=\frac{1}{(t;q)_{n+1}}\sum_{\pi\in\mathcal{L}(P,\omega_n)}t^{\operatorname{des}(\pi)}q^{\operatorname{maj}(\pi)}.$$

Proof. By Theorem 9.3, we have

$$E_q^*(\Delta(P),t)=\sum_{m\ge 0}t^m\sum_{\pi\in\mathcal{L}(P,\omega_n)}q^{\operatorname{maj}(\pi)}\begin{bmatrix}m+n-\operatorname{des}(\pi)\\ n\end{bmatrix}_q.$$

Thus, it suffices to show that

$$\sum_{m\ge 0}t^m\begin{bmatrix}m+n-\operatorname{des}(\pi)\\ n\end{bmatrix}_q=\frac{t^{\operatorname{des}(\pi)}}{(t;q)_{n+1}}. \tag{27}$$

The summand in the left hand side of (27) is zero unless $m\ge\operatorname{des}(\pi)$. By shifting the index m by $m+\operatorname{des}(\pi)$, the left hand side of (27) becomes

$$t^{\operatorname{des}(\pi)}\sum_{m\ge 0}t^m\begin{bmatrix}m+n\\ n\end{bmatrix}_q=t^{\operatorname{des}(\pi)}\cdot\frac{1}{(t;q)_{n+1}},$$

where the q-binomial theorem is used. This completes the proof. □

Let us consider the special case of Corollary 9.7 when P is the anti-chain on $\{x_1,\dots,x_n\}$. In this case we have $\Delta(P)=[0,1]^n$ and $\mathcal{L}(P,\omega_n)=S_n$. Since $E_q([0,1]^n,m)=(1+q+\cdots+q^m)^n$, Corollary 9.7 reduces to MacMahon's identity

$$\sum_{\pi\in S_n} t^{\operatorname{des}(\pi)}q^{\operatorname{maj}(\pi)}=\sum_{m\ge 0} t^m(1+q+\cdots+q^m)^n.$$

We note that Beck and Braun [4] also gave a proof this identity by decomposing $[0,1]^n$ into simplices.

We finish this section by showing that the q-Ehrhart series of $\Delta(P)$ has a q-integral representation. Note that if $0<t<1$ and $0<q<1$, then $\log_q t>0$.

Corollary 9.8. *Let $0<t<1$ and $0<q<1$. For a poset P on $\{x_1,\dots,x_n\}$, we have*

$$E_q^*(\Delta(P),t)=\frac{1}{(1-q)^{n+1}}\int_{\mathcal{O}(P')} x_0^{\log_q t-1}d_qx_0d_qx_1\cdots d_qx_n.$$

where P' is the poset obtained from P by adding a new element x_0 which is smaller than all elements in P.

Proof. By Theorems 9.3 and 4.1,

$$E_q^*(\Delta(P),t)=\sum_{m\ge 0} t^m\cdot\frac{1}{(1-q)^n}V_q(\mathcal{O}_{[q^{m+1},1]^n}(P))=\sum_{m\ge 0}t^m\sum_{\sigma}q^{|\sigma|},$$

where the second sum in the rightmost side is over all (P,ω_n)-partitions σ with $\max(\sigma)\le m$. This can be rewritten as

$$\sum_{\sigma'} t^{\sigma'(x_0)}q^{|\sigma'|-\sigma'(x_0)}=\sum_{\sigma'}(q^{\sigma'(x_0)})^{\log_q t-1}q^{|\sigma'|},$$

where the sum is over all (P',ω')-partitions σ' for the labeling ω' of P' given by $\omega'(x_i)=i$ for $0\le i\le n$. By applying Theorem 4.1 again, we obtain the desired identity. $\square$

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