THE COMBINATORICS OF AL-SALAM-CHIHARA q-LAGUERRE POLYNOMIALS

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Abstract. We decribe various aspects of the Al-Salam-Chihara q-Laguerre polynomials. These include combinatorial descriptions of the polynomials, the moments, the orthogonality relation and a combinatorial interpretation of the linearization coefficients. It is remarkable that the corresponding moment sequence appears also in the recent work of Postnikov and Williams on enumeration of totally positive Grassmann cells.

Contents

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1. Introduction

The monic simple Laguerre polynomials $L_n(x)$ may be defined by the explicit formula:

$$L_n(x) = \sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k!} \binom{n}{k} x^k, \tag{1}$$

or by the three-term recurrence relation

$$L_{n+1}(x) = (x - (2n+1))L_n(x) - n^2 L_{n-1}(x). \tag{2}$$

The moments are

$$\mu_n = \mathcal{L}(x^n) = \int_0^\infty x^n e^{-x} dx = n!. \tag{3}$$

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The linearization formula reads as follows:

$$L_{n_1}(x)L_{n_2}(x) = \sum_{n_3} C_{n_1\,n_2}^{n_3} L_{n_3}(x),$$

where

$$C_{n_1\,n_2}^{n_3} = \sum_{s\geq 0} \frac{n_1!\,n_2!\,2^{N_2+n_3-2s}\,s!}{(s-n_1)!(s-n_2)!(s-n_3)!(N_2+n_3-2s)!n_3!}.$$

Equivalently we have

$$\mathcal{L}(L_{n_1}(x)L_{n_2}(x)L_{n_3}(x)) = \sum_{s\geq 0} \frac{n_1!\,n_2!\,n_3!\,2^{N_2+n_3-2s}\,s!}{(s-n_1)!(s-n_2)!(s-n_3)!(N_2+n_3-2s)!}. \tag{4}$$

Given positive integers $n_1, n_2, \ldots, n_k$ such that $n = n_1 + \cdots + n_k$, let S_i be the consecutive integer segment $\{n_1 + \cdots n_{i-1} + 1, \ldots, n_1 + \cdots + n_i\}$ with $n_0 = 0$, then $S_1 \cup \ldots \cup S_k = [n]$. A permutation σ of $[n]$ is said to be a *generalized derangement* of specification $(n_1, \ldots, n_k)$ if i and $\sigma(i)$ do not belong to a same segment S_j for all $i \in [n]$. Let $\mathcal{D}(n_1, n_2, \ldots, n_k)$ be the set of generalized derangements of specification $(n_1, \ldots, n_k)$ then we have

$$\mathcal{L}(L_{n_1}(x)\ldots L_{n_k}(x)) = \sum_{\sigma\in\mathcal{D}(n_1,n_2,\ldots,n_k)} 1. \tag{5}$$

A q-version of (1) was studied by Garsia and Remmel [9] in 1980. Several q-analogues of the recurrence relation (2) and moments (3) were investigated in the last two decades (see [2,18,19]) in order to obtain new *mahonian* statistics on the symmetric groups. On the other hand, in view of the unified combinatorial interpretations of several aspects of Sheffer orthogonal polynomials (moments, polynomials, and the linearization coefficients)(see [14, 20, 22]) it is natural to seek for a q-version of this picture.

As one can expect, the first result in this direction was the linearization formula for q-Hermite polynomials due to Ismail, Stanton and Viennot [12], dated back to 1987. In particular, their formula provides a combinatorial evaluation of the Askey-Wilson integral. However, a similar formula for q-Charlier polynomials was discovered only recently by Anshelevich [1], who used the machinery of q-Levy stochastic processes. Short later, Kim, Stanton and Zeng [15] gave a combinatorial proof of Anshelevich's result.

The object of this paper is to give a q-version of all the above formulas for simple Laguerre polynomials. It is interesting to note that the corresponding moment sequence appears in the recent work on enumeration of totally positive Grassmann cells [3, 21].

The rest of this paper is organized as follows: We recall the definition of Al-Salam-Chihara polynomials, prove their linearization formula introduce the new q-Laguerre polynomials in Section 2. In Section 3 we study the moment sequence of the q-Laguerre polynomials. In particular we shall give a new proof of Williams' formula for the corresponding moment sequence. We derive then the linearization coefficients of our q-Laguerre polynomials in Section 4. Finally two technical lemmas will be proved in Sections 5 and 6, respectively.

2. Al-Salam-Chihara polynomials revisited

The Al-Salam-Chihara polynomials $Q_n(x) := Q_n(x;\alpha,\beta|q)$ may be defined by the recurrence relation [16, Chapter 3]:

$$\begin{cases} Q_0(x) = 1, \quad Q_{-1}(x) = 0, \\ Q_{n+1}(x) = (2x-(\alpha+\beta)q^n)Q_n(x) - (1-q^n)(1-\alpha\beta q^{n-1})Q_{n-1}(x), \quad n \geq 0. \end{cases} \tag{6}$$

Let $Q_n(x) = 2^n p_n(x)$ then

$$xp_n(x) = p_{n+1}(x) + \frac{1}{2}(\alpha+\beta)q^n p_n(x) + \frac{1}{4}(1-q^n)(1-\alpha\beta q^{n-1})p_{n-1}(x). \tag{7}$$

They also have the following explicit expressions:

$$\begin{aligned} Q_n(x;\alpha,\beta|q) &= \frac{(\alpha\beta;\, q)_n}{\alpha^n}\, {}_3\phi_2\left(\begin{matrix} q^{-n}, \alpha u, \alpha u^{-1} \\ \alpha\beta, 0 \end{matrix} \,\middle|\, q;\, q\right) \\ &= (\alpha u;\, q)_n u^{-n}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, \beta u^{-1} \\ \alpha^{-1}q^{-n+1}u^{-1} \end{matrix} \,\middle|\, q; \alpha^{-1}qu\right) \\ &= (\beta u^{-1};\, q)_n u^{n}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, \alpha u \\ \beta^{-1}q^{-n+1}u \end{matrix} \,\middle|\, q; \beta^{-1}qu^{-1}\right), \end{aligned}$$

where $x = \frac{u+u^{-1}}{2}$ or $x = \cos\theta$ if $u = e^{i\theta}$.

The Al-Salam-Chihara polynomials have the following generating function

$$G(t,x) = \sum_{n=0}^{\infty} Q_n(x;\alpha,\beta|q)\frac{t^n}{(q;\, q)_n} = \frac{(\alpha t, \beta t;\, q)_\infty}{(te^{i\theta}, te^{-i\theta};\, q)_\infty}.$$

They are orthogonal with respect to the linear functional $\hat{\mathcal{L}}_q$:

$$\hat{\mathcal{L}}_q(x^n) = \frac{1}{2\pi}\int_0^{\pi} (\cos\theta)^n \frac{(q, \alpha\beta, e^{2i\theta}, e^{-2i\theta};\, q)_\infty}{(\alpha e^{i\theta}, \alpha e^{-i\theta}, \beta e^{i\theta}, \beta e^{-i\theta};\, q)_\infty} d\theta, \tag{8}$$

where $x = \cos\theta$. Note that

$$\hat{\mathcal{L}}_q(Q_n(x)^2) = (q;\, q)_n(\alpha\beta;\, q)_n.$$

Theorem 1. . *We have*

$$Q_{n_1}(x)Q_{n_2}(x) = \sum_{n_3\geq 0} C_{n_1,n_2}^{n_3}(\alpha,\beta;\, q)Q_{n_3}(x), \tag{9}$$

where

$$\begin{aligned} C_{n_1,n_2}^{n_3}(\alpha,\beta;\, q) &= (-1)^{N_2+n_3}\frac{(q;\, q)_{n_1}(q;\, q)_{n_2}}{(\alpha\beta;\, q)_{n_3}} \\ &\times \sum_{m_2,m_3} \frac{(\alpha\beta;\, q)_{n_1+m_3}\alpha^{m_2}\beta^{n_3+n_2-n_1-m_2-2m_3}q^{\binom{m_2}{2}+\binom{n_3+n_2-n_1-m_2-2m_3}{2}}}{(q;\, q)_{n_3+n_2-n_1-m_2-2m_3}(q;\, q)_{m_2}(q;\, q)_{m_3+n_1-n_3}(q;\, q)_{m_3+n_1-n_2}(q;\, q)_{m_3}}. \end{aligned}$$

Proof. Clearly $C^{n_3}_{n_1,n_2}(\alpha,\beta;\,q) = \hat{\mathcal{L}}_q(Q_{n_1}(x)Q_{n_2}(x)Q_{n_3}(x))/\hat{\mathcal{L}}_q(Q_{n_3}(x)Q_{n_3}(x))$. Using the Askey-Wilson integral:

$$\frac{(q;\,q)_\infty}{2\pi}\int_0^\pi \frac{(e^{2i\theta},e^{-2i\theta};\,q)_\infty}{\prod_{j=1}^4(t_je^{i\theta},t_je^{-i\theta};\,q)_\infty}d\theta = \frac{(t_1t_2t_3t_4;\,q)_\infty}{\prod_{1\le j<k\le 4}(t_jt_k;\,q)_\infty},$$

one can prove [12, Theorem 3.5] that

$$\begin{aligned}&\hat{\mathcal{L}}_q(G(t_1,x)G(t_2,x)G(t_3,x))\\ &\qquad = \frac{(\alpha t_1t_2t_3,\beta qt_1t_2t_3,\alpha\beta q;\,q)_\infty}{(t_1t_2,t_1t_3,t_2t_3;\,q)_\infty}\,{}_3\phi_2\left(\begin{matrix} t_1t_2, & t_1t_3, & t_2t_3\\ & \alpha t_1t_2t_3, & \beta t_1t_2t_3\end{matrix}\,|q;\alpha\beta\right).\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{n_1,n_2,n_3}\hat{\mathcal{L}}_q(Q_{n_1}(x)Q_{n_2}(x)Q_{n_3}(x))\frac{t_1^{n_1}}{(q;\,q)_{n_1}}\frac{t_2^{n_2}}{(q;\,q)_{n_2}}\frac{t_3^{n_3}}{(q;\,q)_{n_3}}\\ &\qquad = \sum_{k\ge 0}\frac{(\alpha t_1t_2t_3q^k,\,\beta t_1t_2t_3q^k,\,\alpha\beta;q)_\infty}{(t_1t_2q^k,\,t_1t_3q^k,\,t_2t_3q^k;q)_\infty}\frac{(\alpha\beta)^k}{(q;\,q)_k}.\end{aligned}\tag{10}$$

Using the Euler formulas:

$$(t;\,q)_\infty = \sum_{n\ge 0}\frac{(-1)^nq^{\binom{n}{2}}}{(q;\,q)_n}t^n;\qquad \frac{1}{(t;\,q)_\infty} = \sum_{n\ge 0}\frac{1}{(q;\,q)_n}t^n,$$

we can rewrite the sum in (10) as follows:

$$\begin{aligned}(\alpha\beta;\,q)_\infty\sum_{k\ge 0}\frac{(\alpha\beta)^k}{(q;\,q)_k}&\sum_{l_1,l_2\ge 0}\frac{\alpha^{l_1}\beta^{l_2}q^{k(l_1+l_2)}(-t_1t_2t_3)^{l_1+l_2}q^{\binom{l_1}{2}+\binom{l_2}{2}}}{(q;\,q)_{l_1}(q;\,q)_{l_2}}\\ &\times\sum_{m_1,m_2,m_3\ge 0}\frac{q^{(m_1+m_2+m_3)k}t_1^{m_1+m_2}t_2^{m_1+m_3}t_3^{m_1+m_3}}{(q;\,q)_{m_1}(q;\,q)_{m_2}(q;\,q)_{m_3}}.\end{aligned}\tag{11}$$

Substituting

$$\sum_{k\ge 0}\frac{(\alpha\beta q^{l_1+l_2+m_1+m_2+m_3})^k}{(q;\,q)_k} = \frac{1}{(\alpha\beta q^{l_1+l_2+m_1+m_2+m_3};q)_\infty}$$

in (11), we get

$$\sum_{l_1,l_2,m_1,m_2,m_3}t_1^{n_1}t_2^{n_2}t_3^{n_3}\frac{(\alpha\beta)_{n_1+m_3}\alpha^{l_1}\beta^{l_2}q^{\binom{l_1}{2}+\binom{l_2}{2}}}{(q;\,q)_{m_1}(q;\,q)_{m_2}(q;\,q)_{m_3}(q;\,q)_{l_1}(q;\,q)_{l_2}}(-1)^{l_1+l_2},\tag{12}$$

where $l_1+l_2+m_1+m_2=n_1$, $l_1+l_2+m_1+m_3=n_2$ and $l_1+l_2+m_2+m_3=n_3$.

Since $l_1+l_2\equiv N_2+n_3 \pmod 2$, extracting the coefficient of $\frac{t_1^{n_1}t_2^{n_2}t_3^{n_3}}{(q;q)_{n_1}(q;q)_{n_2}(q;q)_{n_3}}$ in (12) and dividing by $(q,\alpha\beta;\,q)_{n_3}$ we obtain (9) where l_1 is replaced by m_2. □

We define the new q-Laguerre polynomials $L_n(x;\,q)$ by re-scaling Al-Salam-Chihara polynomials:

$$L_n(x;\,q) = \left(\frac{\sqrt{y}}{q-1}\right)^n Q_n\left(\frac{(q-1)x+y+1}{2\sqrt{y}};\,\frac{1}{\sqrt{y}},\sqrt{y}q|q\right).\tag{13}$$

It follows from (7) that the polynomials $L_n(x;\, q)$ satisfy the recurrence:

$$L_{n+1}(x;\, q) = (x - y[n+1]_q - [n]_q)L_n(x;\, q) - y[n]_q^2 L_{n-1}(x;\, q). \tag{14}$$

We derive then the explicit formula for $L_n(x)$:

$$L_n(x;\, q) = \sum_{k=0}^{n} (-1)^{n-k} \frac{n!_q}{k!_q} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} y^{n-k} \prod_{j=0}^{k-1} \left(x - (1 - yq^{-j})[j]_q \right). \tag{15}$$

Thus

$$\begin{aligned} L_1(x;\, q) &= x - y, \\ L_2(x;\, q) &= x^2 - (1 + 2y + qy)x + (1+q)y^2, \\ L_3(x;\, q) &= x^3 - (q^2 y + 3\,y + q + 2 + 2\,qy)x^2 \\ &\qquad + (q^3 y^2 + yq^2 + q + 2\,qy + 3\,q^2 y^2 + 1 + 4\,qy^2 + 2\,y + 3\,y^2)x \\ &\qquad\quad - (2\,q^2 + 2\,q + q^3 + 1)y^3. \end{aligned}$$

A combinatorial interpretation of these q-Laguerres polynomials can be derived from the Simion and Stanton's combinatorial model for the $a = s = u = 1$ and $r = t = q$ special case of the quadrabasic Laguerre polynomials [19, p.313].

3. Moments of the q-Laguerre polynomials

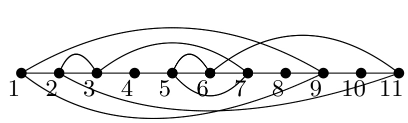
Let $\mathcal{S}_n$ be the set of permutations of $[n] := \{1, 2, \ldots, n\}$. For $\sigma \in \mathcal{S}_n$ the *number of crossings* of σ is defined by

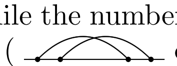
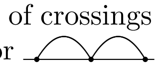
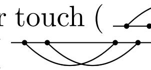
$$cr(\sigma) = \sum_{i=1}^{n} \#\{j | j < i \le \sigma(j) < \sigma(i)\} + \sum_{i=1}^{n} \#\{j | j > i > \sigma(j) > \sigma(i)\},$$

while the number of *weak excedances* of σ is defined by

$$wex(\sigma) = \#\{i | 1 \le i \le n \text{ and } i \le \sigma(i)\}.$$

It is useful to have a geometric interpretation of these statistics by associating with each permutation σ of $[n]$ a diagram as follows: arrange the integers $1, 2, \ldots, n$ on a line in increasing order from left to right and draw an arc $i \to \sigma(i)$ above (resp. under) the line if $i < \sigma(i)$ (resp. $i > \sigma(i)$). For example, the permutation $\sigma = 9\,3\,7\,4\,6\,11\,5\,8\,1\,10\,2$ can be depicted as follows:



Thus, the number of weak excedances of σ is the number of edges drawn above the line plus the number of isolated points, while the number of crossings of σ is the number of pairs of edges above the line that cross or touch ( or ), plus the number of pairs of edges under the line that cross ().

Let $\mu_n^{(\ell)}(y,q)$ be the enumerating polynomial of permutations in $\mathcal{S}_n$ with respect to numbers of weak excedances and crossings:

$$\mu_n^{(\ell)}(y,q) := \sum_{\sigma\in S_n} y^{wex(\sigma)} q^{cr(\sigma)}. \tag{16}$$

It has been proved in [3, 17, 19] that the generating function of the moment sequence has the following continued fraction expansion:

$$E(y,q,t) := \sum_{n\geq 0} \mu_n^{(\ell)}(y,q)t^n = \cfrac{1}{1-b_0t-\cfrac{\lambda_1 t^2}{1-b_1t-\cfrac{\lambda_2t^2}{\ddots}}}, \tag{17}$$

where $b_n = y[n+1]_q + [n]_q$ and $\lambda_n = y[n]_q^2$.

We derive then from the classical theory of orthogonal polynomials the following interpretation for the moments of the q-Laguerre polynomials.

Theorem 2. *The n-th moment of the q-Laguerre polynomials is equal to $\mu_n^{(\ell)}(y,q)$. More precisely, let $\mathcal{L}_q$ be the linear functional defined by $\mathcal{L}_q(x^n) = \mu_n^{(\ell)}(y,q)$, then*

$$\mathcal{L}_q(L_{n_1}(x;\,q)L_{n_2}(x;\,q)) = y^{n_1}(n_1!_q)^2\delta_{n_1\,n_2}. \tag{18}$$

The first values of the moment sequence are as follows:

$$\begin{aligned}
\mu_1^{(\ell)}(y,q) &= y,\\
\mu_2^{(\ell)}(y,q) &= y+y^2,\\
\mu_3^{(\ell)}(y,q) &= y+(3+q)y^2+y^3,\\
\mu_4^{(\ell)}(y,q) &= y+(6+4q+q^2)y^2+(6+4q+q^2)y^3+y^4.
\end{aligned}$$

Combining the results of Corteel [3], Williams [21, Proposition 4.11] and the classical theory of orthogonal polynomials, one can write the moments of the above q-Laguerre polynomials as a finite double sum (cf. (33)). Here we propose a direct proof of this result. Actually we shall give such a formula for the moments of Al-Salam-Chihara polynomials.

Definition 3. *Define the y-versions of the q-Stirling numbers of the second kind by*

$$X^n = \sum_{k=1}^{n} S_q(n,k,y)\prod_{j=0}^{k-1}(X-[j]_q(1-yq^{-j})). \tag{19}$$

The y-versions of q-Stirling numbers of the first kind can be defined by the inverse matrix or equivalently

$$\prod_{j=0}^{n-1}(X-[j]_q(1-yq^{-j})) = \sum_{k=1}^{n} s_q(n,k,y)X^k.$$

Remark 1. *We have*

$$S_q(n,k,y)|_{q=1} = S(n,k)(1-y)^{n-k}, \quad S_q(n,k,0) = S_q(n,k),$$

where $S(n,k)$ and $S_q(n,k)$ are, respectively, the Stirling numbers of the second kind and their well-known q-analogues, see [11].

Consider the rescaled Al-Salam-Chihara polynomials $P_n(x)$:

$$\begin{aligned}P_n(X) &= Q_n(((q-1)X+1/\alpha^2+1)\alpha/2;\ \alpha,\beta|q)\\ &= \alpha^{-n}\sum_{k=0}^{n}\frac{(q^{-n};\ q)_k}{(q;\ q)_k}q^k(\alpha\beta q^k;\ q)_{n-k}(1-q)^k q^{\binom{k}{2}}\alpha^{2k}\\ &\qquad\times\prod_{j=0}^{k-1}\left(X-[i]_q(1-q^{-i}/\alpha^2)\right).\end{aligned}\tag{20}$$

Lemma 1. *The moments of the rescaled Al-Salam-Chihara polynomials $P_n(X)$ are*

$$\mu_n(\alpha,\beta)=\sum_{k=1}^{n}S_q(n,k,1/\alpha^2)(\alpha\beta;\ q)_k q^{-\binom{k}{2}}(1-q)^{-k}\alpha^{-2k}.\tag{21}$$

Proof. Let $L: X^n\mapsto\mu_n(\alpha,\beta)$ be the linear functional. We check that these moments do satisfy $L(P_n(X))=0$ for $n>0$. Let a_k be the coefficients in front of the product in (20), then we have, using y-Stirling orthogonality,

$$\begin{aligned}L(P_n(X)) &= \sum_{k=0}^{n}a_k\sum_{j=1}^{k}s_q(k,j,1/\alpha^2)\sum_{t=1}^{j}S_q(j,t,1/\alpha^2)(\alpha\beta;\ q)_t q^{-\binom{t}{2}}(1-q)^{-t}\alpha^{-2t}\\ &= \sum_{k=0}^{n}a_k(\alpha\beta;\ q)_k q^{-\binom{k}{2}}(1-q)^{-k}\alpha^{-2k}\\ &= \alpha^{-n}(\alpha\beta;\ q)_n\sum_{k=0}^{n}\frac{(q^{-n};\ q)_k}{(q;\ q)_k}q^k=0.\end{aligned}$$

Note that the last equality follows by applying the q-binomial formula. □

Lemma 2. *Let $p=1/q$. We have*

$$\sum_{k=0}^{\infty}\frac{(\alpha\beta;\ q)_k q^{-\binom{k}{2}}(1-q)^{-k}\alpha^{-2k}t^k}{\prod_{i=1}^{k}(1-[i]_q t(1-q^{-i}/\alpha^2))}=\sum_{i\geq 0}\frac{c_i(\alpha,\beta)}{1-[i]_q t(1-q^{-i}/\alpha^2)},\tag{22}$$

where

$$c_i(\alpha,\beta)=\frac{(\alpha\beta;q)_i}{(q;q)_i}\frac{q^{i-i^2}\alpha^{-2i}}{(q^{1-2i}/\alpha^2;q)_i}\frac{(p^{1+i}\alpha\beta/\alpha^2;p)_\infty}{(p^{1+2i}/\alpha^2;p)_\infty}.$$

Proof. Note the following partial fraction decomposition formula:

$$\frac{t^k}{(1-a_1t)(1-a_2t)\dots(1-a_kt)}=\frac{(-1)^k}{a_1\cdots a_k}+\sum_{i=1}^{k}\frac{a_i^{-1}\prod_{j=1,j\neq i}^{k}(a_i-a_j)^{-1}}{1-a_it}.$$

Therefore

$$\frac{t^k}{\prod_{i=1}^{k}(1-[i]_q t(1-q^{-i}/\alpha^2))}=\sum_{i=0}^{k}\frac{\gamma_k(i)}{1-[i]_q t(1-q^{-i}/\alpha^2)},\tag{23}$$

where

$$\gamma_k(i)=\frac{1}{k!_q}\begin{bmatrix}k\\i\end{bmatrix}_q\frac{\alpha^{2(k-i)}q^{\binom{k}{2}+k-i^2}}{(q^{1-2i}/\alpha^2;q)_i(q^{1+2i}\alpha^2;q)_{k-i}}\qquad(0\le i\le k).$$

Substituting this in (22) yields

$$\begin{aligned}c_i(\alpha,\beta)&=\sum_{k\ge i}\frac{(\alpha\beta;q)_k}{(q;q)_k}\begin{bmatrix}k\\i\end{bmatrix}_q\frac{q^{k-i^2}\alpha^{-2i}}{(q^{1-2i}/\alpha^2;q)_i(q^{1+2i}\alpha^2;q)_{k-i}}\\&=\frac{(\alpha\beta;q)_i}{(q;q)_i}\frac{q^{i-i^2}\alpha^{-2i}}{(q^{1-2i}/\alpha^2;q)_i}\sum_{k\ge0}\frac{(\alpha\beta q^i;q)_k}{(q;q)_k}\frac{q^k}{(q^{1+2i}\alpha^2;q)_k}.\end{aligned}$$

The result follows then by applying the ${}_1\Phi_1$ summation formula (see [10, II.5]). ☐

Theorem 4. *The moments $\mu_n(\alpha,\beta)$ have the explicit formula*

$$\mu_n(\alpha,\beta)=\sum_{k=1}^{n}\sum_{i=1}^{k}\begin{bmatrix}k\\i\end{bmatrix}_q\frac{q^{k-i^2}\alpha^{-2i}}{(q;q)_k}\frac{([i]_q(1-q^{-i}/\alpha^2))^n(\alpha\beta;q)_k}{(q^{1-2i}/\alpha^2;q)_i(q^{1+2i}\alpha^2;q)_{k-i}}.\qquad(24)$$

Proof. By definition (19) we have

$$S_q(n,k,y)=S_q(n-1,k-1,y)+[k]_q(1-yq^{-k})S_q(n-1,k,y).$$

Therefore

$$\sum_{n\ge k}S_q(n,k,y)t^n=\frac{t^k}{\prod_{i=1}^k(1-[i]_qt(1-q^{-i}y))}.\qquad(25)$$

It follows from (23) and (25) that

$$S_q(n,k,y)=\frac{q^{-\binom{k}{2}}}{k!_q}\sum_{i=1}^{k}\begin{bmatrix}k\\i\end{bmatrix}_q y^{i-k}q^{k^2-i^2}\frac{([i]_q(1-q^{-i}y))^n}{(q^{1-2i}y;q)_i(q^{1+2i}/y;q)_{k-i}}.\qquad(26)$$

Substituting this into (21) yields the desired formula. ☐

By Lemma 1 and (25) we obtain the generating function for the moments $\mu_n(\alpha,\beta)$:

$$\sum_{n=0}^{\infty}\mu_n(\alpha,\beta)t^n=\sum_{k=0}^{\infty}\frac{(\alpha\beta;\,q)_kq^{-\binom{k}{2}}(1-q)^{-k}\alpha^{-2k}t^k}{\prod_{i=1}^k(1-[i]_qt(1-q^{-i}/\alpha^2))}.\qquad(27)$$

The moment of q-Charlier polynomials corresponds to the $\beta=0$, $\alpha=-1/\sqrt{a(1-q)}$ case, while that of q-Laguerre polynomials corresponds to the $\alpha=1/\sqrt{y}$, $\alpha\beta=q$ case. Therefore,

$$\sum_{n=0}^{\infty}\mu_n^{(c)}(a,q)t^n=\sum_{k=0}^{\infty}\frac{(aqt)^k}{\prod_{i=1}^k(q^i-q^i[i]_qt+a(1-q)[i]_qt)};\qquad(28)$$

$$\sum_{n=0}^{\infty}\mu_n^{(\ell)}(y,q)t^n=\sum_{k=0}^{\infty}\frac{k!_q(qty)^k}{\prod_{i=1}^k(q^i-q^i[i]_qt+[i]_qty)}.\qquad(29)$$

By Lemma 2, we obtain, setting $p = 1/q$,

$$\sum_{n=0}^{\infty} \mu_n^{(c)}(a,q)t^n = \sum_{i\geq 0} \frac{a^i q^{2i}(1-a(1-q)p^{2i})/(a(1-q)p^i;p)_\infty}{i!_q q^{i^2}(q^i - q^i[i]_q t + a[i]_q t(1-q))}, \tag{30}$$

$$\sum_{n=0}^{\infty} \mu_n^{(\ell)}(y,q)t^n = \sum_{i\geq 0} \frac{y^i(q^{2i}-y)}{q^{i^2}(q^i - q^i[i]_q t + [i]_q ty)}. \tag{31}$$

We derive then the following polynomial formulae in a and y for the corresponding moments:

$$\mu_n^{(c)}(a,q) = \sum_{k=1}^{n} a^k \sum_{l=0}^{k} \frac{[k-l]_q^n(-1)^l}{(k-l)!_q} \sum_{j=0}^{l} \frac{(1-q)^j}{(l-j)!_q} q^{\binom{l-j+1}{2}-k(k-l)} \left(\binom{n}{j}q^{k-l} + \binom{n}{j-1}\right); \tag{32}$$

$$\mu_n^{(\ell)}(y,q) = \sum_{k=1}^{n} y^k \sum_{i=0}^{k-1} (-1)^i [k-i]_q^n q^{k(i-k)} \left(\binom{n}{i}q^{k-i} + \binom{n}{i-1}\right). \tag{33}$$

Note that (32) is simpler than the formula given in [15, Proposition 5].

4. Linearization coefficients of the q-Laguerre polynomials

Define the linearization coefficients of the q-Laguerre polynomials by

$$I(n_1,\ldots,n_k) = \mathcal{L}_q(L_{n_1}(x;q)\ldots L_{n_k}(x;q)) \quad (k\geq 1, n_1,\ldots,n_k \geq 0).$$

The following is our main result of this section.

Theorem 5. *We have*

$$I(n_1,\ldots,n_k) = \sum_{\sigma\in\mathcal{D}(n_1,\ldots,n_k)} y^{wex(\sigma)} q^{cr(\sigma)}. \tag{34}$$

For brevity, if $n_1 = \ldots = n_k = 1$, we shall write $(1^k) := (n_1,\ldots,n_k)$ and $\mathcal{D}_k := \mathcal{D}(1^k)$. Hence $\mathcal{D}_n$ is just the set of usual derangements of $[n]$. Define also

$$d_n(y,q) = \sum_{\sigma\in\mathcal{D}_n} y^{wex(\sigma)} q^{cr(\sigma)}.$$

A proof à la Viennot (cf. [12,15]) of (34) would use the combinatorial interpretations for the moments and q-Laguerre polynomials to rewrite the left-hand side of (34) and then construct an adequate *killing involution* on the resulting set. For the time being we do not have such a proof to offer, instead we provide an inductive proof.

Since $L_1(x;q) = x - y$, writing (14) as

$$L_1(x;q)L_n(x;q) = L_{n+1}(x;q) + (yq+1)[n]_q L_n(x;q) + y[n]_q^2 L_{n-1}(x;q),$$

we see immediately that

$$\begin{aligned} I(1,n,n_1,\ldots,n_k) &= I(n+1,n_1,\ldots,n_k) \\ &\quad + (yq+1)[n]_q\, I(n,n_1,\ldots,n_k) + y[n]_q^2\, I(n-1,n_1,\ldots,n_k). \end{aligned} \tag{35}$$

Therefore, the sequence $(I(n_1,\ldots,n_k))$ $(k\geq 1, n_1,\ldots,n_k\geq 0)$ is completely determined by the recurrence relation (35) and the following items:

(i) the special values $I(1^k)$ for all $k \geq 1$,

(ii) the symmetry of $I(n_1, \dots, n_k)$ with respect to the indices $n_1, \dots, n_k$.

Our proof of Theorem 5 will consist in verifying that the right-hand side of (34) has the same special values at (1^k) as the right-hand side, is invariant by rearrangement of the indices and satisfies the same recurrence relation.

Lemma 3. *We have $I(1^n) = d_n(y, q)$ for all $n \geq 1$.*

Proof. Since $L_1(x; q) = x - y$, by definition,

$$I(1^n) = \mathcal{L}_q((x-y)^n) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} y^{n-k} \mu_k^{(\ell)}(y, q).$$

By binomial inversion and (16), it suffices to prove that

$$\sum_{\sigma \in S_n} y^{wex(\sigma)} q^{cr(\sigma)} = \sum_{k=0}^{n} \binom{n}{k} y^k d_{n-k}(y, q).$$

But the latter identity is obvious. □

Since the two cyclic permutations $(1, 2)$ and $(1, 2, 3, \dots, k)$ generate the symmetric group $\mathcal{S}_k$, the invariance of $\sum_{\sigma \in \mathcal{D}(n_1, n_2, \dots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)}$ by permuting the $n_i' s$ will be a consequence of the following two special cases.

Lemma 4. *We have*

$$\sum_{\sigma \in \mathcal{D}(n_1, n_2, \dots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)} = \sum_{\sigma \in \mathcal{D}(n_2, n_3, \dots, n_k, n_1)} y^{wex(\sigma)} q^{cr(\sigma)}. \tag{36}$$

Lemma 5. *We have*

$$\sum_{\sigma \in \mathcal{D}(n_1, n_2, \dots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)} = \sum_{\sigma \in \mathcal{D}(n_2, n_1, n_3 \dots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)}. \tag{37}$$

We postpone the proof of the above two lemmas to the next two sections.

Proof of Theorem 5. By Lemmas 3, 4 and 5, it suffices to check that

$$\begin{aligned} \sum_{\sigma \in \mathcal{D}(1, n, n_1, \dots, n_k)} w(\sigma) = \sum_{\sigma \in \mathcal{D}(n+1, n_1, \dots, n_k)} w(\sigma) + (yq+1)[n]_q \sum_{\sigma \in \mathcal{D}(n, n_1, \dots, n_k)} w(\sigma) \\ + y[n]_q^2 \sum_{\sigma \in \mathcal{D}(n-1, n_1, \dots, n_k)} w(\sigma), \end{aligned} \tag{38}$$

where $w(\sigma) = y^{wex(\sigma)} q^{cr(\sigma)}$.

For derangements $\sigma \in \mathcal{D}(1, n, n_1, \dots, n_k)$ we will distinguish four cases. In each case, we shall describe a mapping to compute the corresponding enumerative polynomial. The reader is refereed to Table 1 and Table 4 in Section 5 for an illustration of these mappings in order to have a better understanding of their properties.

a) $\sigma(1), \sigma^{-1}(1) > n + 1$. We can identify such a derangement in $\mathcal{D}(1, n, n_1, \dots, n_k)$ with a derangement in $\mathcal{D}(n + 1, n_1, \dots, n_k)$. So the corresponding enumerative polynomial is

$$\sum_{\sigma \in \mathcal{D}(n+1, n_1, \dots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)}.$$

b) $\sigma(1) \in \{2, \ldots, n+1\}$ and $\sigma^{-1}(1) > n+1$. Let $\sigma(1) = \ell$. We define the mapping $\sigma \mapsto \sigma' \in \mathcal{D}(n, n_1, \ldots, n_k)$ by

$$\begin{cases} \sigma'(i) = \sigma(i+1) - 1, & \text{if } 1 \le i \le n; \\ \sigma'^{-1}(i) = \sigma^{-1}(i) - 1 & \text{if } 1 \le i \le \ell - 1; \\ \sigma'^{-1}(i) = \sigma^{-1}(i+1) - 1 & \text{if } \ell \le i \le n; \\ \sigma'(i-1) = \sigma(i) - 1 & \text{if } \sigma(i) > i > n+1; \\ \sigma'^{-1}(i-1) = \sigma^{-1}(i) - 1 & \text{if } \sigma^{-1}(i) > i > n+1; \end{cases}$$

Clearly $w(\sigma) = yq^{\ell-1}w(\sigma')$. Moreover, for each given $\ell \in \{2, \ldots, n+1\}$, the above mapping is a bijection from permutations $\sigma \in \mathcal{D}(1, n, n_1, \ldots, n_k)$ satisfying $\sigma(1) = \ell$ and $\sigma^{-1}(1) > n+1$ to permutations in $\mathcal{D}(n, n_1, \ldots, n_k)$. Summing over all $\ell = 2, \ldots, n+1$ yields the generating function:

$$qy[n]_q \sum_{\sigma \in \mathcal{D}(n, n_1, \ldots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)}.$$

c) $\sigma(1) > n+1$ and $\sigma^{-1}(1) \in \{2, \ldots, n+1\}$. Let $\sigma^{-1}(1) = \ell$. We define the mapping $\sigma \mapsto \sigma' \in \mathcal{D}(n, n_1, \ldots, n_k)$ by

$$\begin{cases} \sigma'(i) = \sigma(i) - 1, & \text{if } 1 \le i \le \ell - 1; \\ \sigma'(i) = \sigma(i+1) - 1 & \text{if } \ell \le i \le n; \\ \sigma'^{-1}(i) = \sigma^{-1}(i+1) - 1 & \text{if } 1 \le i \le n; \\ \sigma'(i-1) = \sigma(i) - 1 & \text{if } \sigma(i) > i > n+1; \\ \sigma'^{-1}(i-1) = \sigma^{-1}(i) - 1 & \text{if } \sigma^{-1}(i) > i > n+1; \end{cases}$$

Clearly $w(\sigma) = q^{\ell-2}w(\sigma')$. Moreover, for each given $\ell \in \{2, \ldots, n+1\}$, the above mapping is a bijection from permutations $\sigma \in \mathcal{D}(1, n, n_1, \ldots, n_k)$ satisfying $\sigma^{-1}(1) = \ell$ and $\sigma(1) > n+1$ to permutations in $\mathcal{D}(n, n_1, \ldots, n_k)$. Summing over all $\ell = 2, \ldots, n+1$ yields the generating function:

$$[n]_q \sum_{\sigma \in \mathcal{D}(n, n_1, \ldots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)}.$$

d) $\sigma(1), \sigma^{-1}(1) \in \{2, \ldots, n+1\}$. Let $\sigma(1) = \ell_1$ and $\sigma^{-1}(1) = \ell_2$. Then we define the mapping $\sigma \mapsto \sigma' \in \mathcal{D}(n-1, n_1, \ldots, n_k)$ by

$$\begin{cases} \sigma'(i) = \sigma(i+1) - 2, & \text{if } 1 \le i \le \ell_2 - 2; \\ \sigma'(i) = \sigma(i+2) - 2 & \text{if } \ell_2 - 1 \le i \le n-1; \\ \sigma'^{-1}(i) = \sigma^{-1}(i+1) - 2 & \text{if } 1 \le i \le \ell_1 - 2; \\ \sigma'^{-1}(i) = \sigma^{-1}(i+2) - 2 & \text{if } \ell_1 - 1 \le i \le n-1; \\ \sigma'(i-2) = \sigma(i) - 2 & \text{if } \sigma(i) > i > n+1; \\ \sigma'^{-1}(i-2) = \sigma^{-1}(i) - 2 & \text{if } \sigma^{-1}(i) > i > n+1; \end{cases}$$

Clearly $w(\sigma) = yq^{(\ell_1+\ell_2-4)}w(\sigma')$. Moreover, for each given $\ell_1, \ell_2 \in \{2, \ldots, n+1\}$, the above mapping is a bijection from permutations $\sigma \in \mathcal{D}(1, n, n_1, \ldots, n_k)$ satisfying $\sigma(1) = \ell_1$ and $\sigma^{-1}(1) = \ell_2$ to permutations in $\mathcal{D}(n-1, n_1, \ldots, n_k)$. Summing over all $\ell_1, \ell_2 \in \{2, \ldots, n+1\}$ yields the generating function:

$$y[n]_q^2 \sum_{\sigma \in \mathcal{D}(n-1, n_1, \ldots, n_k)} y^{wex(\sigma)} q^{cr(\sigma)}.$$

Summing up we obtain (38). □

When $k=2$, Theorem 5 reduces to the orthogonality of the q-Laguerre polynomials (18). When $k=3$, we can derive the following explicit formula from Theorem 1.

Theorem 6. *We have*

$$I(n_1,n_2,n_3)=\sum_s \frac{n_1!_q\, n_2!_q\, n_3!_q\, s!_q\, y^s}{(n_1+n_2+n_3-2s)!_q(s-n_3)!_q(s-n_2)!_q(s-n_1)!_q} \times \sum_k \begin{bmatrix} n_1+n_2+n_3-2s \\ k\end{bmatrix}_q y^k q^{\binom{k+1}{2}+\binom{n_1+n_2+n_3-2s-k}{2}}.$$

Proof. By Theorem 1 with $a=\frac{1}{\sqrt{y}}$ and $b=\sqrt{y}q$ we have

$$\begin{aligned} I(n_1,n_2,n_3) &= \mathcal{L}_q(L_{n_3}(x;\,q)^2)\left(\frac{\sqrt{y}}{q-1}\right)^{n_1+n_2-n_3} C^{n_3}_{n_1,n_2}(a,b;\,q)\\ &= \sum_{m_2,m_3}\frac{n_1!_q\, n_2!_q\, n_3!_q\,(n_1+m_3)!_q\, y^{n_2+n_3-m_2-m_3}\, q^{\binom{m_2}{2}+\binom{M+1}{2}}}{M!_q\, m_2!_q\,(m_3+n_1-n_3)!_q\,(m_3+n_1-n_2)!_q\, m_3!_q}, \end{aligned}$$

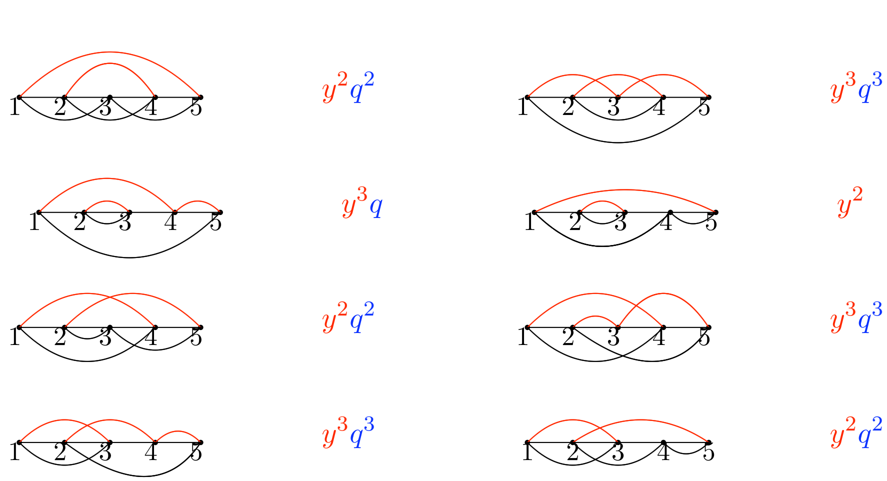
where $M=n_3+n_2-n_1-m_2-2m_3$. Substituting $s=n_1+m_3$ and $k=n_3+n_2-n_1-m_2-2m_3$ in the last sum yields the desired formula. □

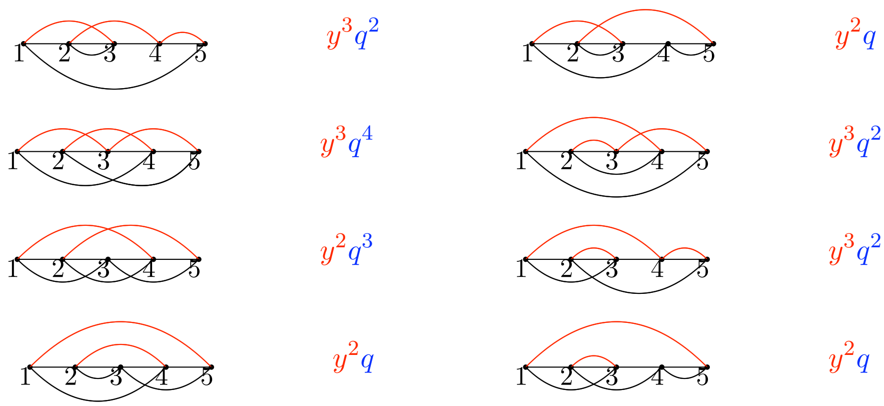
Remark 2. *It would be interesting to give a combinatorial proof of the above result as in [12,15]. When $q=1$ such a proof was given in [23].*

We end this section with an example. If $\mathbf{n}=(2,2,1)$, by Theorem 6 we have

$$\begin{aligned} I(2,2,1) &= \sum_s \frac{2!_q 2!_q 1!_q s!_q y^s}{(5-2s)!_q(s-1)!_q(s-2)!_q(s-2)!_q}\sum_{k\geq 0}\begin{bmatrix} 5-2s\\ k\end{bmatrix}_q y^k q^{\binom{k+1}{2}+\binom{5-2s-k}{2}}\\ &= (1+q)^3(1+qy)y^2. \end{aligned} \tag{39}$$

On the other hand, the sixteen generalized derangements in $\mathcal{D}(2,2,1)$, depicted by their diagrams and the corresponding weights are tabulated as follows:





Summing up we get $\sum_{\sigma\in\mathcal{D}(2,2,1)} y^{wex\,\sigma}q^{cr\,\sigma} = y^2(1+qy)(1+q)^3$, which coincides with (39).

5. Proof of Lemma 4

For each fixed $k\in[n]$ define the two subsets of $\mathcal{S}_n$:

$$
\begin{aligned}
{}^k\mathcal{S}_n &= \{\sigma\ \in S_n \,|\, \sigma(i) > k \quad \text{for } 1\le i\le k\},\\
\mathcal{S}_n^k &= \{\sigma\ \in S_n \,|\, \sigma(n+1-i) < n+1-k \quad \text{for } 1\le i\le k\}.
\end{aligned}
$$

We first define a simple bijection $\Phi_k : \sigma\mapsto\sigma'$ from ${}^k\mathcal{S}_n$ to $\mathcal{S}_n^k$ as follows: for $1\le i\le n$,

$$
\sigma'(i) = \begin{cases} \sigma(i+k)-k, & \text{if } 1\le i\le n-k \text{ and } \sigma(i+k)>k;\\ \sigma(i+k)+n-k, & \text{if } 1\le i\le n-k \text{ and } \sigma(i+k)\le k;\\ \sigma(i+k-n)-k, & \text{if } n-k+1\le i\le n. \end{cases}
$$

The map is illustrated by the diagrams of permutations in Table 1.

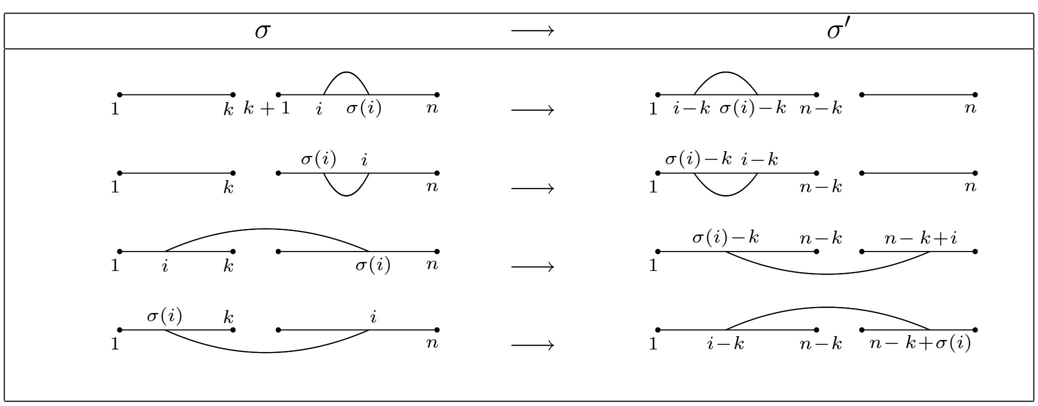
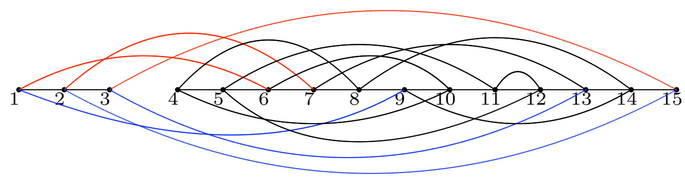
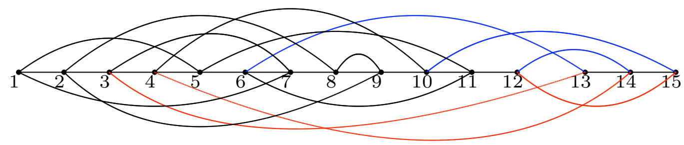


Table 1. The mapping $\Phi_k : \sigma\mapsto\sigma'$.

For example, consider the permutation $\sigma\in{}^3\mathcal{S}_{15}$, whose diagram is given below.



Then the diagram of $\Phi_3(\sigma)$ is given by



The main properties of Φ_k are summarized in following proposition.

Proposition 7. *For each positive integer $k \in [n]$, the map $\Phi_k : {}^k\mathcal{S}_n \to \mathcal{S}_n^k$ is a bijection such that for any $\sigma \in {}^k\mathcal{S}_n$ there holds*

$$(wex, cr)\Phi_k(\sigma) = (wex, cr)\sigma. \tag{40}$$

We first show how to derive Lemma 4 from Proposition 7. Let $n = n_1 + \cdots + n_k$. Then $\mathcal{D}(n_1, n_2, \ldots, n_k) \subseteq {}^{n_1}\mathcal{S}_n$. By definition of Φ_{n_1}, for any $\sigma \in {}^{n_1}\mathcal{S}_n$ and $i \in [n - n_1]$ satisfying $\sigma(i + n_1) > n_1$, we have $i - \Phi_{n_1}(\sigma)(i) = i + n_1 - \sigma(i + n_1)$, so $\Phi_{n_1}(\mathcal{D}(n_1, n_2, \ldots, n_k)) \subseteq \mathcal{D}(n_2, n_3, \ldots, n_k, n_1)$. Since the cardinality of $\mathcal{D}(n_1, n_2, \ldots, n_k)$ is invariant by permutations of the n_i's and Φ_{n_1} is bijective, we have $\Phi_{n_1}(\mathcal{D}(n_1, n_2, \ldots, n_k)) = \mathcal{D}(n_2, n_3, \ldots, n_k, n_1)$. The result follows then by applying (40).

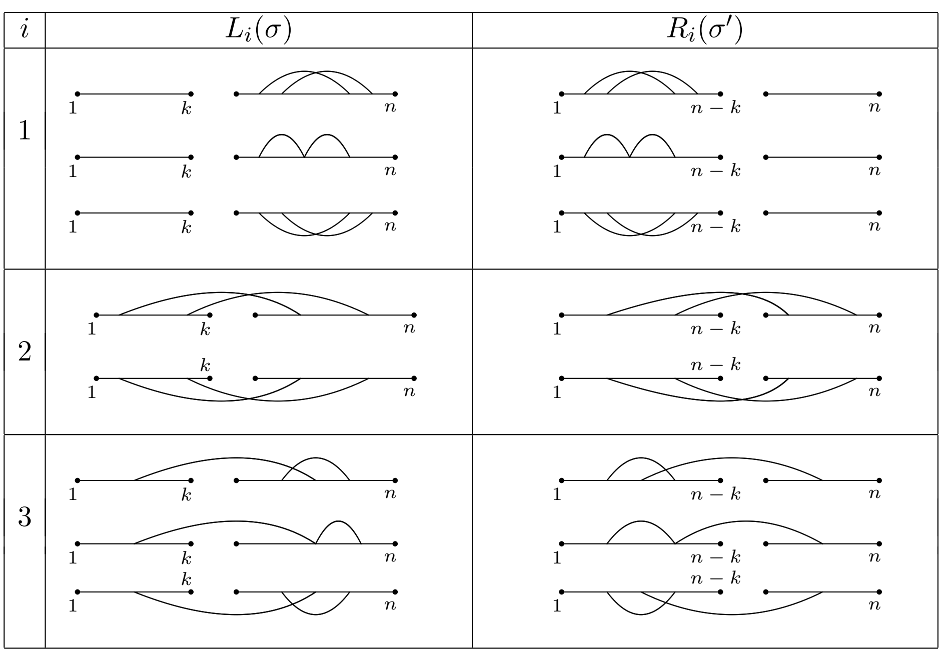


TABLE 2. Forms of crossings in $L_i(\sigma)$ and $R_i(\sigma')$.

Proof of Proposition 7. It is easy to see that Φ_k is a bijection. Let $\sigma \in {}^k\mathcal{S}_n$ and $\sigma' = \Phi_k(\sigma)$. The equality $wex(\sigma') = wex(\sigma)$ follows directly from the definition of Φ_k. It then remains to prove that $cr(\sigma') = cr(\sigma)$. We first decompose the crossings of σ and σ' into three subsets. Set

$$\begin{aligned}
L_1(\sigma) &= \{(i,j) \mid k < i < j \le \sigma(i) < \sigma(j) \quad \text{or} \quad i > j > \sigma(i) > \sigma(j) > k\},\\
L_2(\sigma) &= \{(i,j) \mid i < j \le k < \sigma(i) < \sigma(j) \quad \text{or} \quad i > j > k \ge \sigma(i) > \sigma(j)\},\\
L_3(\sigma) &= \{(i,j) \mid i \le k < j \le \sigma(i) < \sigma(j) \quad \text{or} \quad i > j > \sigma(i) > k \ge \sigma(j)\},
\end{aligned}$$

and

$$\begin{aligned}
R_1(\sigma') &= \{(i,j) \mid i < j \le \sigma'(i) < \sigma'(j) \le n-k \quad \text{or} \quad n-k \ge i > j > \sigma'(i) > \sigma'(j)\},\\
R_2(\sigma') &= \{(i,j) \mid i < j \le n-k < \sigma'(i) < \sigma'(j) \quad \text{or} \quad i > j > n-k \ge \sigma'(i) > \sigma'(j)\},\\
R_3(\sigma') &= \{(i,j) \mid i < j \le \sigma'(i) \le n-k < \sigma'(j) \quad \text{or} \quad i > n-k \ge j > \sigma(i) > \sigma(j)\}.
\end{aligned}$$

The crossings in L_i's and R_i's are illustrated in Table 2. Clearly, we have $cr(\sigma) = \sum_{i=1}^{3} |L_i(\sigma)|$ and $cr(\sigma') = \sum_{i=1}^{3} |R_i(\sigma')|$ since $\sigma \in {}^k\mathcal{S}_n$ and $\sigma' \in \mathcal{S}_n^k$.

By the definition of Φ_k, it is readily seen (see Row 1 in Table 3) that $(i,j) \in L_1(\sigma)$ if and only if $(i-k, j-k) \in R_1(\sigma')$, and thus $|L_1(\sigma)| = |R_1(\sigma')|$. Similarly, we have (see Row 2 in Table 3) that $|L_2(\sigma)| = |R_2(\sigma')|$. It then remains to prove that $|L_3(\sigma)| = |R_3(\sigma')|$. Let

$$L_4(\sigma) = \{(i,j) \mid \sigma(i) \le k < j < i \le \sigma(j) \quad \text{or} \quad i \le k < \sigma(j) < \sigma(i) < j\}.$$

Then it is not difficult to show (see Row 4 of Table 3) that $|R_3(\sigma')| = |L_4(\sigma)|$. The result will thus follow from the following Lemma.

Lemma 6. *For all $\sigma \in {}^k\mathcal{S}_n$ we have $|L_3(\sigma)| = |L_4(\sigma)|$.*

Proof. Suppose $\sigma([1,k]) = \{i_1, i_2, \dots, i_k\}_<$ and $\sigma^{-1}([1,k]) = \{j_1, j_2, \dots, j_k\}_<$. Then

$$\begin{aligned}
|L_3(\sigma)| &= \sum_{s=1}^{k} (|\{\ell \mid k < \ell \le i_s < \sigma(\ell)\}| + |\{\ell \mid \ell > j_s > \sigma(\ell) > k\}|),\\
|L_4(\sigma)| &= \sum_{s=1}^{k} (|\{\ell \mid \ell > i_s > \sigma(\ell) > k\}| + |\{\ell \mid k < \ell < j_s \le \sigma(\ell)\}|).
\end{aligned}$$

For $i \in [n]$ define the set $A_i(\sigma) = \{j \mid j \le i < \sigma(j)\}$. Then it is easily seen that

$$|A_i(\sigma)| = |\{j \mid j > i \ge \sigma(j)\}| = |A_i(\sigma^{-1})|. \tag{41}$$

Noticing that, for $s \in [k]$,

$$\begin{aligned}
|\{\ell \mid k < \ell \le i_s < \sigma(\ell)\}| &= |\{\ell \mid \ell \le i_s < \sigma(\ell)\}| - |\{\ell \mid \ell \le k < i_s < \sigma(\ell)\}|\\
&= |A_{i_s}(\sigma)| - |\{t \mid i_t > i_s\}|,
\end{aligned}$$

$$\begin{aligned}
|\{\ell \mid \ell > j_s > \sigma(\ell) > k\}| &= |\{\ell \mid \ell > j_s > \sigma(\ell)\}| - |\{\ell \mid \ell > j_s > k \ge \sigma(\ell)\}|\\
&= |\{\ell \mid \ell > j_s > \sigma(\ell)\}| - |\{t \mid j_t > j_s\}|\\
&= |A_{j_s}(\sigma^{-1})| - \chi(\sigma^{-1}(j_s) > j_s) - |\{t \mid j_t > j_s\}|,
\end{aligned}$$

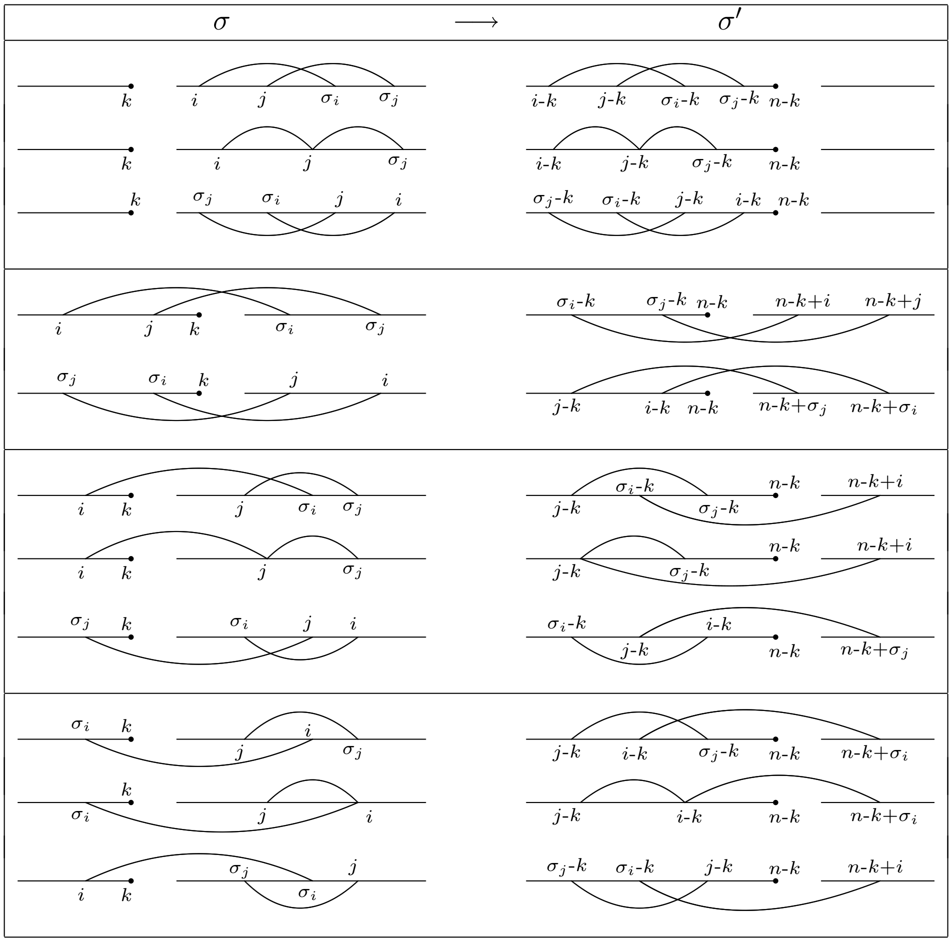


TABLE 3. Effects of the mapping Φ_k on the crossings of σ and σ'.

and

$$|\{\ell \mid \ell > i_s > \sigma(\ell) > k\}| = |A_{i_s}(\sigma^{-1})| - |\{t \mid j_t > i_s\}|,$$
$$|\{\ell \mid k \le \ell < j_s \le \sigma(\ell)\}| = |A_{j_s}(\sigma)| + \chi(\sigma^{-1}(j_s) < j_s) - |\{t \mid i_t \ge j_s\}|,$$

we can rewrite $|L_3(\sigma)|$ and $|L_4(\sigma)|$, using (41), as follows:

$$|L_3(\sigma)| = A - \sum_{s=1}^{k}(\chi(\sigma^{-1}(j_s) > j_s) + |\{t \mid i_t > i_s\}| + |\{t \mid j_t > j_s\}|), \tag{42}$$

$$|L_4(\sigma)| = A + \sum_{s=1}^{k}(\chi(\sigma^{-1}(j_s) < j_s) - |\{t \mid j_t > i_s\}| - |\{t \mid i_t \ge j_s\}|), \tag{43}$$

where $A = \sum_{s=1}^{k}(|A_{i_s}(\sigma)| + |A_{j_s}(\sigma)|$.

Since $|\{t \mid i_t > i_s\}| = |\{t \mid j_t > j_s\}| = k - s$, we have

$$\sum_{s=1}^{k}(|\{t \mid i_t > i_s\}| + |\{t \mid j_t > j_s\}|) = k(k-1).$$

Also,

$$\sum_{s=1}^{k}(|\{t \mid j_t > i_s\}| + |\{t \mid i_t \geq j_s\}|) = \sum_{s,t=1}^{k}(\chi(j_t > i_s) + \chi(i_t \geq j_s)) = k^2.$$

Substituting the above values into (42) and (43) leads to

$$|L_3(\sigma)| - | L_4(\sigma)| = k - \sum_{s=1}^{k}(\chi(\sigma^{-1}(j_s) > j_s) + \chi(\sigma^{-1}(j_s) < j_s)) = 0,$$

where the last equality follows from the fact that $\sigma^{-1}(j_s) \neq j_s$ for all $s \in [k]$. □

6. Proof of Lemma 5

Let $N_2 := n_1 + n_2 \leq n$ and define

$$\mathcal{S}_n^{(n_1,n_2)} := \{\sigma \in \mathcal{S}_n : (i, \sigma(i)) \notin [1, n_1]^2 \cup [n_1 + 1, N_2]^2\}.$$

Hence, in the graph of any permutation in $\mathcal{S}_n^{(n_1,n_2)}$ there is no arc between any two integers in $[1, n_1]$ or $[n_1 + 1,\ N_2]$.

We now construct a mapping $\Gamma^{(n_1,n_2)} : \sigma \mapsto \sigma'$ from $\mathcal{S}_n^{(n_1,n_2)}$ to $\mathcal{S}_n^{(n_2,n_1)}$ as follows. For $i = 1, \dots, n$,

(1) If $i > N_2$ and $\sigma(i) > N_2$, set $\sigma'(i) = \sigma(i)$.
(2) Suppose

$$\{(i, \sigma(i)) \mid i < \sigma(i) \leq N_2\} = \{(i_1, N_2 + 1 - j_1), \dots, (i_p, N_2 + 1 - j_p)\},$$
$$\{(\sigma(i), i) \mid \sigma(i) < i \leq N_2\} = \{(k_1, N_2 + 1 - \ell_1), \dots, (k_q, N_2 + 1 - \ell_q)\}.$$

Then set $\sigma'(j_s) = N_2 + 1 - i_s$ and $\sigma'(N_2 + 1 - k_t) = \ell_t$ for any $s \in [p]$ and $t \in [q]$.
(3) Let $C = \{i \in [1, N_2] : \sigma(i) > N_2\}$ and $D = \{i \in [1, N_2] : \sigma^{-1}(i) > N_2\}$. It is clear that $|C|$=$|D|$. Suppose $C = \{c_1, c_2, \dots, c_u\}_<$, $D = \{d_1, d_2, \dots, d_u\}_<$, $\sigma(C) = \{r_1, r_2, \dots, r_u\}_<$ and $\sigma^{-1}(D) = \{s_1, s_2, \dots, s_u\}_<$. Then, there are (unique) permutations $\alpha, \beta \in \mathcal{S}_u$ satisfying $\sigma(c_i) = r_{\alpha(i)}$ and $\sigma^{-1}(d_i) = s_{\beta(i)}$ for each $1 \leq i \leq u$. Let

$$E = [1, N_2] \setminus \{j_1, \dots, j_p, N_2 + 1 - k_1, \dots, N_2 + 1 - k_q\},$$
$$F = [1, N_2] \setminus \{N_2 + 1 - i_1, \dots, N_2 + 1 - i_p, \ell_1, \dots, \ell_q\}.$$

Clearly, we have $|E| = |C|$ and $|F| = |D|$. Suppose $E = \{e_1, \dots, e_u\}_<$ and $F = \{f_1, \dots, f_u\}_<$. Then set $\sigma'(e_i) = r_{\alpha(i)}$ and $\sigma'(s_i) = f_{\beta(i)}$ for each $1 \leq i \leq u$.

The mapping is illustrated in Table 4.

For example, if we consider the permutation in $\mathcal{S}_{15}^{(3,4)}$ whose diagram is given by

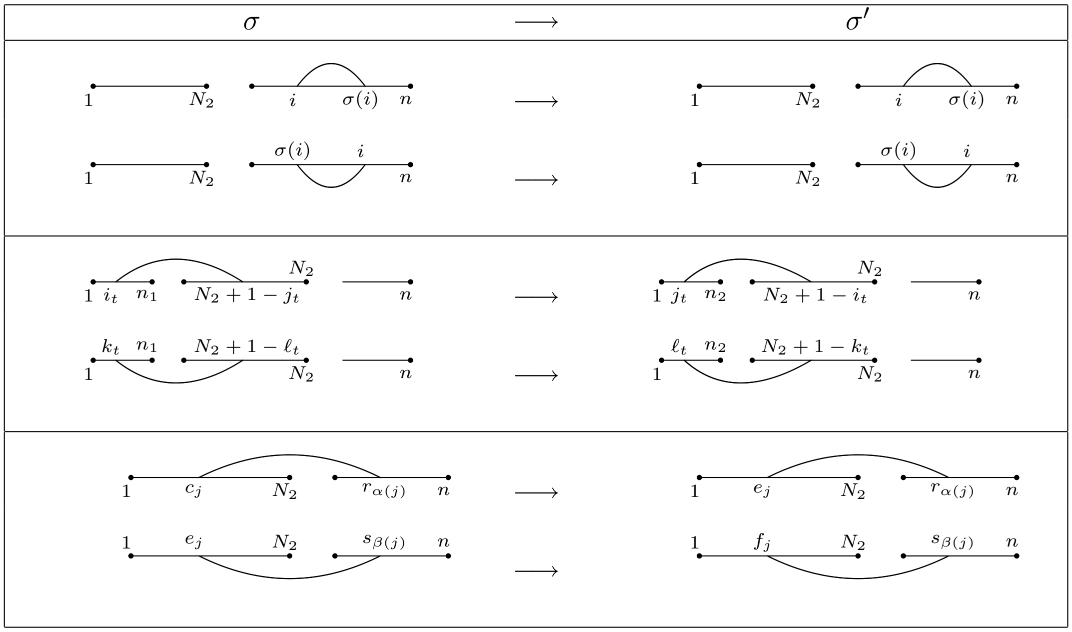
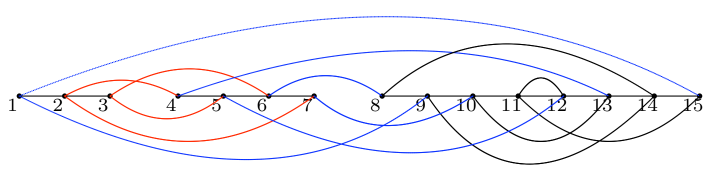
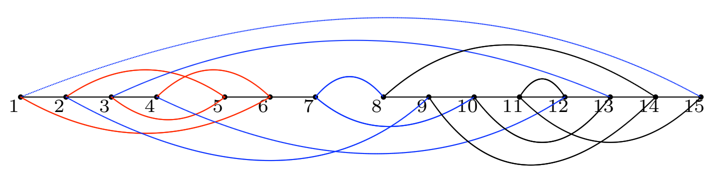


TABLE 4. The mapping $\Gamma^{(n_1,n_2)} : \sigma \mapsto \sigma'$



then the diagram of $\Gamma^{(n_1,n_2)}(\sigma)$ is given by



It is not hard to check that $\Gamma^{(n_1,n_2)} : \mathcal{S}_n^{(n_1,n_2)} \to \mathcal{S}_n^{(n_2,n_1)}$ is well defined and bijective because each step of the construction is reversible, Actually we can prove, the details are left to the reader, that $(\Gamma^{(n_1,n_2)})^{-1} = \Gamma^{(n_2,n_1)}$.

Proposition 8. *For each positive integers n_1, n_2, n, with $N_2 \le n$, the map $\Gamma^{(n_1,n_2)}$ is a bijection from $\mathcal{S}_n^{(n_1,n_2)}$ to $\mathcal{S}_n^{(n_2,n_1)}$ such that for each $\sigma \in \mathcal{S}_n^{(n_1,n_2)}$, we have*

$$(wex, cr)\Gamma^{(n_1,n_2)}(\sigma) = (wex, cr)\sigma. \tag{44}$$

We first derive Lemma 5 from the above proposition. Let $n = n_1 + n_2 + \cdots + n_k$. Then $\mathcal{D}(n_1, n_2, \ldots, n_k) \subseteq \mathcal{S}_n^{(n_1,n_2)}$. By definition of $\Gamma^{(n_1,n_2)}$, for any $\sigma \in \mathcal{S}_n^{(n_1,n_2)}$ and $i > N_2$ satisfying $\sigma(i) > N_2$, we have $i - \Gamma^{(n_1,n_2)}(\sigma)(i) = i - \sigma(i)$, so $\Gamma^{(n_1,n_2)}(\mathcal{D}(n_1, n_2, \ldots, n_k)) \subseteq$

$\mathcal{D}(n_2, n_3, \ldots, n_k, n_1)$. Since the cardinality of $\mathcal{D}(n_1, n_2, \ldots, n_k)$ doesn't depend on the order of the n_i's and $\Gamma^{(n_1,n_2)}$ is a bijection, we have

$$\Gamma^{(n_1,n_2)}(\mathcal{D}(n_1, n_2, \ldots, n_k)) = \mathcal{D}(n_2, n_3, \ldots, n_k, n_1).$$

Lemma 5 then follows from (44).

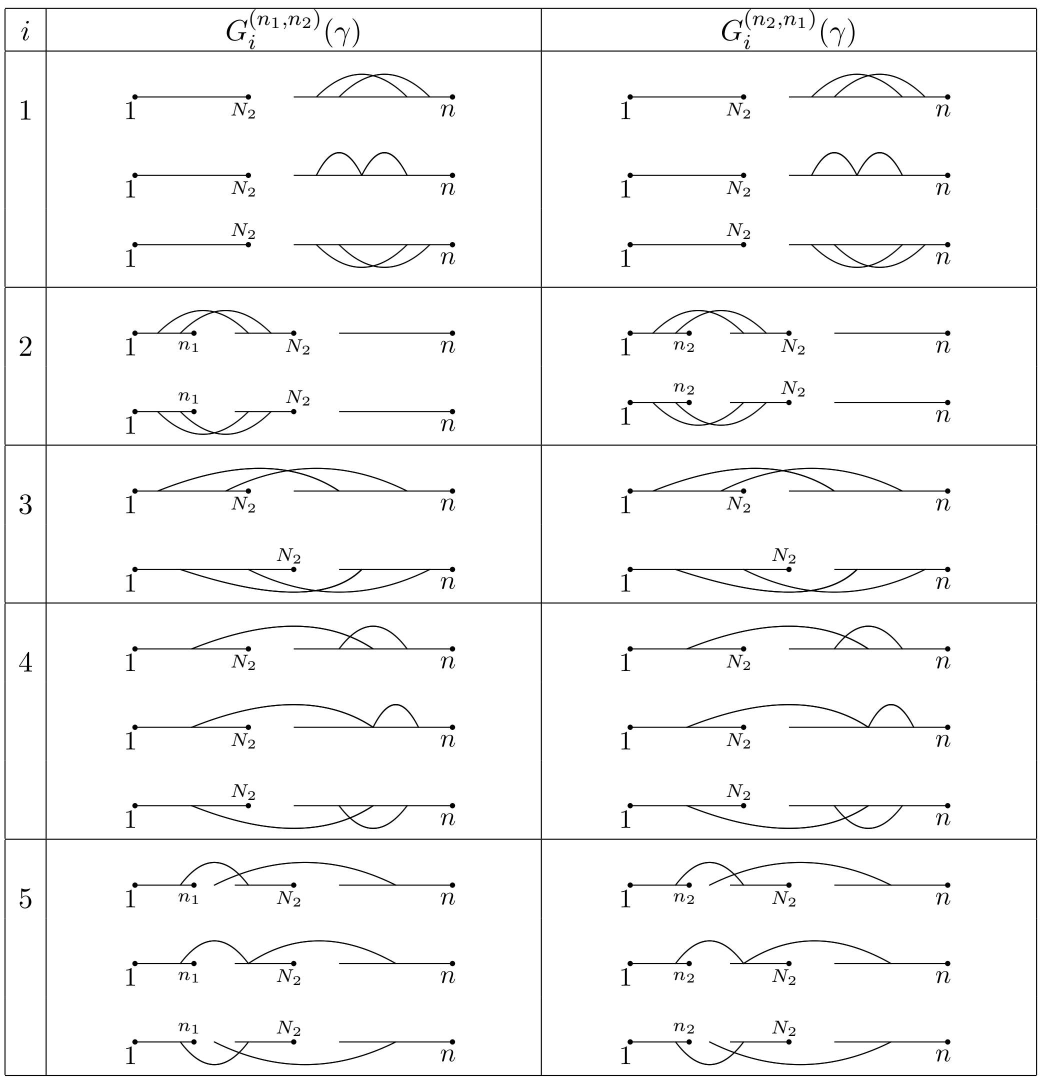


TABLE 5. Forms of the crossings in $G_i^{(n_1,n_2)}(\gamma)$ and $G_i^{(n_2,n_1)}(\gamma)$.

Proof of Proposition 8. It was shown above that $\Gamma^{(n_1,n_2)}$ is bijective. Let $\sigma \in \mathcal{S}_n^{(n_1,n_2)}$ and $\sigma' := \Gamma^{(n_1,n_2)}(\sigma)$. The equality $wex(\sigma') = wex(\sigma)$ is an immediate consequence of the definition of $\Gamma^{(n_1,n_2)}$. It then remains to prove that $cr(\sigma') = cr(\sigma)$. The idea is the same as for the proof of Eq. (4). We first decompose the number of crossings of σ and σ'. For each permutation $\gamma \in \mathcal{S}_n$, set

$$\begin{aligned}
G_1^{(n_1,n_2)}(\gamma) &= \{(i,j) \mid N_2 < i < j \le \gamma(i) < \gamma(j) \quad \text{or} \quad i > j > \gamma(i) > \gamma(j) > N_2\},\\
G_2^{(n_1,n_2)}(\gamma) &= \{(i,j) \mid i < j < \gamma(i) < \gamma(j) \le N_2 \quad \text{or} \quad N_2 \ge i > j > \gamma(i) > \gamma(j)\},\\
G_3^{(n_1,n_2)}(\gamma) &= \{(i,j) \mid i < j \le N_2 < \gamma(i) < \gamma(j) \quad \text{or} \quad i > j > N_2 \ge \gamma(i) > \gamma(j)\},\\
G_4^{(n_1,n_2)}(\gamma) &= \{(i,j) \mid i \le N_2 < j \le \gamma(i) < \gamma(j) \quad \text{or} \quad i > j > \gamma(i) > N_2 \ge \gamma(j)\},\\
G_5^{(n_1,n_2)}(\gamma) &= \{(i,j) \mid i < j \le \gamma(i) \le N_2 < \gamma(j) \quad \text{or} \quad i > N_2 \ge j > \gamma(i) > \gamma(j)\}.
\end{aligned}$$

Clearly, for any $\gamma \in \mathcal{S}_n^{(n_1,n_2)}$, we have $cr(\gamma) = \sum_{i=1}^5 |G_i^{(n_1,n_2)}(\gamma)|$. In particular,

$$cr(\sigma) = \sum_{i=1}^{5} |G_i^{(n_1,n_2)}(\sigma)| \quad \text{and} \quad cr(\sigma') = \sum_{i=1}^{5} |G_i^{(n_2,n_1)}(\sigma')|. \tag{45}$$

The crossings of $G_i^{(n_1,n_2)}$'s and $G_i^{(n_2,n_1)}$'s are illustrated in Table 5. By the definition of $\Gamma^{(n_1,n_2)}$, it is readily seen (see Row 1 in Table 6) that $G_1^{(n_1,n_2)}(\sigma) = G_1^{(n_2,n_1)}(\sigma')$ and thus $|G_1^{(n_1,n_2)}(\sigma)| = |G_1^{(n_2,n_1)}(\sigma')|$. Similarly, we can prove (see Table 6) that $|G_i^{(n_1,n_2)}(\sigma)| = |G_i^{(n_2,n_1)}(\sigma')|$ for $i = 2, 3, 4$. It remains to prove that $|G_5^{(n_1,n_2)}(\sigma)| = |G_5^{(n_2,n_1)}(\sigma')|$. This will follow from the following lemma.

Lemma 7. *Let n_1, n_2 and n be positive integers with $N_2 \le n$ and $\gamma \in \mathcal{S}_n^{(n_1,n_2)}$. Suppose that*

$$B(\gamma) := \{(i, \gamma(i)) \mid i < \gamma(i) \le N_2\} = \{(i_1, j_1), (i_2, j_2), \dots, (i_p, j_p)\}, \tag{46}$$

$$B(\gamma^{-1}) = \{(\gamma(i), i) \mid \gamma(i) < i \le N_2\} = \{(k_1, \ell_1), (k_2, \ell_2), \dots, (k_q, \ell_q)\}, \tag{47}$$

with $i_1 < i_2 < \cdots < i_p$ and $k_1 < k_2 < \cdots < k_q$. Then we have

$$|G_5^{(n_1,n_2)}(\gamma)| = \sum_{r=1}^{p} (j_r - i_r) + \sum_{r=1}^{q} (\ell_r - k_r - 1) - \binom{p+q}{2}. \tag{48}$$

Indeed, suppose

$$\begin{aligned}
B(\sigma) &= \{(i_1, N_2 + 1 - j_1), \dots, (i_p, N_2 + 1 - j_p)\},\\
B(\sigma^{-1}) &= \{(k_1, N_2 + 1 - \ell_1), \dots, (k_q, N_2 + 1 - \ell_q)\},
\end{aligned}$$

then, by construction of σ', we have

$$\begin{aligned}
B(\sigma') &= \{(j_1, N_2 + 1 - i_1), \dots, (j_p, N_2 + 1 - i_p)\},\\
B(\sigma'^{-1}) &= \{(\ell_1, N_2 + 1 - k_1), \dots, (\ell_q, N_2 + 1 - k_q)\}.
\end{aligned}$$

By symmetry, the identity (48) is also valid on $\mathcal{S}_n^{(n_2,n_1)}$. Applying (48) to σ' and σ leads to $|G_5^{(n_1,n_2)}(\sigma)| = |G_5^{(n_2,n_1)}(\sigma')|$. The proof of Lemma 8 is thus completed.

Proof of Lemma 7. For any $\gamma \in \mathcal{S}_n^{(n_2,n_1)}$, by definition, we have

$$\begin{aligned}
|G_5^{(n_1,n_2)}(\gamma)| = {} & |\{(i,j) \mid i < j < \gamma(i) \le N_2 < \gamma(j)\}| + |\{(i,j) \mid \gamma(j) < \gamma(i) < j \le N_2 < i\}|\\
& + |\{i \mid i < \gamma(i) \le N_2 < \gamma^2(i)\}|.
\end{aligned} \tag{49}$$

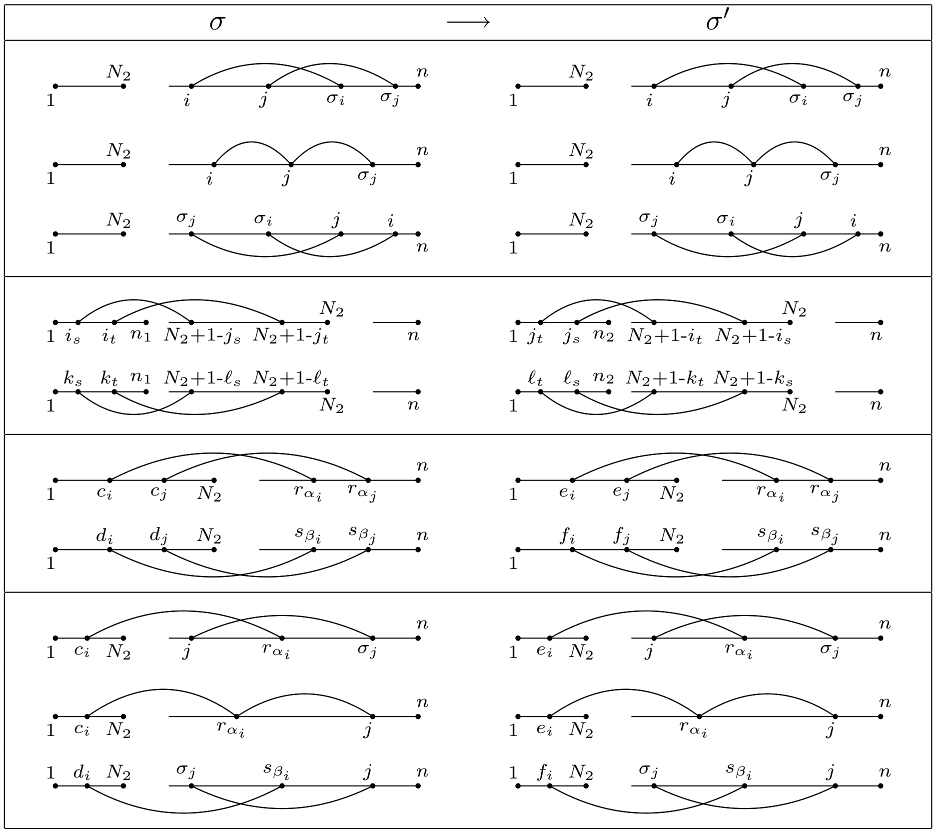


TABLE 6. Effects of the mapping $\Gamma^{(n_1,n_2)}$ on the crossings of σ and σ'.

Now, by the definition of $B(\gamma)$ we get

$$\begin{aligned}
&|\{(i,j) \mid i<j<\gamma(i)\leq N_2<\gamma(j)\}|\\
&=\sum_{r=1}^{p}|\{x \mid i_r<x<j_r\leq N_2<\gamma(x)\}|\\
&=\sum_{r=1}^{p}(|\{x \mid i_r<x<j_r\}|-|\{x \mid i_r<x<j_r\,,\,\gamma(x)\leq N_2\}|).
\end{aligned}$$

For any $r\in[1,p]$, we have $|\{x \mid i_r<x<j_r\}|=j_r-i_r-1$ and

$$\begin{aligned}
&|\{x \mid i_r<x<j_r\,,\,\gamma(x)\leq N_2\}|\\
=&|\{x \mid i_r<x<j_r\,,\,x<\gamma(x)\leq N_2\}|+|\{x \mid i_r<x<j_r\,,\,\gamma(x)<x\leq N_2\}|\\
=&|\{t \mid i_r<i_t<j_r\}|+|\{t \mid i_r<\ell_t<j_r\}| \quad \text{(by definition of } B(\gamma) \text{ and } B(\gamma^{-1}))\\
=&|\{t \mid i_r<i_t\}|+|\{t \mid \ell_t<j_r\}|,
\end{aligned}$$

because, by definition of $\mathcal{S}_n^{(n_1,n_2)}$, (46) and (47), for any integers r and t, we have $i_t \leq n_1$, $k_t \leq n_1$, $j_r > n_1$ and $\ell_t > n_1$, therefore $i_t < j_r$ and $i_r < \ell_t$.

Summing over all r yields

$$|\{(i,j) \mid i < j < \gamma(i) \leq N_2 < \gamma(j)\}| = \sum_{r=1}^{p} (j_r - i_r - 1 - |\{t \mid i_r < i_t\}| - |\{t \mid \ell_t < j_r\}|). \tag{50}$$

It follows that

$$\begin{aligned} &|\{(i,j) \mid \gamma(j) < \gamma(i) < j \leq N_2 < i\}| \\ &= |\{(i,j) \mid i < j < \gamma^{-1}(i) \leq N_2 < \gamma^{-1}(j)\}| \\ &= \sum_{r=1}^{q} (\ell_r - k_r - 1 - |\{t \mid k_r < k_t\}| - |\{t \mid j_t < \ell_r\}|). \end{aligned} \tag{51}$$

As $|\{i \mid i < \gamma(i) \leq N_2 < \gamma^2(i)\}| = |\{t \mid \gamma(j_t) > N_2\}|$, plugging (50) and (51) into (49) leads to

$$\begin{aligned} |G_5^{(n_1,n_2)}(\gamma)| = &\sum_{r=1}^{p} (j_r - i_r - 1) + \sum_{r=1}^{q} (\ell_r - k_r - 1) + |\{t \mid \gamma(j_t) > N_2\}| - \sum_{r=1}^{p} |\{t \mid i_r < i_t\}| \\ &- \sum_{r=1}^{p} |\{t \mid \ell_t < j_r\}| - \sum_{r=1}^{q} |\{t \mid k_r < k_t\}| - \sum_{r=1}^{q} |\{t \mid j_t < \ell_r\}|. \end{aligned} \tag{52}$$

Since the i_r's and k_r's are distinct we have

$$\sum_{r=1}^{p} |\{t \mid i_r < i_t\}| = \binom{p}{2} \quad \text{and} \quad \sum_{r=1}^{q} |\{t \mid k_r < k_t\}| = \binom{q}{2}. \tag{53}$$

On the other hand,

$$\begin{aligned} \sum_{r=1}^{p} |\{t \mid \ell_t < j_r\}| + \sum_{r=1}^{q} |\{t \mid j_t < \ell_r\}| &= \sum_{r=1}^{p} |\{t \mid \ell_t \neq j_r\}| \\ &= pq - |\{t \mid j_t \in \{\ell_1, \ell_2, \ldots, \ell_q\}| \\ &= pq - \sum_{s=1}^{k} |\{t \mid \gamma(j_t) \leq N_2\}|, \end{aligned} \tag{54}$$

where the last identity follows from the definitions of $B(\gamma)$ and $B(\gamma^{-1})$. Inserting (53) and (54) in (52) we get (48). This concludes the proof of Lemma 7. □

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