QUADRATIC q-EXPONENTIALS AND CONNECTION COEFFICIENT PROBLEMS

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ABSTRACT. We establish expansion formulas of q-exponential functions in terms of continuous q-ultraspherical polynomials, continuous q-Hermite polynomials and Askey-Wilson polynomials. The proofs are based on solving connection coefficient problems.

1. Introduction.

The q-exponential function on a q-quadratic grid is

(1.1)
$$\mathcal{E}_q(x;a,b) := \sum_{n=0}^{\infty} \frac{(aq^{(1-n)/2}e^{i\theta}, aq^{(1-n)/2}e^{-i\theta};q)_n}{(q;q)_n}\, q^{n^2/4}\, b^n,$$

where $x = \cos\theta$, [Is:Zh], [At:Su]. Ismail and Zhang [Is:Zh] gave a q-analogue of the expansion of the plane wave in spherical harmonics. Their formula is

(1.2)
$$\mathcal{E}_q(x;-i,b/2) = \frac{(q;q)_\infty\, (2/b)^\nu}{(-b^2/4;q^2)_\infty\, (q^\nu;q)_\infty} \sum_{n=0}^{\infty} i^n (1-q^{n+\nu}) q^{n^2/4} J_{\nu+n}^{(2)}(b;q) C_n(x;q^\nu|q),$$

where $J_{\nu+n}^{(2)}$ are q-Bessel functions [Is], [Ga:Ra] and $C_n(x;\beta|q)$ are the continuous q-ultraspherical polynomials [As:Is], [Ga:Ra]. This formula has attracted some attention and two different proofs were given in [Fl:Vi] and [Is:Ra:Zh]. The proof by Floreanini and Vinet [Fl:Vi] is group theoretic and is of independent interest. Equation (1.2) was extended to continuous q-Jacobi polynomials in [Is:Ra:Zh], where the following expansion was established

(1.3a)
$$\mathcal{E}_q(x;-i,r) = \sum_{m=0}^{\infty} a_m p_m(x;b,bq^{1/2},-c,-cq^{1/2}),$$

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with the a_m's given by

$$a_m = \frac{(b^2c^2, b^2q^{1/2}; q)_m \ (irq^{1/2}; q)_\infty}{(q, bcq^{1/2}, bc; q)_m \ (-ir; q)_\infty} (ir/b)^m q^{m^2/4} \times {}_2\phi_1\left(\begin{matrix} cq^{m/2+1/4}, -bq^{m/2+1/4} \\ bcq^{m+1/2} \end{matrix} \middle| q^{1/2}, ir \right). \tag{1.3b}$$

The purpose of this paper is to extend (1.3) to an Askey-Wilson polynomial expansion (see (2.7)), and thereby also several special cases: (4.1), (4.4), (4.6). We also specialize (2.7) to find expansions of q-Bessel functions instead of the quadratic q-exponential function, see (3.2), (3.3). The fundamental technique is a connection coefficient result for Askey-Wilson polynomials (Theorem 1), which is established from the Nassrallah-Rahman integral [Ga:Ra,(6.3.9)]

It is known that expansions of the type treated in this paper are equivalent to inversions of certain lower triangular matrices [Fi:Is] and to Lagrange and q-Lagrange inversion [Ge:St]. In §6 we give the matrices and inverse relations lying behind the expansions established in this paper. We explain a positivity result for the connection coefficients for certain Askey-Wilson polynomials in §5.

We record here the definitions and properties of the q-Bessel functions that will be required. The q-Bessel functions $J_\nu^{(1)}(x;q)$ and $J_\nu^{(2)}(x;q)$ are defined by

$$J_\nu^{(1)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n=0}^\infty \frac{(-1)^n (x/2)^{\nu+2n}}{(q, q^{\nu+1}; q)_n}, \tag{1.4a}$$

$$J_\nu^{(2)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n=0}^\infty \frac{(-1)^n (x/2)^{\nu+2n}}{(q, q^{\nu+1}; q)_n} q^{n(\nu+n)}. \tag{1.4b}$$

They are related through

$$J_\nu^{(2)}(x;q) = (-x^2/4;q)_\infty J_\nu^{(1)}(x;q). \tag{1.5}$$

We will also need

$${}_2\phi_1(-q^{\nu+1}, -q^{\nu+2}; q^{2\nu+2}; q^2, -b^2/4) = \frac{(q;q)_\infty}{(q^{\nu+1};q)_\infty} \frac{(2/b)^\nu}{(-b^2/4;q^2)_\infty} J_\nu^{(2)}(b;q), \tag{1.6}$$

which follows from (1.23) of [Ra] (note a misprint) via [Ga:Ra,(III.4)].

2. A Basic Expansion Formula.

We prove Theorem 1, which is the fundamental theorem in this paper. It allows one to expand any function of $(\alpha e^{i\theta}, \alpha e^{-i\theta}; q)_n$ in terms of Askey-Wilson polynomials which are defined by (see [As:Wi] or [Ga:Ra])

$$p_n(\cos\theta; a, b, c, d) := (ab, ac, ad; q)_n a^{-n} r_n(\cos\theta; a, b, c, d),$$

where

$$r_n(\cos\theta; a, b, c, d) := {}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix} \middle| q, q \right).$$

The main difference between p_n and r_n is that p_n is symmetric in all four parameters while r_n is symmetric in b, c, d but not in a. The basic expansion formula that we shall prove in this section is

$$(\alpha e^{i\theta}, \alpha e^{-i\theta}; q)_n = \sum_{j=0}^{n} A_{n,j}\, p_j(x; a, b, c, d). \tag{2.1}$$

Theorem 1. *The coefficients $A_{n,j}$ in (2.1) are given by*

$$\begin{aligned} A_{n,j} =& \frac{(q, \alpha d; q)_n\, (\alpha/d; q)_{n-j}\, (abcd/q; q)_j}{(q, \alpha d; q)_j\, (q; q)_{n-j}\, (abcd/q; q)_{2j}} \\ & \times (-\alpha)^j q^{j(j-1)/2} {}_4\phi_3 \left(\begin{matrix} q^{j-n}, adq^j, bdq^j, cdq^j \\ \alpha dq^j, dq^{1-n+j}/\alpha, abcdq^{2j} \end{matrix} \middle| q, q \right). \end{aligned} \tag{2.2}$$

Proof. From the orthogonality relation of the Askey-Wilson polynomials, (7.5.15)-(7.5.17) in [Ga:Ra], we find that

$$\begin{aligned} \kappa A_{n,j} =& h_j \int_{-1}^{1} \frac{h(x; 1, -1, q^{1/2}, -q^{1/2}, \alpha)}{h(x; a, b, c, d, \alpha q^n)}\, p_j(x; a, b, c, d)\, \frac{dx}{\sqrt{1-x^2}} \\ =& (ab, ac, ad; q)_j a^{-j} h_j \sum_{k=0}^{j} \frac{(q^{-j}, abcdq^{j-1}; q)_k\, q^k}{(q, ab, ac, ad; q)_k} \\ & \times \int_{-1}^{1} \frac{h(x; 1, -1, q^{1/2}, -q^{1/2}, \alpha)}{h(x; aq^k, b, c, d, \alpha q^n)}\, \frac{dx}{\sqrt{1-x^2}}, \end{aligned} \tag{2.3}$$

where

$$h(\cos\theta; \alpha) := (\alpha e^{i\theta}, \alpha e^{-i\theta}; q)_\infty, \quad h(x; a_1, \ldots, a_n) := \prod_{j=1}^{n} h(x; a_j),$$

the h_j's are

$$h_j = \frac{1 - abcdq^{2j-1}}{1 - abcd/q}\, \frac{(abcd/q; q)_j}{(q, ab, ac, ad, bc, bd, cd; q)_j}.$$

and κ is

$$\kappa := \frac{2\pi (abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}.$$

This is valid if $\max(|a|, |b|, |c|, |d|) < 1$. The integral in (2.3) can be evaluated by the Nassrallah-Rahman integral, [Ga:Ra,(6.3.8)], and is equal to

$$\begin{aligned} & \frac{2\pi (\alpha b, \alpha c, \alpha d, abcdq^k, \alpha bcdq^n; q)_\infty}{(q, bc, bd, cd, abq^k, acq^k, adq^k, \alpha bq^n, \alpha cq^n, \alpha dq^n, \alpha bcd; q)_\infty} \\ & \quad \times {}_8W_7(\alpha bcd/q; bc, bd, cd, \alpha q^{-k}/a, q^{-n}; q, a\alpha q^{n+k}) \\ & = \kappa\, (\alpha d, \alpha/d; q)_n (ab, ac; q)_k \\ & \quad \times \sum_{m=0}^{n} \frac{(q^{-n}, bd, cd; q)_m\, q^m}{(q, \alpha d, dq^{1-n}/\alpha; q)_m}\, \frac{(ad; q)_{k+m}}{(abcd; q)_{k+m}}, \end{aligned}$$

where (III.17) of [Ga:Ra] was used in the last step. Thus the $A_{n,j}$'s have the representation

$$A_{n,j} =(ab,ac,ad;q)_j a^{-j} h_j\,(\alpha d,\alpha/d;q)_n \sum_{m=0}^{n} \frac{(q^{-n},ad,bd,cd;q)_m}{(q,d\alpha,dq^{1-n}/\alpha,abcd;q)_m}\, q^m \times {}_3\phi_2\left(\begin{matrix} q^{-j},abcdq^{j-1},adq^m \\ abcdq^m,ad \end{matrix}\,\middle|\, q,q\right). \tag{2.4}$$

The ${}_3\phi_2$ in (2.4) is now summed by (II.12) in [Ga:Ra]. After some simplification we establish (2.2), completing the proof of Theorem 1. □

Application of the Sears transformation [Ga:Ra,(III.15)] gives an alternate representation of $A_{n,j}$, which we will find useful later in this work. It is

$$A_{n,j} =\frac{(q,c\alpha,d\alpha,ab;q)_n\,(abcd/q;q)_j}{(q,c\alpha,d\alpha,ab;q)_j\,(q;q)_{n-j}\,(abcd;q)_{n+j}}\,\frac{1-abcdq^{2j-1}}{1-abcd/q}\,(-\alpha)^j\,q^{j(j-1)/2} \times {}_4\phi_3\left(\begin{matrix} q^{j-n},cdq^j,\alpha/a,\alpha/b \\ \alpha cq^j,\alpha dq^j,q^{1-n}/ab \end{matrix}\,\middle|\, q,q\right). \tag{2.5}$$

It is clear from (2.1) that $A_{n,j}$ must be symmetric in a,b,c,d which, however, is not obvious in either of the two forms (2.2) or (2.5). The symmetry becomes explicit when we apply the Watson transformation [Ga:Ra,(III.17)] to obtain the following representation

$$A_{n,j} =\frac{(q,a\alpha,b\alpha,c\alpha,d\alpha;q)_n\,(abcd/q,\alpha^2;q)_j}{(q,a\alpha,b\alpha,c\alpha,d\alpha;q)_j\,(\alpha^2;q)_n\,(q;q)_{n-j}}\,\frac{1-abcdq^{2j-1}}{1-abcd/q}\,\frac{(-\alpha)^j q^{j(j-1)/2}}{(abcd;q)_{n+j}} \times {}_8W_7(\alpha^2q^{j-1};q^{j-n},\alpha/a,\alpha/b,\alpha/c,\alpha/d;abcdq^{n+j}). \tag{2.6}$$

In the subsequent sections we shall use Theorem 1 in the following way. We have

$$\sum_{n=0}^{\infty} c_n(\alpha e^{i\theta},\alpha e^{-i\theta};q)_n = \sum_{j=0}^{\infty} p_j(\cos\theta;a,b,c,d) \sum_{n=0}^{\infty} A_{n+j,j}c_{n+j}, \tag{2.7}$$

which is obtained from (2.1) by multiplying by c_n and and summing over n, where $\{c_n\}$ is an arbitrary sequence, provided that the left-hand side of (2.7) converges and interchanging the sums is justifiable. If we choose $\alpha = eq^{(1-n)/2}$ for some constant e and c_n in accordance with (1.1), then (2.7) is an expansion for a q-exponential function $\mathcal{E}_q$. We will specialize the parameters in such a way that the ${}_4\phi_3$ series in (2.2) and (2.5), or the ${}_8W_7$ series in (2.6) can be summed. We will also choose α independent of n, obtaining basic hypergeometric series expansions.

3. Expansions in continuous q-ultraspherical polynomials.

In this section we first find two general Askey-Wilson polynomial expansions with q-Bessel functions, (3.2) and (3.3). First, if $\alpha = a$, then $A_{n,j}$ is summable by (2.5). Next we specialize to $\alpha=-iq^{(1-n)/2}$ and the continuous q-ultraspherical polynomials, and prove (1.2).

If $\alpha=a$, and

$$c_n=\frac{(E_1,E_2;q)_n}{(q,ab,ac,ad;q)_n}(-B^2/4)^n,$$

then (2.5) and (2.7) imply

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{(ae^{i\theta},ae^{-i\theta},E_1,E_2;q)_n}{(q,ab,ac,ad;q)_n}(-B^2/4)^n\\ &=\sum_{j=0}^{\infty}\frac{(abcd/q,E_1,E_2;q)_j}{(q,ab,ac,ad;q)_j}\frac{1-abcdq^{2j-1}}{1-abcd/q}(aB^2/4)^j q^{j(j-1)/2}p_j(x;a,b,c,d)\\ &\quad\times{}_2\phi_1\left(\begin{matrix}E_1q^j,E_2q^j\\ abcdq^{2j}\end{matrix}\,\middle|\,q,-B^2/4\right).\end{aligned}\tag{3.1}$$

Setting $abcd=q^{2\nu+1}$, $E_1=-q^{\nu+1/2}$, and $E_2=-q^{\nu+1}$, in (3.1) and using (1.6) we find

(3.2)

$$\begin{aligned}&{}_4\phi_3\left(\begin{matrix}ae^{i\theta},ae^{-i\theta},-q^{\nu+1/2},-q^{\nu+1}\\ ab,ac,ad\end{matrix}\,\middle|\,q,-\frac{B^2}{4}\right)\\ &=\frac{(q^{1/2};q^{1/2})_\infty}{(q^{\nu+1/2};q^{1/2})_\infty\,(-b^2/4;q)_\infty}\sum_{j=0}^{\infty}\frac{(q^{2\nu};q)_j(q^{\nu+1/2};q^{1/2})_{2j}}{(q,ab,ac,ad;q)_j}\frac{1-q^{2\nu+2j}}{1-q^{2\nu}}q^{j(j-1)/2}a^j\\ &\quad\times(2/b)^{2\nu}J^{(2)}_{2\nu+2j}(B;q^{1/2})p_j(x;a,b,c,d).\end{aligned}$$

We may find another q-Bessel expansion by choosing $E_1=E_2=0$ in (3.1), and using (1.4a) and (1.5). The result (again for $abcd=q^{2\nu+1}$) is

$$\begin{aligned}&{}_4\phi_3\left(\begin{matrix}ae^{i\theta},ae^{-i\theta},0,0\\ ab,ac,ad\end{matrix}\,\middle|\,q,-\frac{B^2}{4}\right)\\ &=\frac{(q;q)_\infty(2/B)^{2\nu}}{(q^{2\nu+1},-B^2/4;q)_\infty}\sum_{j=0}^{\infty}\frac{(q^{2\nu};q)_j(q^{2\nu+1};q)_{2j}}{(q,ab,ac,ad;q)_j}\frac{1-q^{2\nu+2j}}{1-q^{2\nu}}q^{j(j-1)/2}a^j\\ &\quad\times J^{(2)}_{2\nu+2j}(B;q)p_j(x;a,b,c,d).\end{aligned}\tag{3.3}$$

The continuous q-ultraspherical polynomials $C_n(x;a^2|q)$ are obtained from the Askey-Wilson polynomials by

$$p_n(x;a,a\sqrt{q},-a,-a\sqrt{q})=\frac{(q;q)_n\,(a^4;q)_{2n}}{(a^2,a^4;q)_n}C_n(x;a^2|q).$$

Thus q-ultraspherical versions of (3.2) and (3.3) can be found be putting

$$\{a,b,c,d\}=\{q^{\nu/2},q^{(\nu+1)/2},-q^{\nu/2},-q^{(\nu+1)/2}\}.$$

The left-hand sides then become other q-analogues of the Bessel function $J_{\nu+1/2}$.

Next we consider (2.7) for the continuous q-ultraspherical polynomials, ($b = a\sqrt{q}, c = -a, d = -a\sqrt{q}$) for an arbitrary α. Using (3.4.7) in [Ga:Ra], we have

$$
\begin{aligned}
&{}_8W_7(\alpha^2 q^{j-1}; q^{j-n}, \alpha q^{-1/2}/a, -\alpha q^{-1/2}/a, \alpha/a, -\alpha/a; q, a^4 q^{n+j+1}) \\
&= \frac{(\alpha^2 q^j, a^4 q^{2j+1}; q)_\infty}{(a^2\alpha^2 q^{2j}, a^2 q^{j+1}; q)_\infty} \, {}_2\phi_1\left(\begin{matrix} \alpha^2/qa^2, q^{-n}/a^2 \\ \alpha^2 q^n \end{matrix} \middle| q, a^4 q^{n+j+1} \right),
\end{aligned}
$$

According to (III.2) in [Ga:Ra] the above ${}_2\phi_1$ can be transformed to

$$
\frac{(\alpha^2 q^j, a^2\alpha^2 q^{2n}, a^4 q^{2j+1}; q)_\infty}{(\alpha^2 q^n, a^2\alpha^2 q^{2j}, a^4 q^{n+j+1}; q)_\infty} {}_2\phi_1\left(\begin{matrix} q^{j-n}, q^{-n}/a^2 \\ a^2 q^{j+1} \end{matrix} \middle| q, a^2\alpha^2 q^{2n} \right).
$$

From this we find that Theorem 1 reduces to

$$
\begin{aligned}
(\alpha e^{i\theta}, \alpha e^{-i\theta}; q)_n &= \sum_{j=0}^{n} \frac{(q;q)_n \, (-\alpha)^j \, q^{j(j-1)/2}}{(q;q)_{n-j} \, (a^2;q)_j} \, C_j(x; a^2|q) \\
&\times {}_2\phi_1\left(\begin{matrix} q^{j-n}, q^{-n}/a^2 \\ a^2 q^{j+1} \end{matrix} \middle| q, a^2\alpha^2 q^{2n} \right).
\end{aligned}
\tag{3.4}
$$

There are several choices for α^2 for which the ${}_2\phi_1$ series in (3.4) is summable. The choice of interest here is $\alpha = -iq^{(1-n)/2}$, so that the Bailey-Daum formula [Ga:Ra,(II.9)] implies

$$
{}_2\phi_1\left(\begin{matrix} q^{j-n}, q^{-n}/a^2 \\ a^2 q^{j+1} \end{matrix} \middle| q, -a^2 q^{n+1} \right) = \begin{cases} 0 & \text{if } n-j \text{ is odd,} \\ \frac{(q, q^{-2n}/a^4; q^2)_m}{(q^{-n}/a^2, q^{1-n}/a^2; q^2)_m} & \text{if } n-j = 2m. \end{cases}
$$

The appropriate choice of c_n leads to

$$
\begin{aligned}
\mathcal{E}_q(x; -i, b/2) &= \sum_{j=0}^{\infty} \frac{q^{j^2/4} (ib/2)^j}{(a^2;q)_j} C_j(x; a^2|q) \\
&\times {}_2\phi_1\left(\begin{matrix} -a^2 q^{j+1}, -a^2 q^{j+2} \\ a^4 q^{2j+2} \end{matrix} \middle| q^2, -\frac{b^2}{4} \right).
\end{aligned}
\tag{3.5}
$$

Setting $a = q^{\nu/2}$ in (3.5) and using (1.6), we see that (1.2) follows from (3.5).

By taking the limit $\nu \to \infty$ in (1.3) we find that

$$
\mathcal{E}_q(x; -i, b) = \frac{1}{(-b^2; q^2)_\infty} \sum_{n=0}^{\infty} (ib)^n \, q^{n^2/4} \, \frac{H_n(x|q)}{(q;q)_n}, \tag{3.6}
$$

where

$$
H_n(x|q) = (q;q)_n C_n(x; 0|q),
$$

are the continuous q-Hermite polynomials.

4. $\mathcal{E}_q$ expansions in special Askey-Wilson polynomials.

In this section we evaluate $A_{n,j}$ using special balanced ${}_4\phi_3$'s. This naturally leads to expansions in special families of Askey-Wilson polynomials, see (4.1), (4.2), (4.3), (4.4). We also choose two specializations so that $A_{n,j}$ is transformable, yielding $\mathcal{E}_q$ expansions, including (1.3). At the end of the section we also prove the Al-Salam-Chihara polynomial result by generating functions.

Andrews' terminating q-analogue of the Watson's ${}_3F_2$ sum is, [Ga:Ra,(II.17)]

$$ {}_4\phi_3\left(\begin{matrix} q^{-n}, Aq^n, C, -C \\ \sqrt{Aq}, -\sqrt{Aq}, C^2 \end{matrix}\,\middle|\, q, q\right) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \frac{C^n (q, qA/C^2; q^2)_{n/2}}{(qA, qC^2; q^2)_{n/2}} & \text{if } n \text{ is even.} \end{cases} $$

This suggests that after replacing n by $n+j$ in (2.5), we take

$$ \alpha = -iq^{(1-n)/2}, \quad a = -b, \quad c = -d, \quad C = \pm iq^{(1-n-j)/2}/a, \quad A = -c^2 q^{j-n}. $$

Therefore $A_{2m+j+1,j} = 0$, and

$$ \frac{A_{2m+j,j}}{(q;q)_{2m+j}} = \frac{(a^2c^2/q;q)_j\,(q;q)_{j+2m}(-a^2q^{j+1}, -c^2q^{j+1}; q^2)_m}{(q;q)_j\,(a^2c^2/q;q)_{2j}(q^2, a^2c^2q^{2j+1}; q^2)_m}\, i^j(-1)^m q^{-m^2-mj}. $$

This leads to the expansion

$$ \begin{aligned} \mathcal{E}_q(x; -i, b) = & \sum_{j=0}^{\infty} \frac{(a^2c^2/q;q)_j}{(q;q)_j\,(a^2c^2/q;q)_j}\, p_j(x; a, -a, c, -c) \\ & \times (ib)^j\, q^{j^2/4}\, {}_2\phi_1\left(\begin{matrix} -a^2q^{j+1}, -c^2q^{j+1} \\ a^2c^2q^{2j+1} \end{matrix}\,\middle|\, q^2, -b^2\right). \end{aligned} \tag{4.1} $$

In [Is:Ma:Su] it was pointed out that

$$ \left(\frac{r}{2}\right)^{\nu} \frac{(q^{\nu+1}, -r^2/4; q)_\infty}{(q;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} q^{(\nu+1)/2}e^{i\theta},\ q^{(\nu+1)/2}e^{-i\theta} \\ q^{\nu+1} \end{matrix}\,\middle|\, q, -\frac{r^2}{4}\right), $$

is a q-analogue of $J_\nu(xr)$, $x = \cos\theta$. The ${}_2\phi_1$'s in (4.1) can be expressed in terms of the above q-Bessel function.

If we set $c = 0$ then the ${}_2\phi_1$ in (4.1) becomes a ${}_1\phi_0$, which can be summed by the q-binomial theorem [Ga:Ra,(II.3)]. We find

$$ \mathcal{E}_q(x; -i, b) = \sum_{n=0}^{\infty} \frac{(-a^2;q)_n\,(a^2b^2q^{n+1};q^2)_\infty}{(q;q)_n\,(-b^2;q^2)_\infty} \left(\frac{ib}{a}\right)^n q^{n^2/4}\, r_n(x; a, -a, 0, 0). \tag{4.2} $$

Since (7.5.34) of [Ga:Ra] implies

$$ r_n(x; q^{1/2}, -q^{1/2}, 0, 0) = \frac{q^{n/2}}{(-q;q)_n} H_n(x|q^2), $$

we find that

$$\mathcal{E}_q(x;-i,b) = \sum_{n=0}^{\infty} \frac{(b^2q^{n+2};q^2)_\infty}{(-b^2;q^2)_\infty\,(q;q)_n}(ib)^n q^{n(n-2)/4} H_n(x|q^2), \tag{4.3}$$

which may be compared with (3.6).

Another balanced ${}_4\phi_3$ evaluation is [An,(4.3)]

$$ {}_4\phi_3\left(\begin{matrix} q^{-2n}, A^2q^{2n}, B, Bq \\ Aq, Aq^2, B^2 \end{matrix}\middle| q^2, q^2\right) = B^n \frac{(Aq/B,-q;q)_n}{(A,-B;q)_n}\frac{1-A}{1-Aq^{2n}}.$$

So (2.2) is summable if $b = aq^{1/2}$, $d = cq^{1/2}$, and $\alpha^2 = q^{1/2-n}$, or $\alpha^2 = q^{3/2-n}$.

After replacing c by $-c$, the $\alpha = q^{(1-2n)/4}$ result with $c_n = q^{n(n+1)/4}(it/2)^n/(q;q)_n$ is

$$\begin{aligned} \mathcal{E}_q(x;q^{-1/4},itq^{1/4}/2) =& \sum_{n=0}^{\infty} \frac{(a^2c^2;q)_n}{(q;q)_n\,(a^2c^2;q)_{2n}} q^{n^2/4}\left(-\frac{it}{2}\right)^n \\ &\times p_n(x;a,aq^{1/2},-c,-cq^{1/2}) \\ &\times {}_2\phi_1\left(\begin{matrix} aq^{(2n+1)/4}, -cq^{(2n+1)/4} \\ acq^{n+1/2} \end{matrix}\middle| q^{1/2}, \frac{it}{2}\right). \end{aligned} \tag{4.4}$$

The choice of $\alpha = q^{(3-2n)/4}$ (thus expanding $\mathcal{E}_q(x;q^{1/4},itq^{-1/4}/2)$) gives the same expansion as (4.4). These two $\mathcal{E}_q$ functions are identical from [Su,(3.4)].

If, instead, we take $a = -iq^{(1-n)/2}$ then the ${}_4\phi_3$ series in (2.5) can no longer be summed. However the ${}_4\phi_3$ series in (2.5) can still be transformed to another ${}_4\phi_3$ series in base $q^{1/2}$ by [Ga:Ra,(III.21)]. It turns out that the series over n in (2.7), a double sum, can be simplified further by an interchange of the order of summation and finally reduced to a single sum. The result is equivalent to (1.3),

$$\begin{aligned} \mathcal{E}_q(x;-i,t/2) =& \frac{(itq^{1/2}/2;q)_\infty}{(-it/2;q)_\infty} \sum_{n=0}^{\infty} \frac{(a^2c^2;q)_n q^{n^2/4}}{(q;q)_n\,(a^2c^2;q)_{2n}} \left(-\frac{it}{2}\right)^n \\ &\times p_n(x;a,aq^{1/2},-c,-cq^{1/2}) \\ &\times {}_2\phi_1\left(\begin{matrix} aq^{(2n+1)/4}, -cq^{(2n+1)/4} \\ acq^{n+1/2} \end{matrix}\middle| q^{1/2}, \frac{it}{2}\right). \end{aligned} \tag{4.5}$$

The right sides of (4.4) and (4.5) are identical. This follows from Suslov's addition theorem for the $\mathcal{E}_q$ functions, see [Su, Theorem 3.1].

We have another example when a ${}_4\phi_3$ transformation, not a summation, leads to an $\mathcal{E}_q$ expansion. The Al-Salam-Chihara polynomials are defined by

$$p_n(x;a,b) := r_n(x;a,b,0,0).$$

We have

$$\mathcal{E}_q(x;-i,b)=\sum_{n=0}^{\infty}\frac{(-\gamma^2b^2q^n;q^2)_\infty}{(-b^2;q^2)_\infty}(ib)^nq^{n^2/4}p_n(x;\gamma e^{i\phi},\gamma e^{-i\phi}) \times \gamma^{-n}e^{-in\phi}\frac{(\gamma^2;q)_n}{(q;q)_n}\mathcal{E}_q(\cos\phi;-i,\gamma bq^{n/2}). \tag{4.6}$$

To prove (4.6) we find the coefficient of $b^n p_j(x;\gamma e^{-i\phi},\gamma e^{i\phi})$ on both sides. The left side coefficient (by (2.7) and (2.5)) is a ${}_3\phi_2$ on base q with one denominator parameter equal to 0. The right side coefficient (by the q-binomial theorem) gives a ${}_3\phi_2$ on base q^2. These are equal using [Ga:Ra,(3.10.13), (3.2.2)].

The preceding sketch of the proof of (4.6) is not transparent, so we also prove (4.6) from earlier results in this paper.

The Al-Salam-Chihara polynomials have the generating function [As:Is2]

$$\sum_{n=0}^{\infty}\frac{(t_1t_2;q)_n}{(q;q)_n}(t/t_1)^np_n(\cos\theta,t_1,t_2)=\frac{(tt_1,tt_2;q)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty}.$$

The continuous q-Hermite polynomials, see (3.10), have the generating function [As:Is]

$$\sum_{n=0}^{\infty}H_n(\cos\theta|q)t^n/(q;q)_n=1/(te^{i\theta},te^{-i\theta};q)_\infty.$$

Evidently the above two generating functions imply

$$\frac{H_n(x|q)}{(q;q)_n}=\sum_{j=0}^{n}\frac{H_j(\cos\phi|q)}{(q;q)_j}p_{n-j}(x;\gamma e^{i\phi},\gamma e^{-i\phi})\gamma^{2j-n}e^{-i(n-j)\phi}\frac{(\gamma^2;q)_{n-j}}{(q;q)_{n-j}}. \tag{4.7}$$

We then combine (4.7) and (3.6) to obtain

$$\mathcal{E}_q(x;-i,b)=\frac{1}{(-b^2;q^2)_\infty}\sum_{n=0}^{\infty}(ib)^nq^{n^2/4}p_n(x;\gamma e^{i\phi},\gamma e^{-i\phi})\gamma^{-n}e^{-in\phi}\frac{(\gamma^2;q)_n}{(q;q)_n} \times\sum_{j=0}^{\infty}\frac{H_j(\cos\phi|q)}{(q;q)_j})\gamma^jq^{j^2/4}(ibq^{n/2})^j.$$

The j sum can be expressed in terms of $\mathcal{E}_q$ by (3.6) to obtain (4.6).

It is not hard to see that choosing $\phi=\pi/2$, $\gamma=-i\sqrt{q}$ in (4.6) gives (4.3).

5. Connection coefficients for Askey-Wilson polynomials.

From (2.2) and (2.7) we have

$$r_m(x;\alpha,\beta,\gamma,\delta)=\sum_{j=0}^{m}B_{m,j}r_j(x;a,b,c,d),$$

where

$$B_{m,j} = \frac{(ab, ac, ad, q^{-m}, \alpha\beta\gamma\delta\, q^{m-1}, abcd/q; q)_j}{(\alpha\beta, \alpha\gamma, q, abcd/q; q)_j} \times \left(-\frac{\alpha}{a}\right)^{j(j+1)/2} \sum_{n=0}^{m-j} \frac{(q^{j-m}, \alpha\beta\gamma\delta q^{m+j-1}, \alpha d q^j, \alpha/d; q)_n}{(q, \alpha\beta q^j, \alpha\gamma q^j, \alpha\delta q^j; q)_n} q^n \times {}_4\phi_3\left(\begin{matrix} q^{-n}, adq^j, bdq^j, cdq^j \\ dq^{1-n}/\alpha, \alpha d q^j, abcdq^{2j} \end{matrix} \middle| q, q\right). \tag{5.1}$$

We shall assume that

$$\max(|a|, |b|, |c|, |d|) < 1$$

and that α and a are of the same sign. Askey and Wilson [As:Wi] have shown that the connection coefficients are positive when

$$0 < \alpha < a < 1, \quad b = \beta, \quad c = \gamma, \quad d = \delta.$$

This can be iterated using the symmetry of $p_n(x; a, b, c, d)$ in its parameters. Thus we have proven that the $B_{m,j}$ of (5.1) is nonnegative for

$$0 < \alpha < a < 1, \quad 0 < \beta < b < 1, \quad 0 < \gamma < c < 1, \quad 0 < \delta < d < 1.$$

6. Matrix Inversion.

In this section we prove Theorem 1 from an explicit matrix inversion.

Let

$$\phi_n(x; a) = (ae^{i\theta}, ae^{-i\theta}; q)_n, \quad x = \cos\theta.$$

It follows from [Ga:Ra,(II.12)] that

$$\phi_n(x, \alpha) = \sum_{k=0}^{n} C_{nk}\phi_k(x, a).$$

where

$$C_{nk} = \frac{q^k\,(q^{-n}; q)_k(a\alpha, \alpha/a; q)_n}{(q, a\alpha, aq^{1-n}/\alpha; q)_k}.$$

To establish Theorem 1, it remains to expand $\phi_k(x, a)$ in terms of the Askey-Wilson polynomials $p_j(x; a, b, c, d)$ by inverting

$$p_n(x; a, b, c, d) = \sum_{j=0}^{n} D_{nj}(t)c_j\phi_j(x, a),$$

where

$$D_{nj}(t) = (q^{-n}, tq^n; q)_j, \quad t = abcd/q, \quad 1/c_j = q^{-j}(q, ab, ac, ad; q)_j.$$

It is known (see for example (3.6.19) and (3.6.20) in [Ga:Ra]) that

$$D^{-1}_{jk}(t) = \frac{t^{-k}q^{j-k^2}}{(q, q^{1-2k}/t; q)_k(q, tq^{1+2k}; q)_{j-k}}. \tag{6.1}$$

Thus Theorem 1 follows from the matrix inversion (6.1). Since there is bibasic version of (6.1), there is also a bibasic version of Theorem 1.

We can also find the inverse of the lower triangular matrix $A_{n,j}$ of Theorem 1 by expanding the polynomials $p_j(x;a,b,c,d)$ in terms of $\phi_l(x;\alpha)$. Using (6.1) we find

$$\begin{aligned}A_{j,l}^{-1} =& q^l a^{l-j} \alpha^{-l} (abq^l, acq^l, adq^l;q)_{j-l} \frac{(abcdq^{j-1}, q^{-j};q)_l}{(q;q)_l} \\ &\times {}_4\phi_3 \left(\begin{matrix} q^{l-j}, abcdq^{j+l-1}, a/\alpha, a\alpha q^l \\ abq^l, acq^l, adq^l \end{matrix} \middle| q, q \right),\end{aligned}$$

which gives a discrete orthogonality relation for a ${}_4\phi_3$.

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