SPECIALIZATIONS OF GENERALIZED LAGUERRE POLYNOMIALS

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The second author dedicates this paper to the one and only Dick Askey, his mathematical father.

Abstract. Three specializations of a set of orthogonal polynomials with "8 different q's" are given. The polynomials are identified as q-analogues of Laguerre polynomials, and the combinatorial interpretation of the moments give infinitely many new Mahonian statistics on permutations.

1. Introduction.

The Laguerre polynomials $L_n^\alpha(x)$ have been extensively studied, analytically [E] and combinatorially [F-S],[V1]. There is also a classical set of q-Laguerre polynomials, [M]. Recently, a set of orthogonal polynomials generalizing the Laguerre polynomials has been studied [Si-St]. These polynomials in some sense have "8 different q's". Various specializations of them give orthogonal polynomials associated with many types of combinatorial objects. The purpose of this paper is to present the specializations which are true q-analogues of $L_n^0(x)$. By this we mean that the nth moments, instead of being $n!$, are basically $n!_q$.

We present here three specializations whose moments lead to new Mahonian statistics on permutations (Theorems 2, 3, and 4). In fact, infinitely many Mahonian statistics can be derived from those presented here. Moreover, the theorems obtained from our specializations follow easily from classical analytic facts, but are combinatorially non-trivial.

We shall use the terminology and notation found in [G-R], and let

$$[n]_q = \frac{1-q^n}{1-q}, \quad [n]_{r,s} = \frac{r^n - s^n}{r-s}.$$

2. The polynomials and their moments.

Any set of monic orthogonal polynomials satisfies the three term recurrence relation

$$p_{n+1}(x) = (x - b_n)p_n(x) - \lambda_n p_{n-1}(x). \tag{2.1a}$$

For the set of orthogonal polynomials with 8 different "q's" considered in [Si-St], the coefficients are

$$b_n = a[n+1]_{r,s} + b[n]_{t,u}, \quad \lambda_n = ab[n]_{p,q}[n]_{v,w}. \tag{2.1b}$$

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We refer to the polynomials defined by (2.1) as the *octabasic Laguerre polynomials.* They generalize the Laguerre polynomials and are the polynomials whose specializations we consider in this paper.

The fundamental combinatorial fact (Theorem 1) that we need here concerns the moments for these polynomials. They are generating functions for permutations according to certain statistics.

For the definition of these statistics, it is convenient to represent a permutation σ as a word $\sigma(1)\sigma(2)\cdots\sigma(n)$ consisting of increasing runs, separated by the descents of the permutation. For example, the permutation $\sigma = 26|357|4|189$ has 4 runs separated by 3 descents, and we write $\mathrm{run}(\sigma) = 3$. The runs of length 2 or more will be called *proper runs* and those of length 1 will be called *singleton runs.*

The elements $\sigma(i)$ of σ fall into four classes: the elements which begin proper runs (openers), the elements which close proper runs (closers), the elements which form singleton runs (singletons), and the elements which continue runs (continuators). We shall abbreviate these classes of elements "op", "clos", "sing", and "cont" respectively. In the example, $op(\sigma) = \{3, 2, 1\}$, $clos(\sigma) = \{6, 7, 9\}$, $sing(\sigma) = \{4\}$, and $cont(\sigma) = \{5, 8\}$.

Definition 1. *For $\sigma \in S_n$, the statistics $lsg(\sigma)$ and $rsg(\sigma)$ are defined by*

$$lsg(\sigma) = \sum_{i=1}^{n} lsg(i), \qquad rsg(\sigma) = \sum_{i=1}^{n} rsg(i),$$

where $lsg(i)$ = the number of runs of σ strictly to the left of i which contain elements smaller and greater than i, and $rsg(i)$ = the number of runs of σ strictly to the right of i which contain elements smaller and greater than i.

We also define the lsg and rsg on the openers of σ

$$lsg(op)(\sigma) = \sum_{i \in op(\sigma)} lsg(i), \qquad rsg(op)(\sigma) = \sum_{i \in op(\sigma)} lsg(i).$$

The statistics lsg and rsg have analogous definitions on each of the remaining three classes of elements.

For example, if $\sigma = 26|357|4|189$, then $lsg(7) = 0$, $rsg(7) = 1$, $lsg(op)(\sigma) = 0 + 1 + 0 = 1$, $rsg(op)(\sigma) = 1 + 1 + 0 = 2$, $lsg(clos)(\sigma) = 0$, $rsg(clos)(\sigma) = 2 + 1 + 0 = 3$, etc.

Theorem 1. *The nth moment μ_n for the octabasic Laguerre polynomials is*

$$\mu_n = \sum_{\sigma \in S_n} r^{\mathrm{lsg(sing)}(\sigma)} s^{\mathrm{rsg(sing)}(\sigma)} t^{\mathrm{lsg(cont)}(\sigma)} u^{\mathrm{rsg(cont)}(\sigma)} p^{\mathrm{lsg(op)}(\sigma)} q^{\mathrm{rsg(op)}(\sigma)} v^{\mathrm{lsg(clos)}(\sigma)} w^{\mathrm{rsg(clos)}(\sigma)} a^{\mathrm{run}(\sigma)} b^{n-\mathrm{run}(\sigma)}.$$

Proof. The Viennot theory [V1], [V2] gives μ_n as a generating function for Motzkin paths from $(0, 0)$ to $(n, 0)$. From (2.1b) the paths have 4 types of edges (or steps):

(1) northeast (NE) edges starting at level k with weight in $a[k+1]_{p,q}$,
(2) southeast (SE) edges starting at level k with weight in $b[k]_{v,w}$,
(3) east (E solid) edges starting at level k with weight in $a[k+1]_{r,s}$,
(4) east (E dotted) edges starting at level k with weight in $b[k]_{t,u}$,

where "weight in $\alpha[m]_{c,d}$" means that the weight of the edge is one of the monomials $\alpha c^{m-1}, \alpha c^{m-2}d, \ldots, \alpha d^{m-1}$ which appear in $\alpha[m]_{c,d}$.

The weight of a Motzkin path is then the product of the weights of its edges. An example of such a weighted path is given in Figure 1.

In the statement of Theorem 1 we claim that μ_n is the sum over all permutations in S_n of monomial weights defined in terms of our permutation statistics. To prove the theorem we construct a bijection between the weighted Motzkin paths of length n and permutations in S_n, so that the weight of a path is equal to the weight of its corresponding permutation.

Given a weighted Motzkin path of length n, its corresponding permutation will be constructed in n stages. We begin at the origin and with the empty permutation. We traverse the path from left to right, and the ith step will determine where to insert i in the current (partial) permutation $\sigma_{i-1} \in S_{i-1}$. The end result will be a permutation $\sigma_n = \sigma \in S_n$. Depending on the type of the ith step of the path, i will belong to one or another of the four classes of elements of σ:

(1) NE: $i \in \ op(\sigma)$,
(2) SE: $i \in \ clos(\sigma)$,
(3) E solid: $i \in \ sing(\sigma)$,
(4) E dotted: $i \in \ cont(\sigma)$.

The weight $\alpha c^j d^k$ of the ith step of the path determines, as described below, the exact position in σ_{i-1} where we insert i as a point in the appropriate class of elements.

A run in σ_{i-1} will be called an *active run* if its maximum is a (future) opener or continuator in σ. Notice that the first step of the path starts at level 0 and that σ_0, being the empty permutation, has no active runs. Inductively, assume that the level h at which the ith step starts is equal to the number of active runs in σ_{i-1}, and recall the relation between the weight $\alpha c^j d^k$ of a step and its starting level h: for NE and E solid steps we have $j+k = h$, while for other steps we have $j+k = h-1$.

If the ith step is E dotted or SE, then the element i is adjoined to the $(j+1)^{st}$ active run of σ_{i-1}, as a continuator or closer, respectively. This is well-defined since, as discussed above, $j+k$ is one unit less than the number of active runs in σ_{i-1}. It follows as well that the new partial permutation, σ_i will have as many active runs as the starting level of the $(i+1)^{st}$ step of the path.

If the ith step is NE or E solid, then $j+k$ is equal to the number of active runs in σ_{i-1} and we insert i in the leftmost position possible such that it will have j of the *active* runs of σ_{i-1} strictly to its left and k of the *active* runs of σ_{i-1} strictly to its right. The position where i is inserted in this case ensures that i will be the initial element of a run, and it is again true that σ_i will have as many active runs as the starting level of the $(i+1)^{st}$ step of the path.

Note that in the final permutation σ the values of $lsg(i)$ and $rsg(i)$ are completely determined by the runs which were active in σ_{i-1} and the position, relative to these runs, where i was inserted. So, if the ith step has weight $\alpha c^j d^k$, then $lsg(i) = j$ because each of the j open runs of σ_{i-1} which remain to the left of i upon its insertion will eventually be extended by at least one element greater than i. Similarly, $rsg(i) = k$. Finally, $\alpha = a$ in the weight of the ith step corresponds precisely to i being the initial (possibly the only) element of a run. Consequently, the weight of a Motzkin path is equal to our intended weight for the corresponding permutation σ.

It remains to verify that this correspondence is bijective. We claim that, given $\sigma \in S_n$, we can reconstruct its associated Motzkin path, step by step, beginning at the right end of the path, $(n, 0)$, since each stage of our construction is reversible.

Using our rules (1)-(4), each permutation in S_n will produce a Motzkin path of length n (not yet weighted), since $|op(\sigma)| = |clos(\sigma)|$ and each closer is larger than its corresponding opener. We must check that the weight $\alpha c^{lsg(i)} d^{rsg(i)}$ (where $\alpha = a$ if i is the initial element of a run, and $\alpha = b$ otherwise) is a valid weight for the ith step of the path. This follows immediately from the equality $lsg(i)+rsg(i) =$ the level of the left endpoint e_{i+1} of the partial path reconstructed from the values $n, n-1, \ldots, i+1$. The equality holds for $i = n$ since $lsg(n) = rsg(n) = 0$ in all permutations of S_n, and the left endpoint of the one-point path consisting just of $(n, 0) = e_{n+1}$ is at level 0. Suppose the equality holds for $i+1$ and we will prove it for i. Observe that the level of e_{i+1} is equal to the number of SE steps minus the number of NE steps in the partial path from e_{i+1} to $(n, 0)$. That is, the level of e_{i+1} is equal to the number of proper runs in σ whose maximum is larger than i, minus the number of proper runs in σ whose minimum is larger than i. But this is equal in turn with the number of proper runs in σ with maximum larger than i and minimum smaller than i, i.e., it is equal to $lsg(i)+rsg(i)$. It now becomes clear that our map from weighted Motzkin paths to permutations is indeed invertible. $\square$

Figure 1 shows the weighted Motzkin path which corresponds with the permutation $\sigma = 10|89\,11|137|5|46|2$. We have also included a binary tree representation of the permutation, deeming it of possible interest to the readers familiar with [V1]. The bijection constructed in the proof above is related to Viennot's correspondence between Motzkin paths and permutations [V1]. As an intermediate step in Viennot's correspondence, Motzkin paths and permutations are encoded by increasingly labeled binary trees.

We will be concerned with specializations of the 8 q's under which the moments in Theorem 1 become multiples of $n!_q = [n]_q[n-1]_q \cdots [1]_q$. It is clear that the parameter b can be rescaled to 1. Also, in considering specializations, we can take advantage of the property — obvious from (2.1) — that the moments are fixed under the interchange of $\{r, s\}$, $\{t, u\}$, $\{p, q\}$, and $\{v, w\}$, and also fixed if p and q are interchanged with v and w.

3. The specializations.

In this section we state three different specializations of the polynomials defined by (2.1). The polynomials which arise are monic little q-Jacobi, sums of two little q-Jacobi, and classical q-Laguerre. Each of these three cases will have moments which are basically $n!_q$.

First we choose the parameters so that the polynomials coincide with the monic form of the little q-Jacobi polynomials [G-R, p. 166], $p_n(xq(1-q); q^\alpha, 0; q)$, which have

$$b_n = q^{n-1}[n+1+\alpha]_q + q^{n+\alpha-1}[n]_q, \quad \lambda_n = q^{2n-3+\alpha}[n]_q[n+\alpha]_q. \tag{3.1}$$

The appropriate specialization occurs only for $\alpha = 0$, $p_n(xq(1-q); 1, 0; q)$, and is $r = t = p = v = q^2$, $s = u = q = w$, $a = 1/q$, $b = 1$. The explicit formula for the polynomials, which is just the definition of the little q-Jacobi polynomials, is

$$p_n(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q [n]_q \cdots [n-k+1]_q (-1)^k x^{n-k} q^{\binom{k-1}{2}-1}. \tag{3.2}$$

The measure for $p_n(x;q^\alpha,q^\beta;q)$ is purely discrete, with masses of

$$\frac{q^{(\alpha+1)i}(q^{\beta+1};q)_i(q^{\alpha+1};q)_\infty}{(q;q)_i(q^{\alpha+\beta+2};q)_\infty}$$

at $x=q^i$. An easy calculation shows that the moments are given by $\mu_n = \frac{(q^{\alpha+1};q)_n}{(q^{\alpha+\beta+2};q)_n}$. Thus for $p_n(xq(1-q);1,0;q)$, we have $\mu_n = q^{-n}n!_q$. Based upon these remarks, Theorem 1 becomes the following theorem. An equivalent theorem has been given in [deM-V, Prop. 5.2].

Theorem 2. *For $\sigma \in S_n$, let*

$$s(\sigma) := n - run(\sigma) + 2lsg(\sigma) + rsg(\sigma).$$

Then

$$\sum_{\sigma\in S_n} q^{s(\sigma)} = n!_q.$$

Moreover, we see from the symmetry of (2.1) with respect to the 4 pairs of "q's" that Theorem 2 holds for 16 statistics related to $s(\sigma)$. These 16 are obtained by choosing the coefficients 1 and 2 for lsg and rsg independently for the four types of elements of σ. This means for example that

$$\begin{aligned} s'(\sigma) = n - \mathrm{run}(\sigma) &+ \mathrm{lsg(sing)}+2\mathrm{rsg(sing)}+\mathrm{lsg(op)}+2\mathrm{rsg(op)} \\ &+ 2\mathrm{lsg(cont)}+\mathrm{rsg(cont)}+2\mathrm{lsg(clos)}+\mathrm{rsg(clos)} \end{aligned}$$

also satisfies Theorem 2. We will see later that in fact there are infinitely many equidistributed statistics related to $s(\sigma)$.

For our second specialization we consider

$$b_n = q^{n+1}[n+1]_q + q^{n-1}[n]_q, \quad \lambda_n = q^{2n-1}[n]_q[n]_q. \tag{3.3}$$

The appropriate values are $r=t=p=v=q^2$, $s=u=q=w$, $a=q$, $b=1$. The polynomials turn out to be a sum of two little q-Jacobi polynomials,

$$\begin{aligned} p_n(x) = n!_q q^{\binom{n}{2}}(-1)^n \Big[{}_2\phi_1\left(\begin{matrix} q^{-n} & 0; & q, & xq(1-q) \\ & q & & \end{matrix}\right) \\ -(1-q^n){}_2\phi_1\left(\begin{matrix} q^{1-n} & 0; & q, & xq(1-q) \\ & q^2 & & \end{matrix}\right)\Big], \end{aligned}$$

or equivalently

$$p_n(x) = x^n + \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q [n]_q \cdots [n-k+2]_q([n-k]_q+q^n)(-1)^k x^{n-k} q^{\binom{k}{2}} + (-1)^n q^{\binom{n+1}{2}} n!_q. \tag{3.4}$$

We omit the proof of these formulas. It is a verification of the recurrence relation (2.1) from the recurrence relation for the little q-Jacobi polynomials.

Since these polynomials do not explicitly appear in the literature, we cannot compute the moments by quoting the relevant facts about their measure. Nonetheless, the moments and measure are easily determined.

Proposition 1. *The moments for the polynomials in (3.3) are $\mu_0 = 1$, and $\mu_n = q\,n!_q, n > 0$. The measure is purely discrete, with masses of $q^i(q;q)_\infty/(q;q)_{i-1}$ at $q^{i-1}/(1-q)$, $i \geq 1$, and a mass of $1-q$ at 0.*

Proof. The q-binomial theorem clearly shows that the total mass $\mu_0 = (1-q)+q = 1$. It also implies

$$\begin{aligned}\mu_n &= \sum_{i=1}^{\infty} \frac{q^{in} q^i (q;q)_\infty}{(1-q)^n (q;q)_{i-1}} + (1-q)\delta_{n,0} \\ &= qn!_q + (1-q)\delta_{n,0}.\end{aligned}$$

Thus the stated measure has the right moments. To show that the polynomials are orthogonal with respect to this measure, note that it is easy to check, from the explicit formula (3.4), that the moments annihilate $p_1, p_2, \ldots$. Hence, the linear functional defined by the measure annihilates $p_1, p_2, \ldots$. Finally, the three terms recurrence now shows that the polynomials are indeed orthogonal. □

We then get a companion theorem to Theorem 2. As in the case of Theorem 2, we have 16 equivalent versions of the statistic $s(\sigma)$, by assigning coefficients 1 and 2 to lsg and rsg independently on openers, continuators, closers and singletons.

Theorem 3. *For $\sigma \in S_n$, let*

$$s(\sigma) := run(\sigma) - 1 + 2lsg(\sigma) + rsg(\sigma).$$

Then

$$\sum_{\sigma \in S_n} q^{s(\sigma)} = n!_q.$$

We remark that Theorems 2 and 3 are valid for an infinite number of variations of the statistic $s(\sigma)$. It is easy to verify that for each $\sigma \in S_n$

$$lsg(op)(\sigma) + rsg(op)(\sigma) = lsg(clos)(\sigma) + rsg(clos)(\sigma). \tag{3.5}$$

(In fact there is a specialization of $\{r, s, t, u, p, q, v, w\}$ with one free parameter giving (3.5).) Therefore the value of $s(\sigma)$ remains the same if the coefficients $\{1, 2\}$ are replaced on the openers with $\{1+c, 2+c\}$, and on the closers with $\{1-c, 2-c\}$. This provides a variation of Theorems 2 and 3 for each choice of the real parameter c. For example, $c = 1$ gives the unusual choice of coefficients $\{2, 3\}$ and $\{0, 1\}$.

Our third choice for specialization is the set of the classical q-Laguerre polynomials $L_n^\alpha(x(1-q);q)$ [M], [G-R, p. 194], whose monic form has

$$b_n = q^{-2n-\alpha}[n]_q + q^{-2n-1-\alpha}[n+1+\alpha]_q, \quad \lambda_n = q^{1-4n-2\alpha}[n]_q[n+\alpha]_q. \tag{3.6}$$

The appropriate values are $r = t = p = v = q^{-2} = b$, $s = u = q = w := q^{-1} = a$ for $L_n^0(x(1-q), q)$. The explicit formula for the monic form of $L_n^\alpha(x(1-q);q)$ is

$$p_n(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q [n+\alpha]_q \cdots [n+\alpha-k+1]_q (-1)^k x^{n-k} q^{k(k-\alpha-2n)}. \tag{3.7}$$

Again a measure of these polynomials is explicitly known [M, Th.1], and the moments for $L_n^\alpha(x(1-q);q)$ can be found as

$$\mu_n = (q^{\alpha+1};q)_n q^{-n\alpha-\binom{n+1}{2}}/(1-q)^n.$$

For $\alpha = 0$ this is $q^{-\binom{n+1}{2}} n!_q$. However, the combinatorial version of this theorem is equivalent to Theorem 2, if q is replaced with $1/q$. Thus no new combinatorial theorem results.

4. The "odd" polynomials.

If $r = p$, $s = q$, $t = v$, and $u = w$, then the polynomials defined by (2.1) are the "even" polynomials for the polynomials defined by (see [C, p.41])

$$b_n = 0, \quad \lambda_{2n} = b[n]_{t,u}, \quad \lambda_{2n+1} = a[n+1]_{r,s}.$$

The "odd" polynomials have the coefficients

$$b_n = a[n+1]_{r,s} + b[n+1]_{r,s}, \quad \lambda_n = ab[n+1]_{r,s}[n]_{t,u}. \tag{4.1}$$

The moments for these "odd" polynomials satisfy $\mu_n(odd) = \mu_{n+1}(even)/\mu_1(even)$. Since all of our specializations in §3 satisfied $r = p$, $s = q$, $t = v$, and $u = w$, these "odd" polynomials also have moments which are multiples of $(n+1)!_q$. There is a version of Theorem 1 for the "odd" polynomials which yields more statistics related to $s(\sigma)$. We do not state this combinatorial theorem here, rather in this section we state what these "odd" polynomials are, give their moments, and state in Theorem 4 what the statistics related to $s(\sigma)$ are. Clearly the "odd" polynomials are analogues of the Laguerre polynomials $L_n^1(x)$.

We keep the parameters r, s, t, u. This specialization gives the "even" and "odd" polynomials a combinatorial interpretation as weighted versions of injective maps (see [F-S], $L_n^0(x)$ and $L_n^1(x)$). This family with "4 q's" also contains other families of orthogonal polynomials of combinatorial interest which are discussed in [Si-St].

We list here the "odd" polynomials for the three cases in §3, and the respective moments. In each case the polynomials are monic forms of the given polynomials.

(1) little q-Jacobi $p_n(xq(1-q); q, 0; q)$, $\mu_n = q^{-n}\,(n+1)!_q$,
(2) little q-Jacobi $p_n(x(1-q); q, 0; q)$, $\mu_n = (n+1)!_q$,
(3) q-Laguerre $L_n^1(x(1-q); q)$, $\mu_n = q^{-(n^2+3n)/2}\,(n+1)!_q$.

The combinatorial theorem that results is Theorem 4 below. To describe the suitable statistic $s(\sigma)$ we shall need an auxillary statistic, $n(\sigma)$, defined as follows. For a given permutation $\sigma \in S_n$, let d be the largest element in the same run as 1, $d \geq 1$. Now partition the elements of σ into three classes: elements to the left of 1, elements in the same run as 1, and those to the right of d. Suppose that over the portion of σ to the right of d no left-to-right minimum constitutes a singleton run. Put $n(\sigma) = 0$. Otherwise, let c be the last (rightmost, smallest) singleton which is a left-to-right minimum on the portion of σ to the right of d. Let $nleft(\sigma) := \#\{i : i < \sigma^{-1}(1),\ c < \sigma(i) < d\}$. In this case put $n(\sigma) = 2(d-c) - nleft(\sigma)$. For example, $n(9 \mid 1\ 5\ 7 \mid 2\ 6 \mid 4 \mid 3\ 8) = 0$, while for $\sigma = 7\ 12 \mid 1\ 6\ 9 \mid 3 \mid 2\ 10\ 11 \mid 5 \mid 4\ 8$ we have $d = 9$, $c = 3$, $nleft(\sigma) = 1$ (from the element 7), and we get $n(\sigma) = 2(9-3)-1 = 11$.

We also need variations lsg^* and rsg^* on the statistics lsg and rsg. These differ from the original statistics in two respects. For closers and singletons (the maxima of the runs), the run containing the element 1 is ignored in the calculation of lsg^* and rsg^*. For openers and continuators, the run containing 1 is always counted in lsg^* (if it is to the left) or in rsg^* (if it is to the right).

For example, if $\sigma = 7\ 12 \mid 1\ 6\ 9 \mid 3 \mid 2\ 10\ 11 \mid 5 \mid 4\ 8$, then $lsg^*(5) = 1$, $lsg^*(10) = 2$, $lsg^*(\sigma) = 10$, $rsg^*(\sigma) = 8$.

Theorem 4. *For $\sigma \in S_n$, let*

$$s(\sigma) := run(\sigma) - 1 + 2lsg^*(\sigma) + rsg^*(\sigma) + n(\sigma).$$

Then

$$\sum_{\sigma \in S_n} q^{s(\sigma)} = n!_q.$$

Theorem 4 also holds if $run(\sigma) - 1$ is replaced by $n - run(\sigma)$. Moreover, the role of closers and openers can be interchanged in Theorem 4, and there are also 16 variations, although complicated ones.

A version of Theorem 4 holds for permutations in S_{n+1} which satisfy the following condition: 1 and $n+1$ belong to the same run, and no left-to-right minimum to the right of $n+1$ constitutes a singleton run. There are $n!$ such permutations in S_{n+1}.

Finally, we remark that these specializations are the only ones we have found for which the moments factor into an analogue of $n!$. They are also the only specializations which give the three sets of polynomials that were considered. A more extensive study of the specializations of (2.1) appears in [Si-St].

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