Applications of q-Taylor theorems *

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Abstract

We establish two new q-analogues of a Taylor series expansion for polynomials using special Askey-Wilson polynomial bases. Combining these expansions with an earlier expansion theorem we derive inverse relations and evaluate certain linearization coefficients. Byproducts include new summation theorems, new results on a q-exponential function, and quadratic transformations for q-series.

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1 Introduction

The Taylor theorem for polynomials $f(x)$ evaluates the coefficients f_k in the expansion

$$f(x) = \sum_{k=0}^{\infty} f_k (x-c)^k, \quad f_k = \frac{f^{(k)}(c)}{k!}. \tag{1.1}$$

It is possible to generalize (1.1) by considering other polynomial bases and suitable operators. One such example, which has been previously considered

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[5], replaces $(x-c)^k$ by

$$\phi_k(x;a) = (ae^{i\theta}, ae^{-i\theta};q)_k = \prod_{i=0}^{k-1}(1-2axq^i+a^2q^{2i}).$$

Since

$$\lim_{q\to 1}\phi_k(x;a) \quad = \quad (1-2ax+a^2)^k$$

we can consider $\phi_k(x;a)$ as a q-analogue of $(x-c)^k$ for $c = a+1/a$. The Taylor theorem for $\phi_k(x;a)$ is stated as Theorem 1.1 below. In this paper we consider two other q-analogues of (1.1): Theorem 2.1 which has a q-analogue of $(x-1)^k$ and Theorem 2.2 for a q-analogue of x^k. We shall follow the notations and terminology in [1] and [4].

All three theorems use polynomial bases and the Askey-Wilson operator. We first define the Askey-Wilson operator $\mathcal{D}_q$. Given a function f we set $\breve{f}(e^{i\theta}) := f(x),\;\; x = \cos\theta$, that is

$$\breve{f}(z) = f((z+1/z)/2), \quad z = e^{i\theta}.$$

In other words we think of $f(\cos\theta)$ as a function of $e^{i\theta}$. In this notation the Askey-Wilson divided difference operator $\mathcal{D}_q$ is defined by

$$(1.2) \qquad (\mathcal{D}_q f)(x) = \frac{\breve{f}(q^{1/2}e^{i\theta}) - \breve{f}(q^{-1/2}e^{i\theta})}{(q^{1/2}-q^{-1/2})\; i\;\sin\theta}, \quad x = \cos\theta.$$

It easy to see that the action of $\mathcal{D}_q$ on Chebyshev polynomials is given by

$$\mathcal{D}_q T_n(x) = \frac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}}U_{n-1}(x),$$

hence $\mathcal{D}_q$ reduces the degree of a polynomial by one and

$$\lim_{q\to 1}\mathcal{D}_q = \frac{d}{dx}.$$

In the calculus of the Askey-Wilson operator [3, p. 32] the basis $\{\phi_n(x;a) : n \geq 0\}$ plays the role played by the monomials $\{(1-2ax+a^2)^n : n \geq 0\}$ in the differential and integral calculus. In fact

$$(1.3) \;\; \mathcal{D}_q(ae^{i\theta}, ae^{-i\theta};q)_n = -\frac{2a(1-q^n)}{1-q}(aq^{1/2}e^{i\theta}, aq^{1/2}e^{-i\theta};q)_{n-1}.$$

Ismail [5] proved the following Taylor theorem for polynomials $f(x)$.

Theorem 1.1. *If $f(x)$ is a polynomial in x of degree n, then*

$$f(x) = \sum_{k=0}^{n} f_k \phi_k(x; a),$$

where

$$f_k = \frac{(q-1)^k}{(2a)^k (q;q)_k} q^{-k(k-1)/4} (\mathcal{D}_q^k f)(x_k)$$

and x_k is given by

$$x_k := \frac{1}{2}\left(aq^{k/2} + q^{-k/2}/a\right).$$

In our recent work [7] it was realized that the basis $\{\phi_n(x) : n \geq 0\}$,

$$\phi_n(\cos\theta) = (q^{1/4}e^{i\theta}, q^{1/4}e^{-i\theta}; q^{1/2})_n, \tag{1.4}$$

plays an important role in the calculus of the Askey-Wilson operators and basic hypergeometric functions. Rahman [11], [3, p. 23] had previously used this basis for expressing continuous q-Jacobi polynomials [3] in the basis $\{\phi_n(x)\}$. Since

$$\lim_{q\to 1} \phi_n(x) = 2^n(1-x)^n,$$

we consider $\phi_n(x)$ as a q-analogue of $(x-1)^n$. We shall also see that the basis $\{\psi_n(x) : n \geq 0\}$,

$$\psi_n(\cos\theta) = (1+e^{2i\theta})(-q^{2-n}e^{2i\theta}; q^2)_{n-1} e^{-in\theta}, \tag{1.5}$$

has a nice relationship to the Askey-Wilson operator. Since

$$\lim_{q\to 1} \psi_n(x) = 2^n x^n$$

we consider $\psi_n(x)$ as a q-analogue of x^n. Note that as functions of θ the polynomials $\{\psi_n(\cos\theta)\}$ are essentially partial theta functions [14] since

$$\begin{aligned} \psi_{2n}(\cos\theta) &= q^{n(1-n)}(-e^{2i\theta}, -e^{-2i\theta}; q^2)_n, \\ \psi_{2n+1}(\cos\theta) &= 2q^{-n^2} \cos\theta\, (-qe^{2i\theta}, -qe^{-2i\theta}; q^2)_n. \end{aligned}$$

In this paper we give q-Taylor theorems for polynomials using $\{\phi_n(x)\}$ and $\{\psi_n(x)\}$ in Theorems 2.1 and 2.2. We then explore consequences of these results to connection coefficient problems. They are applied to obtain connection coefficient results in Theorems 2.3 and 2.4. These two theorems are then used to give a simple proof of our new [7] representation (Corollary 2.5) for $\mathcal{E}_q$, the addition theorem for the q-exponential function $\mathcal{E}_q$, (Corollary 2.6) [6], [13], and another new representation for $\mathcal{E}_q$ (Corollary 2.7). In §3 we explicitly evaluate the coefficients in the expansion of a product $\phi_m(x;a)\phi_n(x;b)$ in terms of $\{\phi_k(x;c)\}$. In §4 all three theorems are used to derive expansions for the continuous q-ultraspherical polynomials (Propositions 4.1, 4.2, and 4.3) which include known quadratic transformations. Section 5 contains remarks on representations of continuous q-ultraspherical polynomials and implications of Theorems 1.1 and 2.1.

We note that the q-Taylor expansions derived and applied here are different from the recent results in [10], [12].

2 More q-Taylor theorems

In this section we give the version of Theorem 1.1 for $\{\phi_n(x)\}$ and $\{\psi_n(x)\}$, which are Theorem 2.1 and Theorem 2.2. We apply the resulting facts to the q-exponential function $\mathcal{E}_q$, in Corollaries 2.5, 2.6 and 2.7.

It is straightforward to see that

$$\mathcal{D}_q \phi_n(x) = -2q^{1/4}\frac{1-q^n}{1-q}\,\phi_{n-1}(x). \tag{2.1}$$

Theorem 2.1. *If $f(x)$ is a polynomial in x of degree n, then*

$$f(x) = \sum_{k=0}^{n} f_k \phi_k(x),$$

where

$$f_k = \frac{(q-1)^k}{2^k q^{k/4}(q;q)_k}(\mathcal{D}_q^k f)(\zeta_0)$$

and $\zeta_0 = (q^{1/4}+q^{-1/4})/2$.

Proof. The proof is an immediate consequence of (2.1), since $\phi_n(\zeta_0) = 0$ for $n \geq 1$. □

It is important to contrast the series expansion of Theorem 1.1 with evaluations at variable points x_k and that of Theorem 2.1, where the coefficients depend on evaluations at a fixed point ζ_0.

For $\{\psi_n(x)\}$ we have the following result, which uses

$$\mathcal{D}_q \psi_n(x) = 2q^{(1-n)/2} \frac{1-q^n}{1-q}\, \psi_{n-1}(x). \tag{2.2}$$

Theorem 2.2. *If $f(x)$ is a polynomial in x of degree n, then*

$$f(x) = \sum_{k=0}^{n} f_k \psi_k(x),$$

where

$$f_k = \frac{q^{(k^2-k)/4}(1-q)^k}{2^k (q;q)_k} (\mathcal{D}_q^k\, f)(0)\ .$$

Proof. The proof is an immediate consequence of (2.2), since if $e^{2i\theta} = -1$, namely $\theta = \pi/2$, $\cos\theta = 0$, $\psi_n(0) = 0$ for $n \geq 1$. □

The first application of Theorem 2.1 is to expand $\phi_n(x;a)$ in terms of $\{\phi_k(x) : 0 \leq k \leq n\}$.

Theorem 2.3. *The following summation theorem holds*

$$\begin{aligned}&\frac{(ae^{i\theta}, ae^{-i\theta};q)_n}{(aq^{-1/4};q^{1/2})_{2n}} \\ &= {}_4\phi_3\left(\begin{matrix} q^{-n/2}, -q^{-n/2}, q^{1/4}e^{i\theta}, q^{1/4}e^{-i\theta} \\ -q^{1/2},\ aq^{-1/4},\ q^{-n+3/4}/a \end{matrix} \,\middle|\, q^{1/2}, q^{1/2} \right).\end{aligned}$$

Proof. Use Theorem 2.1 and (1.3). □

We now give a direct proof of Theorem 2.3.

Proof. Use the Sears transformation [4, (III.15)] with

$$\begin{aligned} &A = -q^{-n/2},\ \ B = q^{1/4}e^{i\theta},\ \ C = q^{1/4}e^{-i\theta} \\ &D = aq^{-1/4},\ \ E = -q^{1/2},\ \ F = q^{-n+3/4}/a. \end{aligned}$$

Then use the quadratic transformation [4, (III.21)] with

$$C = q^{-n/2}, D = -q^{-n/2},\ \ A^2 = aq^{-1/2}e^{-i\theta},\ \ B^2 = aq^{-1/2}e^{i\theta}.$$

The result is that the right-hand side of the equation in Theorem 2.3 is

$$\frac{(q^{(n+1)/2}, -q^{(-n+3/2)/2}/a; q^{1/2})_n}{(-q^{1/2}, q^{-n+3/4}/a; q^{1/2})_n} (-1)^n q^{-n^2/2}$$
$$\times {}_3\phi_2 \left(\begin{matrix} aq^{-1/2}e^{i\theta}, aq^{-1/2}e^{-i\theta}, q^{-n} \\ a^2q^{-1/2}, q^{-n+1/2} \end{matrix} \middle| q, q \right).$$

The ${}_3\phi_2$ is balanced and its sum is $(ae^{i\theta}, ae^{-i\theta}; q)_n/(a^2q^{-1/2}, q^{1/2}; q)_n$, see [4, (II.12)]. □

Although Theorem 2.3 can be proved from the existing literature we nevertheless believe it is interesting and is worth recording as a sum of a special balanced ${}_4\phi_3$.

Since Theorem 2.2 is just an expansion of $\phi_n(x; a)$ in terms of $\{\phi_k(x)\}$, it is natural to record the inverse relation expanding $\phi_n(x)$ in $\{\phi_k(x; a)\}$. The result is

$$\frac{(q^{1/4}e^{i\theta}, q^{1/4}e^{-i\theta}; q^{1/2})_n}{(q^{1/4}a, q^{1/4}/a; q^{1/2})_n} = {}_3\phi_2 \left(\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ aq^{(1-2n)/4}, \ aq^{(3-2n)/4} \end{matrix} \middle| q, q \right). \tag{2.3}$$

The ${}_3\phi_2$ in (2.3) is balanced, hence it can be summed by [4, (II.12)], proving the result.

We next expand $\phi_n(x; a)$ in terms of $\{\psi_k(x) : 0 \le k \le n\}$.

Theorem 2.4. *The following summation theorem holds*

$$(ae^{i\theta}, ae^{-i\theta}; q)_n = \quad = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} (-a)^k (-a^2q^k; q^2)_{n-k} \psi_k(\cos\theta).$$

Proof. Use Theorem 2.2 and (1.3). □

We next give several corollaries to Theorem 2.3 and Theorem 2.4, which concern the q-exponential functions of [9]

$$\mathcal{E}_q(\cos\theta; t) = \frac{(t^2; q^2)_\infty}{(qt^2; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-it)^n}{(q; q)_n} q^{n^2/4} \times (-iq^{(1-n)/2}e^{i\theta}, -iq^{(1-n)/2}e^{-i\theta}; q)_n, \tag{2.4}$$

$$
(2.5) \qquad \begin{aligned}
&\mathcal{E}_q(\cos\theta, \cos\phi; t) \\
&= \frac{(t^2;q^2)_\infty}{(qt^2;q^2)_\infty} \sum_{n=0}^{\infty} (-e^{i(\phi+\theta)} q^{(1-n)/2}, -e^{i(\phi-\theta)} q^{(1-n)/2}; q)_n \\
&\qquad \times \frac{(te^{-i\phi})^n}{(q;q)_n} q^{n^2/4}.
\end{aligned}
$$

Note that (2.4) is an expansion for $\mathcal{E}_q(x;t)$ in terms of $\{\phi_n(x; -iq^{(1-n)/2})\}$, so it reasonable to find the expansion in the bases $\{\phi_n(x)\}$ and $\{\psi_n(x)\}$. Corollary 2.5 gives the $\{\phi_n(x)\}$ expansion, and Corollary 2.7 gives the $\{\psi_n(x)\}$ expansion.

Corollary 2.5. *The function $\mathcal{E}_q(x;t)$ has the representations*

$$
\begin{aligned}
\mathcal{E}_q(\cos\theta;t) &= \frac{(-t;q^{1/2})_\infty}{(qt^2;q^2)_\infty} \, {}_2\phi_1\left(\left. \begin{matrix} q^{1/4}e^{i\theta}, q^{1/4}e^{-i\theta} \\ -q^{1/2} \end{matrix} \right| q^{1/2}, -t \right) \\
&= \frac{(t;q^{1/2})_\infty}{(qt^2;q^2)_\infty} \, {}_2\phi_1\left(\left. \begin{matrix} -q^{1/4}e^{i\theta}, -q^{1/4}e^{-i\theta} \\ -q^{1/2} \end{matrix} \right| q^{1/2}, t \right).
\end{aligned}
$$

Corollary 2.6. *We have*

$$
\mathcal{E}_q(\cos\theta, \cos\phi; t) = \mathcal{E}_q(\cos\theta;t)\mathcal{E}_q(\cos\phi;t).
$$

Corollary 2.7. *The function $\mathcal{E}_q(x;t)$ has the expansion formula*

$$
\mathcal{E}_q(\cos\theta;t) = \sum_{k=0}^{\infty} \frac{(1+e^{2i\theta})(-e^{2i\theta}q^{-k};q^2)_k}{(q;q)_k(1+e^{2i\theta}q^{-k})} q^{k^2/4} t^k e^{-ik\theta}.
$$

It must emphasized that the first equation in Corollary 2.5 says that the q-Taylor expansion of Theorem 2.1 holds for the function $f(x) = \mathcal{E}_q(x;t)$, because

$$
\begin{aligned}
\mathcal{D}_q \mathcal{E}_q(\cos\theta;t) &= \frac{2tq^{1/4}}{1-q} \mathcal{E}_q(\cos\theta;t), \\
\mathcal{E}_q(\zeta_0;t) &= \frac{(-t;q^{1/2})_\infty}{(qt^2;q^2)_\infty}.
\end{aligned}
$$

Similarly Theorem 2.2 holds for $f(x) = \mathcal{E}_q(x;t)$ because

$$
\mathcal{E}_q(0;t) = 1.
$$

Corollary 2.5 is Corollary 4.3 in [7], where two other proofs are given. Corollary 2.6 is the addition theorem for $\mathcal{E}_q$; [13], [6]; and Corollary 2.7 is new.

Proof. We prove Corollary 2.5 and 2.6 simultaneously from Theorem 2.3. We rewrite Theorem 2.3 as

$$\phi_n(x;a) \quad = \sum_{k=0}^{n} \frac{a^k q^{k(k-2)/4}\,(q;q)_n}{(q;q)_k(q;q)_{n-k}}\,(aq^{(2k-1)/4};q^{1/2})_{2n-2k}\phi_k(x).$$

Thus

$$\sum_{n=0}^{\infty} \frac{q^{n^2/4}t^n a^n}{(q;q)_n}\phi_n(x;aq^{(1-n)/2})$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{a^{n+k}t^n q^{(n-k)^2/4}}{(q;q)_k(q;q)_{n-k}}\,(aq^{(k-n+1/2)/2};q^{1/2})_{2n-2k}\;\phi_k(x).$$

After replacing n by $n+k$ the n-sum is

$$\sum_{n=0}^{\infty} \frac{a^{2n}t^n(-1)^n}{(q;q)_n}(aq^{1/4},q^{1/4}/a;q^{1/2})_n.$$

Therefore

$$\text{(2.6)} \qquad \frac{(t^2;q^2)_\infty}{(qt^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2/4}t^n a^n}{(q;q)_n}\phi_n(x;aq^{(1-n)/2})$$
$$= \frac{(t^2;q^2)_\infty}{(qt^2;q^2)_\infty}\left[\sum_{n=0}^{\infty}\frac{(-a^2t)^n}{(q;q)_n}(aq^{1/4},q^{1/4}/a;q^{1/2})_n\right]$$
$$\times\left[\sum_{k=0}^{\infty}\frac{a^{2k}t^k}{(q;q)_k}\phi_k(x)\right]$$

Equation (2.6) proves both Corollary 2.5 and 2.6. If $a=\pm i$, the left side of (2.6) is the definition (2.4) of $\mathcal{E}_q(\cos\theta;t)$, while the n-sum on the right side is evaluable to infinite products by the q-binomial theorem. The k-sum is the ${}_2\phi_1$ for both equations in Corollary 2.5. For Corollary 2.6, replace a by $-e^{i\phi}$ and t by $-e^{-2i\phi}t$ in (2.6) and use Corollary 2.5.

The identical steps may be performed using Theorem 2.4 to find the $\{\psi_n\}$ analogue of (2.6)

$$\text{(2.7)} \qquad \frac{(t^2;q^2)_\infty}{(qt^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2/4}t^n a^n}{(q;q)_n}\phi_n(x;aq^{(1-n)/2})$$
$$= \frac{(t^2;q^2)_\infty}{(qt^2;q^2)_\infty}\left[\sum_{k=0}^{\infty}\frac{(-a^2t)^k q^{k^2/4}}{(q;q)_k}\psi_k(x)\right]$$
$$\times\left[\sum_{n=0}^{\infty}\frac{(-a^2q^{1-n};q^2)_n}{(q;q)_n}a^n t^n q^{n^2/4}\right]$$

This time the choice $a = \pm i$ in (2.7) allows the n-sum to be evaluated, and the result is Corollary 2.7. As before putting $a = -e^{i\phi}$ and $t = -e^{-2i\phi}t$ gives Corollary 2.6 and as bonus $\mathcal{E}_q(0, \cos\theta; t) = \mathcal{E}_q(\cos\theta; t)$. $\square$

Corollary 2.7 may be also proven by splitting the definition (2.4) into the even and odd terms, applying a ${}_2\phi_1$ transformation to each, and recombining the terms.

Because the polynomials $\{\phi_n(x; aq^{(1-n)/2})\}$ are fundamental to the study of $\mathcal{E}_q(\cos\theta; t)$, it is worthwhile to record another connection coefficient result which is equivalent to the addition theorem in Corollary 2.6.

Corollary 2.8. *The polynomials $\{\phi(x; aq^{(1-n)/2})\}$ satisfy the connection relation*

$$\begin{aligned}\phi_n(x; aq^{(1-n)/2}) \quad &= \quad \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1+a^2}{1+a^2q^{-k}} (-a^2q^{-k}; q^2)_k q^{-k(n-k)/2} (ia)^{n-k} \\ &\quad \times \phi_{n-k}(x; -iq^{(1-n+k)/2}).\end{aligned}$$

Proof. If $a = e^{i\phi}$, $\phi_n(x; aq^{(1-n)/2})/a^n$ is a polynomial in $y = \cos\phi$ of degree n due to

$$\phi_n(\cos\theta; e^{i\phi}q^{(1-n)/2})e^{-in\phi} = (-1)^n \phi_n(\cos\phi; e^{i\theta}q^{(1-n)/2})e^{-in\theta}. \tag{2.8}$$

The result then follows from Theorem 2.2 applied to a function of y, $y = \cos\phi$. $\square$

3 Linearization of Products

In this section we use Theorem 1.1 to evaluate the linearization coefficients $c_{k,m,n}(a, b, c)$ in

$$\phi_m(x; b)\phi_n(x; c) = \sum_{k=0}^{m+n} c_{k,m,n}(a, b, c)\phi_k(x; a), \tag{3.1}$$

our main result is Theorem 3.1.

We shall use the q-Leibniz role [5]

$$\mathcal{D}_q^n(fg) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)/2} \left(\eta_q^k \mathcal{D}_q^{n-k} f\right) \left(\eta_q^{k-n} \mathcal{D}_q^k g\right), \tag{3.2}$$

where

$$(\eta_q^a)f(x) = \breve{f}(q^{a/2}e^{i\theta}), \quad x = \cos\theta. \tag{3.3}$$

We see that to apply Theorem 1.1 we need to evaluate $\eta_q^s D_q^{k-s}\phi_m(x;b)$ and $\eta_q^{s-k}D_q^s\phi_n(x;c)$ at $x = x_k$. It is easy to see from (1.3) that if $x = \cos\theta$

$$\begin{aligned}
\eta_q^s D_q^{k-s}\phi_m(x;b) &= \frac{(2b)^{k-s}(q;q)_m q^{\binom{k-s}{2}/2}}{(q-1)^{k-s}(q;q)_{m+s-k}} \\
&\quad \times (bq^{k/2}e^{i\theta}, bq^{-s+k/2}e^{-i\theta};q)_{m+s-k}, \\
\eta_q^{s-k}D_q^s\phi_n(x;c) &= \frac{(2c)^s(q;q)_n q^{\binom{s}{2}/2}}{(q-1)^s(q;q)_{n-s}}(cq^{s-k/2}e^{i\theta}, cq^{k/2}e^{-i\theta};q)_{n-s}.
\end{aligned}$$

This leads to

$$\begin{aligned}
c_{k,m,n}(a,b,c) &= \frac{b^k(q,ab;q)_m(q,ac;q)_n}{a^k(ab;q)_k}\sum_{s=0}^{k}\frac{c^s(abq^m;q)_s}{b^s(q,ac;q)_s} \\
&\quad \times \frac{(c/a;q)_{n-s}(bq^{-s}/a;q)_{s+m-k}}{(q;q)_{k-s}(q;q)_{n-s}(q;q)_{m+s-k}}q^{s(s-k)}.
\end{aligned} \tag{3.4}$$

Thus we have proved the following theorem.

Theorem 3.1. *We have the summation identity*

$$\begin{aligned}
&\frac{(be^{i\theta}, be^{-i\theta};q)_m(ce^{i\theta}, ce^{-i\theta};q)_n}{(q,ab;q)_m(q,ac;q)_n} \\
&= \sum_{k,s\geq 0}\frac{b^{k-s}c^s(abq^m;q)_s(c/a;q)_{n-s}(bq^{-s}/a;q)_{s+m-k}}{a^k(ab;q)_k(q,ac;q)_s(q;q)_{k-s}(q;q)_{n-s}(q;q)_{m+s-k}} \\
&\quad \times q^{s(s-k)}(ae^{i\theta}, ae^{-i\theta};q)_k.
\end{aligned}$$

The sum is so that $0 \leq s \leq \min\{n,k\}$, $0 \leq k \leq s+m$.

Mizan Rahman kindly pointed out that the spcial case $a = b = c$ of Theorem 3.1 is (8.1.2) of [4].

We now give another proof of Theorem 3.1. First compute the k sum by replacing k by $k+s$ then observe that the right-hand side becomes

$$\sum_{s\geq 0} \frac{c^s (abq^m;q)_s (c/a;q)_{n-s}}{a^s (ab,q,ac;q)_s (q;q)_{n-s}} \phi_s(\cos\theta;a)$$

$$\times \sum_{k\geq 0} \frac{b^k a^{-k} (bq^{-s}/a;q)_{m-k}}{a^k (abq^s;q)_k (q;q)_k (q;q)_{m-k}} q^{-sk} (aq^s e^{i\theta}, aq^s e^{-i\theta};q)_k.$$

$$= \sum_{s\geq 0} \frac{c^s (abq^m, ae^{i\theta}, ae^{-i\theta};q)_s (c/a;q)_{n-s} (bq^{-s}/a;q)_m}{a^s (ab,q,ac;q)_s (q;q)_{n-s} (q;q)_m}$$

$$\times {}_3\phi_2 \left(\begin{matrix} q^{-m}, aq^s e^{i\theta}, aq^s e^{-i\theta} \\ abq^s, aq^{s+1-m}/b, \end{matrix} \middle| q, q \right).$$

The ${}_3\phi_2$ is now summable by the q-Pfaff-Saalschütz theorem [4, (II.12)] and the result is that each side in the above expression equals

$$\frac{\phi_m(x;a)}{(ab;q)_m} \sum_{s=0}^{n} \frac{c^s (ae^{i\theta}, ae^{-i\theta};q)_s (c/a;q)_{n-s}}{a^s (q,ac;q)_s (q;q)_{n-s} (q;q)_m},$$

which is again summable by the q-Pfaff-Saalschütz theorem and Theorem 3.1 follows.

In terms of basic hypergeometric functions, Theorem 3.1 can be restated as

$$(3.5)\quad c_{k,m,n}(a,b,c) = \frac{b^k (q,ab;q)_m (ac,c/a;q)_n (b/a;q)_{m-k}}{a^k (q,ab;q)_k (q;q)_{m-k}} \times {}_4\phi_3 \left(\begin{matrix} q^{-n}, q^{-k}, abq^m, qa/b \\ ac, q^{1-n}a/c, q^{m-k+1} \end{matrix} \middle| q, q \right), \quad m \geq k.$$

For $k > m$ the s-sum is now over s, $min\{k,n\} \geq s \geq k-m$, so we replace s by $s+k-m$. The result is

$$(3.6)\quad c_{k,m,n}(a,b,c) = \frac{b^m c^{k-m} (q,ac;q)_n (c/a;q)_{m+n-k}}{a^k (q,ac;q)_{k-m} (q;q)_{n+m-k}} \times {}_4\phi_3 \left(\begin{matrix} q^{-m}, q^{k-m-n}, abq^k, q^{1+k-m}a/b \\ acq^{k-m}, q^{1-n-m+k}a/c, q^{k-m+1} \end{matrix} \middle| q, q \right), \quad m \leq k.$$

These ${}_4\phi_3$'s are terminating and balanced and are in general form because they depend on six parameters. This is an interesting observation because the coefficients must possess the symmetry relation $c_{k,m,n}(a,b,c) =$

$c_{k,n,m}(a,c,b)$. This symmetry leads to the Sears transformation as follows. Fix k as a positive integer and a terminating parameter, then observe that $c_{k,m,n}(a,b,c) = c_{k,n,m}(a,c,b)$ is a rational function identity valid for infinitely many integer values of m and n. This allows us to replace q^m and q^n by two general parameters M and N respectively and we see that the right-hand side of (3.5) is

$$\frac{b^k(q, ab, ac, q^{1-k}M, b/a, c/a; q)_\infty}{a^k(q, ab; q)_k(abM, acN, cN/a, bq^{-k}M/a; q)_\infty} \times {}_4\phi_3\left(\begin{matrix} q^{-k}, 1/N, abM, qa/b \\ ac, qa/(Nc), Mq^{1-k} \end{matrix} \middle| q, q\right),$$

which must be symmetric under the exchange (a, M) and (c, N), hence

$$\begin{aligned} &\frac{b^k(q^{1-k}M; q)_k}{(ab; q)_k(bq^{-k}M/a; q)_k} {}_4\phi_3\left(\begin{matrix} q^{-k}, 1/N, abM, qa/b \\ ac, qa/(Nc), Mq^{1-k} \end{matrix} \middle| q, q\right) \\ &= \frac{c^k(q^{1-k}N; q)_k}{(ac; q)_k(cq^{-k}N/a; q)_k} {}_4\phi_3\left(\begin{matrix} q^{-k}, 1/M, acN, qa/c \\ ab, qa/(Mb), Nq^{1-k} \end{matrix} \middle| q, q\right). \end{aligned} \tag{3.7}$$

Equation (3.7) is the version (III.16) in [4] of the Sears transformation and iterating it leads to the standard form of the Sears transformation [4, (III.15)].

When $c = a$ (3.4) implies

$$\begin{aligned} &\phi_m(x; b)\phi_n(x; a) \\ &= \frac{(q; q)_m(ab; q)_{m+n}}{(ab; q)_n} \sum_{k=0}^{m} \frac{b^k q^{-nk}(bq^{-n}/a; q)_{m-k}}{a^k(q, abq^n; q)_k(q; q)_{m-k}} \phi_{k+n}(x; a). \end{aligned} \tag{3.8}$$

This result is not new since [5, (2.2)] is

$$\phi_m(x; b) = (q, ab; q)_m \sum_{k=0}^{m} \frac{b^k(b/a; q)_{m-k}}{a^k(q, ab; q)_k(q; q)_{m-k}} \phi_k(x; a), \tag{3.9}$$

so we can replace a by aq^n, multiply by $\phi_n(x; a)$ and apply the identity $\phi_{n+k}(x; a) = \phi_k(x; a)\phi_n(x; aq^k)$. This yields (3.8). This indicates that (3.8) is equivalent to the connection coefficients between two ϕ_n's with different a's, which was shown to be equivalent to the q-Pfaff Saalschütz theorem [5]. Having said that (3.8) is not new, nevertheless, the special case $m = n$ and $b = -a$ gives the useful connection coefficient formula

$$\begin{aligned} &(a^2e^{2i\theta}, a^2e^{-2i\theta}; q^2)_n \\ &= (a; q)_n(-a^2; q)_{2n} \sum_{k=0}^{n} \frac{(-1)^{n+k}q^{n(k-n)}(-q^{-n}; q)_k}{(q; q)_k(q; q)_{n-k}(-a^2; q)_{2n-k}} \\ &\times (ae^{i\theta}, ae^{-i\theta}; q)_{2n-k}. \end{aligned} \tag{3.10}$$

4 q-Ultraspherical polynomials

In this section we apply Theorems 1.1, 2.1 and 2.2 to the q-ultraspherical polynomials which have the closed form [2], [4]

$$
\begin{aligned}
(4.1) \qquad C_n(\cos\theta;\beta|q) &= \sum_{k=0}^{n} \frac{(\beta;q)_k(\beta;q)_{n-k}}{(q;q)_k(q;q)_{n-k}} e^{i(n-2k)\theta} \\
&= \frac{(\beta;q)_n}{(q;q)_n} e^{in\theta} {}_2\phi_1\left(\begin{matrix} q^{-n}, \beta \\ q^{1-n}/\beta \end{matrix} \,\middle|\, q, \frac{qe^{-2i\theta}}{\beta} \right),
\end{aligned}
$$

and satisfy

$$
(4.2) \qquad \mathcal{D}_q C_n(x;\beta|q) = \frac{2(1-\beta)}{1-q} q^{(1-n)/2} C_{n-1}(x;q\beta|q).
$$

Furthermore the C_n's have the generating function [2]

$$
(4.3) \qquad \sum_{n=0}^{\infty} C_n(\cos\theta;\beta)t^n = \frac{(\beta te^{i\theta}, \beta te^{-i\theta};q)_\infty}{(te^{i\theta}, te^{-i\theta};q)_\infty}.
$$

We first expand $C_n(x;\beta|q)$ in terms of $\{\phi_k(x)\}$. It is clear from (4.3) that

$$
(4.4) \qquad C_n(\zeta_0;\beta|q) = \frac{(\beta;q^{1/2})_n}{(q^{1/2};q^{1/2})_n} \, q^{-n/4}.
$$

Thus Theorem 2.1, (4.2) and (4.4) imply the following proposition, which is (7.5.34) in [4].

Proposition 4.1. *The continuous q-ultraspherical polynomials have the basic hypergeometric representation*

$$
\begin{aligned}
C_n(\cos\theta;\beta|q) &= q^{-n/4} \frac{(\beta;q^{1/2})_n}{(q^{1/2};q^{1/2})_n} \\
&\quad \times {}_4\phi_3\left(\begin{matrix} q^{-n/2}, \beta q^{n/2}, q^{1/4}e^{i\theta}, q^{1/4}e^{-i\theta} \\ -q^{1/2}, \beta^{1/2}q^{1/4}, -\beta^{1/2}q^{1/4} \end{matrix} \,\middle|\, q^{1/2}, q^{1/2} \right).
\end{aligned}
$$

We next apply Theorem 2.2 using

$$
C_n(0;\beta|q) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \frac{(\beta^2;q^2)_n}{(q^2;q^2)_{n/2}}(-1)^{n/2} & \text{if } n \text{ is even.} \end{cases}
$$

The result is the following proposition.

Proposition 4.2. *We have*

$$\begin{aligned} C_n(\cos\theta;\beta|q) &= \frac{(\beta;q)_n}{(q;q)_n}(-e^{2i\theta}q^{2-n};q)_{n-1}(1+e^{2i\theta})e^{-in\theta} \\ &\quad \times {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{1-n}, -q^{1-n}/\beta, -q^{2-n}/\beta \\ q^{2-2n}/\beta^2, -e^{2i\theta}q^{2-n}, -e^{-2i\theta}q^{2-n} \end{matrix} \middle| q^2, q^2 \right). \end{aligned}$$

Finally we apply Theorem 1.1 to $C_n(x;\beta|q)$. Let

$$C_n(x;\beta|q) = \sum_{k=0}^{n} b_{n,k}(a,\beta)\phi_k(x;a). \tag{4.5}$$

This time (4.1) and (4.2) give

$$\begin{aligned} b_{n,k}(a,\beta) &= \frac{(\beta;q)_k}{(q;q)_k}\frac{q^{(1-n)k/2}}{a^k}(-1)^k C_{n-k}(x_k;\beta q^k|q) \\ &= \frac{(-1)^k(\beta;q)_n\, a^{n-2k}}{(q;q)_k(q;q)_{n-k}}q^{k(1-k)/2} \\ &\quad \times {}_2\phi_1\left(\begin{matrix} q^{k-n}, q^k\beta \\ q^{1-n}/\beta \end{matrix} \middle| q, \frac{q^{1-2k}}{a^2\beta} \right). \end{aligned} \tag{4.6}$$

The ${}_2\phi_1$ in (4.6) may be summed by the q-Gauss theorem [4] if $\beta = a^2$ to obtain a quadratic transformation (see [2] or (7.5.33) in [4]).

Proposition 4.3. *We have*

$$\begin{aligned} &C_n(x;\beta|q) \\ &\quad = \frac{(\beta^2;q)_n}{\beta^{n/2}(q;q)_n}{}_4\phi_3\left(\begin{matrix} q^{-n}, \beta^2 q^n, \sqrt{\beta}e^{i\theta}, \sqrt{\beta}e^{-i\theta} \\ \beta\sqrt{q}, \ -\beta\sqrt{q}, \ -\beta \end{matrix} \middle| q, q \right). \end{aligned}$$

For our last expansion we invert (4.5), let

$$\phi_n(x;a) = \sum_{j=0}^{n} a_{n,j}C_j(x;\beta|q). \tag{4.7}$$

We shall find $a_{n,j}$ using the orthogonality relation of the continuous q-ultraspherical polynomials

$$\begin{aligned} &\int_0^{\pi} C_m(\cos\theta;\beta|q)C_n(\cos\theta;\beta|q)\frac{(e^{2i\theta}, e^{-2i\theta};q)_\infty}{(\beta e^{2i\theta}, \beta e^{-2i\theta};q)_\infty}\, d\theta \\ &\qquad = \frac{2\pi(\beta, q\beta;q)_\infty}{(q,\beta^2;q)_\infty}\frac{(\beta^2;q)_n(1-\beta)}{(q;q)_n(1-\beta q^n)}\delta_{m,n} \end{aligned} \tag{4.8}$$

From (4.8) we get

$$\frac{2\pi(\beta, q\beta; q)_\infty}{(q, \beta^2; q)_\infty} \frac{(\beta^2; q)_j (1-\beta)}{(q; q)_j (1-\beta q^j)} a_{n,j} = \int_0^\pi (ae^{i\theta}, ae^{-i\theta}; q)_n C_j(\cos\theta; \beta|q) \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(\beta e^{2i\theta}, \beta e^{-2i\theta}; q)_\infty}\, d\theta.$$

By writing $(ae^{i\theta}, ae^{-i\theta}; q)_n$ as quotients of infinite products then applying (4.3) to expand it in powers of a, we see that the above expression is

$$\begin{aligned} &= \sum_{m=0}^\infty a^m q^{mn} \int_0^\pi C_m(\cos\theta; q^{-n}|q) C_j(\cos\theta; \beta|q) \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(\beta e^{2i\theta}, \beta e^{-2i\theta}; q)_\infty}\, d\theta \\ &= \sum_{m=0}^\infty a^m q^{mn} \sum_{k=0}^{[m/2]} \frac{\beta^k (q^{-n}/\beta; q)_k (q^{-n}; q)_{m-k}}{(q; q)_k (q\beta; q)_{m-k}} \frac{1-\beta q^{m-2k}}{1-\beta} \\ &\times \int_0^\pi C_{m-2k}(\cos\theta; \beta|q) C_j(\cos\theta; \beta|q) \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(\beta e^{2i\theta}, \beta e^{-2i\theta}; q)_\infty}\, d\theta \end{aligned}$$

Therefore

$$\frac{(1-\beta)}{(1-\beta q^j)} a_{n,j} = a^j q^{nj} \sum_{k=0}^\infty \frac{\beta^k (q^{-n}/\beta; q)_k (q^{-n}; q)_{k+j}}{(q; q)_k (q\beta; q)_{k+j}}\, a^{2k} q^{2kn},$$

that is

$$a_{n,j} = a^j q^{nj} \frac{(q^{-n}; q)_j}{(\beta; q)_j} {}_2\phi_1\left(\left. \begin{matrix} q^{j-n}, q^{-n}/\beta \\ \beta q^{j+1} \end{matrix} \right| q, a^2 \beta q^{2n} \right) \tag{4.9}$$

The ${}_2\phi_1$ can be summed if $a^2 = q\beta$ in which case we get

$$\phi_n(x; \sqrt{q\beta}) = \frac{(\beta^2 q^{n+1}; q)_n}{(q\beta; q)_n} \sum_{j=0}^n \frac{(q^{-n}, q\beta; q)_j}{(\beta^2 q^{n+1}, \beta; q)_j} \beta^{j/2} q^{j(n+1/2)} C_j(x; \beta|q). \tag{4.10}$$

As another example, if $a^2 = \beta$ the ${}_2\phi_1$ is a sum of two terms and the result becomes

$$\phi_n(x; \sqrt{\beta}) = \frac{(1-\beta q^n)(\beta^2 q^n; q)_n}{(1-\beta^2 q^n)(q\beta; q)_n} \times \sum_{j=0}^n \frac{(1-\beta^2 q^{2j})(q^{-n}; q)_j}{(1-\beta)(\beta^2 q^{n+1}; q)_j} \beta^{j/2} q^{nj} C_j(x; \beta|q). \tag{4.11}$$

5 Remarks

It is natural to ask if Theorems 1.1, 2.1, and 2.2 hold for a class of non-polynomial functions. In [8] such theorems are given for entire functions which satisfy growth conditions.

Theorems 1.1 and 2.1 give two alternative forms for the coefficients f_k of $\phi_k(x)$, what results is

$$(5.1)\quad (-q^{1/2};q^{1/2})_k q^{-k(k-1)/8}(\mathcal{D}^k_{q^{1/2}}f)(\zeta_k) = (q^{1/2}+1)^k(\mathcal{D}^k_q f)(\zeta_0),$$

where

$$\zeta_k = (q^{(k+1)/4}+q^{-(k+1)/4})/2.$$

We do not have basic hypergeometric proofs of Theorem 2.4 and Corollary 2.8. It is likely that Theorem 2.4 could be proven by splitting the sum into even and odd terms as a sum of two balanced ${}_4\phi_3$'s. Applying the Sears transformation to each sum could then lead to a recombined single sum that is evaluable- this type of proof establishes Corollary 2.7. Nonetheless these proposed details contrast with the ease of use of Theorems 1.1, 2.1, and 2.2.

The coefficient $a_{n,j}$ in (4.9), the inverse to (4.6), can also be written as a multiple of the q-ultraspherical polynomial

$$C_{n-j}(\cos\phi;q^{-n}/b|q), \quad a=q^{(1-n)/2}e^{-i\phi}.$$

Even polynomials in x may be expanded as functions of $\cos 2\theta = 2x^2-1$, for example [4, (7.5.40)] is

$$(5.2)\quad \begin{aligned} C_{2n}(\cos\theta;\beta|q) &= q^{-n/2}\frac{(\beta;q^{1/2})_{2n}}{(q^{1/2};q^{1/2})_{2n}} \\ &\quad \times {}_4\phi_3\left(\begin{matrix} q^{-n},\beta q^n, q^{1/2}e^{2i\theta}, q^{1/2}e^{-2i\theta} \\ \beta q^{1/2}, -q^{1/2}, -q \end{matrix}\,\middle|\, q,q\right). \end{aligned}$$

However it can be shown that (5.2) is equivalent to the reversal of Proposition 4.2, which is

$$(5.3)\quad \begin{aligned} C_{2n}(\cos\theta;\beta|q) &= (-1)^n\frac{(\beta^2;q^2)_n}{(q^2;q^2)_n} \\ &\quad \times {}_4\phi_3\left(\begin{matrix} q^{-2n},\beta^2 q^{2n}, -e^{2i\theta}, -e^{-2i\theta} \\ -\beta, -\beta q, q \end{matrix}\,\middle|\, q^2,q^2\right). \end{aligned}$$

Apply Singh's quadratic ${}_4\phi_3$ transformation [4, (III.21)] followed by the 1-balanced ${}_4\phi_3$ transformation [4, (III.15)] to show that (5.3) and (5.2) are equivalent.

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