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The Bailey-Rogers-Ramanujan group

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Abstract. A certain group of upper triangular 2×2 matrices is explicitly defined via generators. Any element of this group has an associated multisum identity of Rogers-Ramanujan type. Several infinite families of identities are given as examples. Different expressions for an element in the generators can yield distinct identities. An application to the Borwein polynomials is given.

1. Introduction. The Rogers-Ramanujan identities have many proofs [**5**]. One idea which has been fruitful [**3**], [**16**], [**17**] is the concept of a Bailey pair. This technique allows for iteration to objects called Bailey chains [**3**], and results in multisum generalizations of the Rogers-Ramanujan theorems to arbitrary modulus. The purpose of this paper is to define a group of 2×2 rational matrices, which contains the standard iteration of Bailey chains. Any element of this group has a corresponding identity of Rogers-Ramanujan type, in fact there may be many such identities.

We review Bailey pairs and give the relevant transformations for Bailey pairs in §2. These transformations are written as 2×2 matrices in §3, where the Bailey-Rogers-Ramanujan group is defined in Definition 2. A Rogers-Ramanujan type identity is given for an element of the group in Theorem 1 in §4. Examples of the identities are given in §5 and §6, and an application to the Borwein polynomials is given in §7.

We use standard notation for q-series found in [**2**], [**12**], and we shall also use the Jacobi triple product identity

$$\sum_{n=-\infty}^{\infty} q^{n^2} x^n = (q^2, -qx, -q/x; q^2)_\infty. \tag{1.1}$$

2. Bailey pairs. In this section we review Bailey pairs, and give the versions of the transformations on pairs which are needed for the Bailey-Rogers-Ramanujan group.

Recall [**3**] the definition of a Bailey pair.

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Definition 1. *A pair of sequences* $(\alpha_n(a,q), \beta_n(a,q))$ *is called a Bailey pair with parameters* (a,q) *if*

$$\beta_n(a,q) = \sum_{r=0}^{n} \frac{\alpha_r(a,q)}{(q;q)_{n-r}(aq;q)_{n+r}}$$

for all $n \geq 0$.

The first example of a Bailey pair, which will be used throughout this paper, is the unit Bailey pair

$$\text{(UBP)} \quad \beta_n^{(0)}(a,q) = \begin{cases} 1, & \text{if } n=0 \\ 0, & \text{if } n>0, \end{cases} \quad \alpha_n^{(0)}(a,q) = \frac{(a;q)_n}{(q;q)_n}\frac{(1-aq^{2n})}{(1-a)}(-1)^n q^{\binom{n}{2}}.$$

Bailey's lemma [**3**],[**17**] takes a Bailey pair $(\alpha_n(a,q), \beta_n(a,q))$ and produces another Bailey pair $(\alpha'_n(a,q), \beta'_n(a,q))$ with parameters (a,q). One limiting case of Bailey's lemma is denoted here by (S1)

$$\text{(S1)} \quad \begin{aligned} \alpha'_r(a,q) &= a^r q^{r^2}\alpha_r(a,q), \\ \beta'_n(a,q) &= \sum_{k=0}^{n} \frac{a^k q^{k^2}}{(q;q)_{n-k}}\beta_k(a,q). \end{aligned}$$

If we start with (UBP), apply (S1) twice, we have

$$\begin{aligned} \beta_n^{(2)}(a,q) &= \sum_{r=0}^{n} \frac{a^{2r}q^{2r^2}\alpha_r^{(0)}(a,q)}{(q;q)_{n-r}(aq;q)_{n+r}} \\ &= \sum_{s=0}^{n} \frac{a^s q^{s^2}}{(q;q)_{n-s}(q;q)_s}. \end{aligned} \tag{2.1}$$

The Rogers-Ramanujan identities modulo 5 occur if $a=1$ and $n \to \infty$ in (2.1).

Another limiting case of Bailey's lemma is

$$\text{(S2)} \quad \begin{aligned} \alpha'_r(a,q) &= a^{r/2}q^{r^2/2}\alpha_r(a,q), \\ \beta'_n(a,q) &= \sum_{k=0}^{n} \frac{(-\sqrt{aq};q)_k}{(q;q)_{n-k}(-\sqrt{aq};q)_n} a^{k/2}q^{k^2/2}\beta_k(a,q). \end{aligned}$$

It is clear from the action on $\alpha_n(a,q)$ that applying (S2) twice is equivalent to applying (S1) once.

Some other transformations of Bailey pairs which changed the base q were given in [**11**]. The following three choices, denoted (D1), (D2), and (D3), all have $(\alpha'_n(a,q), \beta'_n(a,q))$ as a Bailey pair with parameters (a,q).

$$\text{(D1)} \quad \begin{aligned} \alpha'_r(a,q) &= \alpha_r(a^2,q^2), \\ \beta'_n(a,q) &= \sum_{k=0}^{n} \frac{(-aq;q)_{2k}}{(q^2;q^2)_{n-k}}\, q^{n-k}\beta_k(a^2,q^2), \end{aligned}$$

$$\text{(D2)} \quad \begin{aligned} \alpha'_r(a,q) &= a^{-r}q^{-r^2}\alpha_r(a^2,q^2), \\ \beta'_n(a,q) &= \sum_{k=0}^{n} \frac{(-aq;q)_{2k}}{(q^2;q^2)_{n-k}}\, q^{k^2+k-2kn-n}(-1)^{n-k}a^{-n}\beta_k(a^2,q^2), \end{aligned}$$

and

$$\text{(D3)}\qquad \begin{aligned} \alpha'_r(a,q) &= a^{-r/2}q^{-r^2/2}\alpha_r(a^2,q^2),\\ \beta'_n(a,q) &= \sum_{k=0}^{n} \frac{(-aq;q)_{2k}(q^{-1/2-k}a^{-1/2},q^{k+3/2}a^{1/2};q)_{n-k}}{(aq^{2k+1};q^2)_{n-k}(q^2;q^2)_{n-k}}\\ &\quad \times q^{-\binom{k}{2}}\,(aq)^{-k/2}\beta_k(a^2,q^2). \end{aligned}$$

The inverse versions of (D1)-(D3), denoted (E1)-(E3), follow from Theorem 2.2 of [**11**]. To avoid fractional powers we choose to write (E1)-(E3) in such a way that $(\alpha'_n(a,q),\beta'_n(a,q))$ is a Bailey pair with parameters (a^4,q^4) in each of these three cases.

$$\text{(E1)}\qquad \begin{aligned} \alpha'_r(a^4,q^4) &= \alpha_r(a^2,q^2),\\ \beta'_n(a^4,q^4) &= \sum_{k=0}^{n} \frac{(-1)^{n-k}q^{2(n-k)^2}}{(-a^2q^2;q^2)_{2n}(q^4;q^4)_{n-k}}\,\beta_k(a^2,q^2), \end{aligned}$$

$$\text{(E2)}\qquad \begin{aligned} \alpha'_r(a^4,q^4) &= a^{2r}q^{2r^2}\alpha_r(a^2,q^2),\\ \beta'_n(a^4,q^4) &= \sum_{k=0}^{n} \frac{a^{2k}q^{2k^2}}{(-a^2q^2;q^2)_{2n}(q^4;q^4)_{n-k}}\,\beta_k(a^2,q^2), \end{aligned}$$

$$\text{(E3)}\qquad \begin{aligned} \alpha'_r(a^4,q^4) &= a^{r}q^{r^2}\alpha_r(a^2,q^2),\\ \beta'_n(a^4,q^4) &= \sum_{k=0}^{n} \frac{(aq;q^2)_{2n-k}(-aq;q^2)_k a^k q^{k^2}}{(-a^2q^2;q^2)_{2n}(q^4;q^4)_{n-k}(a^2q^2;q^4)_n}\,\beta_k(a^2,q^2). \end{aligned}$$

For changing the base q to q^3 we have one possibility and its inverse, denoted (T1) and (T2). In (T1) (α'_n,β'_n) has parameters (a^3,q^3), while in (T2) it has parameters (a,q).

$$\text{(T1)}\qquad \begin{aligned} \alpha'_r(a^3,q^3) =& a^r q^{r^2}\alpha_r(a,q),\\ \beta'_n(a^3,q^3) =& \sum_{k=0}^{n} \frac{(aq;q)_{3n-k}a^k q^{k^2}}{(a^3q^3;q^3)_{2n}(q^3;q^3)_{n-k}}\beta_k(a,q), \end{aligned}$$

$$\text{(T2)}\qquad \begin{aligned} \alpha'_r(a,q) =& a^{-r}q^{-r^2}\alpha_r(a^3,q^3),\\ \beta'_n(a,q) =& \sum_{k=0}^{n} \frac{(aq^{2n+1};q^{-1})_{3k}(a^3q^3;q^3)_{2(n-k)}}{(aq;q)_{2n}(q^3;q^3)_k}\\ &\times (-1)^k q^{3\binom{k}{2}-n^2}a^{-n}\beta_{n-k}(a^3,q^3). \end{aligned}$$

3. 2×2 matrices. In this section we realize the operations of §2 on Bailey pairs as 2×2 matrices. These are the generators of the Bailey-Rogers-Ramanujan group in Definition 2.

We are concerned with iterating the transformations (S), (D), (E), and (T) of §2. Our initial choice is always the unit Bailey pair (UBP) with $a=1$. We shall also assume that each iteration yields a Bailey pair with parameters $(1,q)$. We have

$$\alpha'_r(1,q) = q^{Ar^2}\alpha_r^{(0)}(1,q^B).$$

for some rational numbers A and B. Thus we need only keep track A and B while carrying out the iteration.

We encode a transformation

$$\alpha'_r(1,q) = q^{Ar^2}\alpha_r(1,q^B),$$

where $(\alpha'_n(1,q), \beta'_n(1,q))$ has parameters $(1,q)$, by the 2×2 matrix

$$\begin{bmatrix} 1 & A \\ 0 & B \end{bmatrix}.$$

We next check that matrix multiplication on the right does correspond to the composition of transformations. If

$$\alpha''_r(1,q) = q^{Cr^2}\alpha'_r(1,q^D) = q^{Cr^2+ADr^2}\alpha_r(1,q^{BD}),$$

the corresponding matrix is

$$\begin{bmatrix} 1 & C+AD \\ 0 & BD \end{bmatrix} = \begin{bmatrix} 1 & A \\ 0 & B \end{bmatrix}\begin{bmatrix} 1 & C \\ 0 & D \end{bmatrix}.$$

With this notation we see that

$$(S1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ (S2) = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix},\ (D1) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},\ (D2) = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix},$$

$$(D3) = \begin{bmatrix} 1 & -1/2 \\ 0 & 2 \end{bmatrix},\ (E1) = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix},\ (E2) = \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix},$$

$$(E3) = \begin{bmatrix} 1 & 1/4 \\ 0 & 1/2 \end{bmatrix},\ (T1) = \begin{bmatrix} 1 & 1/3 \\ 0 & 1/3 \end{bmatrix},\ (T2) = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}.$$

DEFINITION 2. *The Bailey-Rogers-Ramanujan group is the subgroup of* 2×2 *upper triangular rational matrices generated by*

$$\{(S1), (S2), (D1), (D2), (D3), (E1), (E2), (E3), (T1), (T2)\}.$$

Even though (S1), (E1), (E2), (E3) and (T2) are unnecessary as generators, it will be convenient in the following sections to keep their designation as generators.

There are other relations amongst the generators, for example

$$(S1)(D1) = (D1)(S1), \quad (E2)(D3) = (S2).$$

4. The Rogers-Ramanujan type identities. Let

$$g = w_1 w_2 \cdots w_{k+1}$$

be an element of the Bailey-Rogers-Ramanujan group, with each w_i a generator from Definition 2. Suppose that the corresponding Bailey pairs are

$$(\alpha_n^{(0)}, \beta_n^{(0)}), (\alpha_n^{(1)}, \beta_n^{(1)}), \cdots, (\alpha_n^{(k+1)}, \beta_n^{(k+1)}).$$

Let the corresponding relations for w_{i+1} between $\beta_n^{(i)}$ and $\beta_n^{(i+1)}$ be expressed as

$$\beta_n^{(i+1)} = \sum_{s_i=0}^{n} M_{n,s_i}^{(i)}\beta_{s_i}^{(i)}, \quad 0 \le i \le k.$$

We say that $M^{(i)}$ is the infinite lower triangular matrix corresponding to w_{i+1}. For example, if $w_1 = (S1)$, then

$$M^{(0)}_{nk} = q^{k^2}/(q;q)_{n-k}.$$

THEOREM 1. *If $w = w_1 w_2 \cdots w_{k+1}$ is an element of the Bailey-Rogers- Ramanujan group, the corresponding finite Rogers-Ramanujan identity is given by*

$$\beta_n^{(k+1)} = \sum_{n \geq s_k \geq \cdots \geq s_1 \geq 0} M^{(k)}_{n,s_k} \cdots M^{(1)}_{s_2,s_1} M^{(0)}_{s_1,0} = \\ = \sum_{r=0}^{n} \frac{\alpha_r^{(k+1)}}{(q;q)_{n+r}(q;q)_{n-r}},$$

where $M^{(i)}$ is the infinite lower triangular matrix corresponding to w_{i+1}.

PROOF. The first equality is the expression for $\beta_n^{(k+1)}$ as a $(k+1)$-fold sum over $\beta_s^{(0)}$. This sum reduces to a k-fold sum because of the (UBP) condition. The right side expresses the fact that $(\alpha_n^{(k+1)}, \beta_n^{(k+1)})$ is a Bailey pair with parameters $(1, q)$. □

Next we consider the $n \to \infty$ limit of Theorem 1. Let

$$w = w_1 w_2 \cdots w_{k+1} = \begin{bmatrix} 1 & A \\ 0 & B \end{bmatrix}.$$

The right-side of Theorem 1, using the (UBP) and the Jacobi triple product identity (1.1), approaches

$$\frac{(q^{2A+B}, q^{A+B}, q^A; q^{2A+B})_\infty}{(q;q)_\infty^2}. \tag{4.1}$$

The left-side will have a termwise limit if

$$\lim_{n \to \infty} M^{(k)}_{n,s_k} = M^{(k)}_{\infty,s_k}$$

exists. This is the case if $w_{k+1} = (S1)$, $(S2)$, $(D3)$, $(E2)$, $(E3)$, or $(T1)$. If $w_{k+1} = (D1)$ or $(E1)$, we see that

$$\lim_{n \to \infty} M^{(k)}_{n,n-s_k}$$

exists, so that the limit may be taken as long as

$$\lim_{n \to \infty} M^{(k-1)}_{n-s_k,s_{k-1}}$$

exists. It will exist if $w_k = (S1)$, $(S2)$, $(D3)$, $(E2)$, $(E3)$, or $(T1)$, otherwise we may need to replace s_{k-1} by $n - s_{k-1}$ and continue.

We also see that $w_k = (D2)$ or $(T2)$ will lead to interesting identities, even though the termwise limit does not exist. Each term will be a Laurent series, yet the sum has a limit with no negative powers of q as $n \to \infty$. Several such examples are given in §5.

5. Single sum identities. In this section we record which single sum Rogers-Ramanujan type identities appear from words of length 2 in Theorem 1. In each case we have taken the $n \to \infty$ limit in Theorem 1 and multiplied by the infinite product occurring in $M^{(k)}_{\infty,s_k}$. Each of these identities has many multisum generalizations by considering longer words, a few are given in §6.

Rogers-Ramanujan identity (Slater (18)):

$$(S1)(S1) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

$$\sum_{s=0}^{\infty} \frac{q^{s^2}}{(q;q)_s} = \frac{(q^5, q^2, q^3; q^5)_\infty}{(q;q)_\infty}.$$

Bailey's mod 9 identity (Slater (42)):

$$(T1)(S1) = \begin{bmatrix} 1 & 4/3 \\ 0 & 1/3 \end{bmatrix},$$

$$\sum_{s=0}^{\infty} \frac{q^{3s^2}(q;q)_{3s}}{(q^3;q^3)_{2s}(q^3;q^3)_s} = \frac{(q^9, q^4, q^5; q^9)_\infty}{(q^3;q^3)_\infty}.$$

Rogers' mod 7 identity (Slater (33)):

$$(E2)(S1) = \begin{bmatrix} 1 & 3/2 \\ 0 & 1/2 \end{bmatrix},$$

$$\sum_{s=0}^{\infty} \frac{q^{2s^2}}{(-q;q)_{2s}(q^2;q^2)_s} = \frac{(q^7, q^4, q^3; q^7)_\infty}{(q^2;q^2)_\infty}.$$

Rogers' mod 5 identity, (Slater (19)):

$$(E1)(S1) = \begin{bmatrix} 1 & 1 \\ 0 & 1/2 \end{bmatrix},$$

$$\sum_{s=0}^{\infty} \frac{(-1)^s q^{3s^2}}{(-q;q)_{2s}(q^2;q^2)_s} = \frac{(q^5, q^2, q^3; q^5)_\infty}{(q^2;q^2)_\infty}.$$

Rogers' mod 5 identity (Slater (20)):

$$(E2)(S2) = \begin{bmatrix} 1 & 1 \\ 0 & 1/2 \end{bmatrix},$$

$$\sum_{s=0}^{\infty} \frac{q^{s^2}}{(q^4;q^4)_s} = \frac{(q^5, q^2, q^3; q^5)_\infty (-q;q^2)_\infty}{(q^2;q^2)_\infty}.$$

Slater's identity (36):

$$(S1)(S2) = \begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix},$$

$$\sum_{s=0}^{\infty} \frac{(-q;q^2)_s}{(q^2;q^2)_s} q^{s^2} = \frac{1}{(q^1, q^4, q^7; q^8)_\infty}.$$

Slater's identity (39):

$$(S2)(S1) = \begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix},$$

$$\sum_{s=0}^{\infty} \frac{q^{2s^2}}{(q^2;q^2)_s(-q;q^2)_s} = \frac{(q^8,q^3,q^5;q^8)_\infty}{(q^2;q^2)_\infty}.$$

Slater's identity (53):

$$(E3)(S1) = \begin{bmatrix} 1 & 5/4 \\ 0 & 1/2 \end{bmatrix},$$

$$\sum_{s=0}^{\infty} \frac{q^{4s^2}(q;q^2)_{2s}}{(q^4;q^4)_{2s}} = \frac{(q^{12},q^5,q^7;q^{12})_\infty}{(q^4;q^4)_\infty}.$$

mod 4 identity:

$$(D2)(S2) = \begin{bmatrix} 1 & -1/2 \\ 0 & 2 \end{bmatrix},$$

$$\sum_{s=0}^{\infty} \frac{(-q;q^2)_s}{(q^4;q^4)_s}(-1)^{s+1}q^{(s-1)^2} = \frac{(q;q)_\infty}{(q^4;q^4)_\infty}.$$

mod 6 identity:

$$(D3)(S1) = \begin{bmatrix} 1 & 1/2 \\ 0 & 2 \end{bmatrix},$$

$$\sum_{s=0}^{\infty} \frac{(q^{-1};q^2)_s(q^3;q^2)_s}{(q^2;q^2)_{2s}}q^{2s^2} = \frac{(q^1,q^5;q^6)_\infty}{(q^2,q^4;q^6)_\infty}.$$

mod 8 identity:

$$(E3)(S2) = \begin{bmatrix} 1 & 3/4 \\ 0 & 1/2 \end{bmatrix},$$

$$\sum_{s=0}^{\infty} \frac{(q;q^2)_{2s}(-q^2;q^4)_s}{(q^4;q^4)_{2s}}q^{2s^2} = \frac{(q^3,q^5;q^8)_\infty}{(q^2,q^6;q^8)_\infty}.$$

mod 12 identity:

$$(D1)(S2) = \begin{bmatrix} 1 & 1/2 \\ 0 & 2 \end{bmatrix},$$

$$\sum_{s=0}^{\infty} \frac{(-q;q^2)_s}{(q^4;q^4)_s}q^{s^2+2s} = \frac{(q^6;q^{12})_\infty}{(q^3,q^4,q^8,q^9;q^{12})_\infty}.$$

mod 12 identity:

$$(T1)(S2) = \begin{bmatrix} 1 & 5/6 \\ 0 & 1/3 \end{bmatrix},$$

$$\sum_{s=0}^{\infty} \frac{(-q^3;q^6)_s(q^2;q^2)_{3s}}{(q^6;q^6)_{2s}(q^6;q^6)_s}q^{3s^2} = \frac{(q^{12},q^5,q^7;q^{12})_\infty(-q^3;q^6)_\infty}{(q^6;q^6)_\infty}.$$

mod 4 identity:

$$(D1)(T2) = \begin{bmatrix} 1 & -1 \\ 0 & 6 \end{bmatrix},$$

$$\lim_{n\to\infty}\sum_{s=0}^{n} \frac{(q^{2n+1};q^{-1})_{3s}(q^3;q^6)_{n-s}}{(q^3;q^3)_s}(-1)^{s+1}q^{3\binom{s}{2}-n^2+3(n-s)+1} = \frac{1}{(q^2;q^4)_\infty}.$$

mod 2 identity:

$$(E3)(T2) = \begin{bmatrix} 1 & -1/4 \\ 0 & 3/2 \end{bmatrix},$$

$$\lim_{n\to\infty}\sum_{s=0}^{n} \frac{(q^{8n+4};q^{-4})_{3s}(q^3;q^6)_{2(n-s)}}{(q^{12};q^{12})_s}(-1)^{s+1}q^{12\binom{s}{2}-4n^2+1} = (q;q^2)_\infty.$$

mod 10 identity:

$$(S1)(D3)=\begin{bmatrix}1 & 3/2\\ 0 & 2\end{bmatrix},$$

$$\lim_{n\to\infty}\sum_{s=0}^{n}\frac{(-q^2;q^2)_{2s}(q^{-1-2s},q^{3+2s};q^2)_{n-s}}{(q^{4s+2};q^4)_{n-s}(q^4;q^4)_s(q^4;q^4)_{n-s}}q^{-s^2}=\frac{(q^{10},q^7,q^3;q^{10})_\infty}{(q^2;q^2)_\infty^2}.$$

mod 7 identity:

$$(S1)(T2)=\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix},$$

$$\lim_{n\to\infty}\sum_{s=0}^{n}\frac{(q^{2n+1};q^{-1})_{3s}(q^3;q^3)_{2(n-s)}}{(q^3;q^3)_s(q^3;q^3)_{n-s}}(-1)^s q^{3\binom{s}{2}-n^2}=\frac{(q^7,q^2,q^5;q^7)_\infty}{(q;q)_\infty}.$$

mod 8 identity:

$$(S2)(T2)=\begin{bmatrix}1 & 1/2\\ 0 & 3\end{bmatrix},$$

$$\lim_{n\to\infty}\sum_{s=0}^{n}\frac{(q^{4n+2};q^{-2})_{3s}(q^6;q^6)_{2(n-s)}(-1)^s q^{6\binom{s}{2}-2n^2}}{(q^6;q^6)_s(-q^3;q^6)_{n-s}(q^6;q^6)_{n-s}}=\frac{(q^8,q^7,q^1;q^8)_\infty}{(q^2;q^2)_\infty}.$$

mod 5 identity:

$$(E2)(T2)=\begin{bmatrix}1 & 1/2\\ 0 & 3/2\end{bmatrix},$$

$$\lim_{n\to\infty}\sum_{s=0}^{n}\frac{(q^{4n+2};q^{-2})_{3s}(q^6;q^6)_{2(n-s)}(-1)^s q^{6\binom{s}{2}-2n^2}}{(q^6;q^6)_s(-q^3;q^3)_{2(n-s)}(q^6;q^6)_{n-s}}=\frac{(q^5,q^4,q^1;q^5)_\infty}{(q^2;q^2)_\infty}.$$

mod 2 identity:

$$(D3)(T2)=\begin{bmatrix}1 & -5/2\\ 0 & 6\end{bmatrix},$$

$$\lim_{n\to\infty}\sum_{s=0}^{n}\frac{(q^{4n+2};q^{-2})_{3s}(q^{-3},q^9;q^6)_{n-s}}{(q^6;q^6)_s}(-1)^{s+1}q^{6\binom{s}{2}-2n^2+9}=(q;q^2)_\infty^2.$$

mod 2 identity:

$$(D3)(D3)=\begin{bmatrix}1 & -3/2\\ 0 & 4\end{bmatrix},$$

$$\lim_{n\to\infty}\sum_{s=0}^{n}\frac{(-q^2;q^2)_{2s}(q^{-1-2s},q^{2s+3};q^2)_{n-s}(q^{-2},q^6;q^4)_s}{(q^{4s+2};q^4)_{n-s}(q^4;q^4)_{n-s}(q^4;q^4)_{2s}}q^{-s^2+4}=\frac{(q;q^2)_\infty^2}{(q^2;q^2)_\infty}.$$

mod 2 identity:

$$(D1)(D2)=\begin{bmatrix}1 & -1\\ 0 & 4\end{bmatrix},$$

$$\lim_{n\to\infty}\sum_{s=0}^{n}\frac{(-q;q)_{2s}(-1)^{n-s+1}q^{s^2+3s-2sn-n+1}}{(q^2;q^2)_{n-s}(q^4;q^4)_s}=\frac{1}{(q^2;q^2)_\infty}.$$

mod 2 identity:

$$(S1)(D2)=\begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix},$$

$$\lim_{n\to\infty}\sum_{s=0}^{n}\frac{(-q;q)_{2s}(-1)^{n-s}q^{s^2+s-2sn-n}}{(q^2;q^2)_{n-s}(q^2;q^2)_s}=\frac{(-q^2;q^2)_\infty}{(q;q)_\infty}.$$

mod 6 identity:

$$(D1)(D3) = \begin{bmatrix} 1 & -1/2 \\ 0 & 4 \end{bmatrix},$$

$$\lim_{n\to\infty} \sum_{s=0}^{n} \frac{(-q^2;q^2)_{2s}(q^{-1-2s},q^{2s+3};q^2)_{n-s}q^{-s^2+4s+1}}{(q^{4s+2};q^4)_{n-s}(q^4;q^4)_{n-s}(q^8;q^8)_s} = \frac{-(q^6,q^1,q^5;q^6)_\infty}{(q^2;q^2)_\infty^2}.$$

mod 6 identity:

$$(S2)(D3) = \begin{bmatrix} 1 & 1/2 \\ 0 & 2 \end{bmatrix},$$

$$\lim_{n\to\infty} \sum_{s=0}^{n} \frac{(-q^2;q^2)_{2s}(q^{-1-2s},q^{2s+3};q^2)_{n-s}q^{-s^2}}{(q^{4s+2};q^4)_{n-s}(q^4;q^4)_{n-s}(q^4;q^4)_s(-q^2;q^4)_s} = \frac{(q^6,q^1,q^5;q^6)_\infty}{(q^2;q^2)_\infty^2}.$$

mod 6 identity:

$$(T1)(D3) = \begin{bmatrix} 1 & 1/6 \\ 0 & 2/3 \end{bmatrix},$$

$$\lim_{n\to\infty} \sum_{s=0}^{n} \frac{(q^{-3-6s},q^{6s+9};q^6)_{n-s}(q^4,q^8;q^{12})_s q^{-3s^2}}{(q^{12s+6};q^{12})_{n-s}(q^{12};q^{12})_{n-s}(q^6;q^6)_{2s}} = \frac{(q^1,q^5;q^6)_\infty}{(q^6;q^6)_\infty}.$$

6. Multisum identities. In this section we give several specific examples of multisum identities which correspond to group elements via Theorem 1.

Andrews-Gordon identities, ($k \geq 1$): Since

$$(S1)^{k+1} = \begin{bmatrix} 1 & k+1 \\ 0 & 1 \end{bmatrix},$$

the result from (4.1) is

$$\sum_{s_1\geq s_2\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+\cdots+s_k^2}}{(q;q)_{s_1-s_2}(q;q)_{s_2-s_3}\cdots(q;q)_{s_k}} = \frac{(q^{2k+3},q^{k+2},q^{k+1};q^{2k+3})_\infty}{(q;q)_\infty}.$$

Bressoud identities, ($k \geq 2$): Choose

$$(D1)(S1)^{k-1} = \begin{bmatrix} 1 & k-1 \\ 0 & 2 \end{bmatrix},$$

the result from (4.1) is

$$\sum_{s_1\geq s_2\geq\cdots\geq s_{k-1}\geq 0} \frac{q^{s_1^2+\cdots+s_{k-1}^2+s_{k-1}}}{(q;q)_{s_1-s_2}(q;q)_{s_2-s_3}\cdots(q;q)_{s_{k-2}-s_{k-1}}(q^2;q^2)_{s_{k-1}}} = \frac{(q^{2k},q^{k-1},q^{k+1};q^{2k})_\infty}{(q;q)_\infty}.$$

mod 2^k+2i identities, ($k+i \geq 2$): Choose

$$(D1)^k(S1)^i = \begin{bmatrix} 1 & i \\ 0 & 2^k \end{bmatrix},$$

$$\sum_{s_{k+i-1}\geq\cdots\geq s_1\geq s_0=0}\frac{q^{s_{k+i-1}^2+\cdots+s_k^2}}{\prod_{j=k}^{k+i-2}(q;q)_{s_{j+1}-s_j}}\prod_{j=0}^{k-1}\frac{q^{2^j(s_{k-j}-s_{k-j-1})}(-q^{2^j};q^{2^j})_{2s_{k-j-1}}}{(q^{2^{j+1}};q^{2^{j+1}})_{s_{k-j}-s_{k-j-1}}}$$
$$=\frac{(q^{2^k+2i},q^{2^k+i},q^i;q^{2^k+2i})_\infty}{(q;q)_\infty}.$$

If $k=0$ these are the Andrews-Gordon identities, while for $k=1$ they are the Bressoud identities. The $i=2$ case was previously given in [**11, Corollary 4.4**].

mod $i2^{k+1}+1$ identities, $(k+i\geq 2)$: Choose

$$(E1)^k(S1)^i=\begin{bmatrix}1 & i\\ 0 & 2^{-k}\end{bmatrix},$$

$$\sum_{s_{k+i-1}\geq\cdots\geq s_1\geq s_0=0}\frac{q^{2^k(s_{k+i-1}^2+\cdots+s_k^2)}}{\prod_{j=k}^{k+i-2}(q^{2^k};q^{2^k})_{s_{j+1}-s_j}}\prod_{j=0}^{k-1}\frac{(-1)^{s_{j+1}-s_j}q^{2^j(s_{j+1}-s_j)^2}}{(q^{2^{j+1}};q^{2^{j+1}})_{s_{j+1}-s_j}(-q^{2^j};q^{2^j})_{2s_{j+1}}}$$
$$=\frac{(q^{i2^{k+1}+1},q^{i2^k+1},q^{i2^k};q^{i2^{k+1}+1})_\infty}{(q^{2^k};q^{2^k})_\infty}.$$

If $k=0$ these are again the Andrews-Gordon identities, for $k=i=1$ they are Rogers' identities for modulus 5.

mod $(i+1)2^{k+1}-1$ identities, $(k+i\geq 2)$: Choose

$$(E2)^k(S1)^i=\begin{bmatrix}1 & 1+i-2^{-k}\\ 0 & 2^{-k}\end{bmatrix},$$

$$\sum_{s_{k+i-1}\geq\cdots\geq s_1\geq s_0=0}\frac{q^{2^k(s_{k+i-1}^2+\cdots+s_k^2)}}{\prod_{j=k}^{k+i-2}(q^{2^k};q^{2^k})_{s_{j+1}-s_j}}\prod_{j=0}^{k-1}\frac{q^{2^j s_j^2}}{(q^{2^{j+1}};q^{2^{j+1}})_{s_{j+1}-s_j}(-q^{2^j};q^{2^j})_{2s_{j+1}}}$$
$$=\frac{(q^{(i+1)2^{k+1}-1},q^{(i+1)2^k},q^{(i+1)2^k-1};q^{(i+1)2^{k+1}-1})_\infty}{(q^{2^k};q^{2^k})_\infty}.$$

If $k=0$ these are again the Andrews-Gordon identities, for $k=i=1$ they are Rogers' identities for modulus 7.

mod $(2i+1)3^k$ identities, $(k+i\geq 2)$: Choose

$$(T1)^k(S1)^i=\begin{bmatrix}1 & i+(1-3^{-i})/2\\ 0 & 2^k3^{-i}\end{bmatrix},$$

$$\sum_{s_{k+i-1}\geq\cdots\geq s_1\geq s_0=0}\frac{q^{3^k(s_{k+i-1}^2+\cdots+s_k^2)}}{\prod_{j=k}^{k+i-2}(q^{3^k};q^{3^k})_{s_{j+1}-s_j}}$$
$$\times\prod_{j=0}^{k-1}\frac{(q^{3^j};q^{3^j})_{3s_{j+1}-s_j}q^{3^js_j^2}}{(q^{3^{j+1}};q^{3^{j+1}})_{2s_{j+1}}(q^{3^{j+1}};q^{3^{j+1}})_{s_{j+1}-s_j}}$$
$$=\frac{(q^{(2i+1)3^k},q^{(2i+1)3^k/2-1/2},q^{(2i+1)3^k/2+1/2};q^{(2i+1)3^k})_\infty}{(q^{3^k};q^{3^k})_\infty}.$$

If $k=i=1$ this is Bailey's mod 9 identity.

mod 3^i+2^k-1 identities ($i\geq 1, k\geq 0$): Choose

$$(D1)^k(T1)^i=\begin{bmatrix}1 & (1-3^{-i})/2\\ 0 & 2^k3^{-i}\end{bmatrix},$$

$$\sum_{s_{k+i-1}\geq\cdots\geq s_1\geq s_0=0} q^{3^{i-1}s_{k+i-1}^2}\prod_{j=k}^{k+i-2}\frac{q^{3^{j-k}s_j^2}(q^{3^{j-k}};q^{3^{j-k}})_{3s_{j+1}-s_j}}{(q^{3^{j-k+1}};q^{3^{j-k+1}})_{2s_{j+1}}(q^{3^{j-k+1}};q^{3^{j-k+1}})_{s_{j+1}-s_j}}$$
$$\times\prod_{j=0}^{k-1}\frac{q^{2^j(s_{k-j}-s_{k-j-1})}(-q^{2^j};q^{2^j})_{2s_{k-j-1}}}{(q^{2^{j+1}};q^{2^{j+1}})_{s_{k-j}-s_{k-j-1}}}$$
$$=\frac{(q^{3^i+2^k-1},q^{2^k+(3^i-1)/2},q^{(3^i-1)/2};q^{3^i+2^k-1})_\infty}{(q^{3^{i-1}};q^{3^{i-1}})_\infty}.$$

If $k=i=1$, this is the special case of the q-binomial theorem, which says that partitions of N into odd parts are equinumerous with partitions of N into distinct parts.

We next give examples of mod 11 identities which are double sums. Any word $w=w_1w_2w_3$

$$w=\begin{bmatrix}1 & A\\ 0 & B\end{bmatrix},$$

with $(2A+B)/B=11$ will give such an identity. A Mathematica run finds all 16 such words. These 16 words give 6 distinct identities. We list these 6 identities along with a representative word.

$$(S1)(T1)(S1)=\begin{bmatrix}1 & 5/3\\ 0 & 1/3\end{bmatrix},$$

$$\sum_{s_1,s_2\geq 0}\frac{q^{3s_1^2+s_2^2}(q;q)_{3s_1-s_2}}{(q^3;q^3)_{2s_1}(q^3;q^3)_{s_1-s_2}(q;q)_{s_2}}=\frac{(q^5,q^6,q^{11};q^{11})_\infty}{(q^3;q^3)_\infty}, \tag{6.1}$$

$$(E2)(E1)(S1)=\begin{bmatrix}1 & 5/4\\ 0 & 1/4\end{bmatrix},$$

$$\sum_{s_1,s_2\geq 0}\frac{q^{4s_1^2+2(s_1-s_2)^2}(-1)^{s_1-s_2}}{(-q^2;q^2)_{2s_1}(q^4;q^4)_{s_1-s_2}(-q;q)_{2s_2}(q^2;q^2)_{s_2}}=\frac{(q^5,q^6,q^{11};q^{11})_\infty}{(q^4;q^4)_\infty}, \tag{6.2}$$

$$(E2)(E2)(S2)=\begin{bmatrix}1 & 5/4\\ 0 & 1/4\end{bmatrix},$$

$$\sum_{s_1,s_2\geq 0}\frac{q^{2s_2^2+2s_1^2}(-q^2;q^4)_{s_1}}{(-q^2;q^2)_{2s_1}(q^4;q^4)_{s_1-s_2}(-q;q)_{2s_2}(q^2;q^2)_{s_2}}=\frac{(q^5,q^6,q^{11};q^{11})_\infty}{(q^8;q^8)_\infty(q^2;q^4)_\infty}, \tag{6.3}$$

$$(E2)(S1)(S1)=\begin{bmatrix}1 & 5/2\\ 0 & 1/2\end{bmatrix},$$

$$\sum_{s_1,s_2\geq 0}\frac{q^{2s_2^2+2s_1^2}}{(q^2;q^2)_{s_1-s_2}(-q;q)_{2s_2}(q^2;q^2)_{s_2}}=\frac{(q^5,q^6,q^{11};q^{11})_\infty}{(q^2;q^2)_\infty}, \tag{6.4}$$

$$(E1)(E3)(S1) = \begin{bmatrix} 1 & 5/4 \\ 0 & 1/4 \end{bmatrix},$$

(6.5)
$$\sum_{s_1,s_2\geq 0} \frac{q^{2s_2^2+4s_1^2}(-1)^{s_2}(-q;q^2)_{s_2}(q;q^2)_{2s_1-s_2}}{(-q^2;q^2)_{2s_1}(q^2;q^4)_{s_1}(q^4;q^4)_{s_1-s_2}(-q;q)_{2s_2}(q^2;q^2)_{s_2}} = \frac{(q^5,q^6,q^{11};q^{11})_\infty}{(q^4;q^4)_\infty},$$

$$(E1)(T1)(S2) = \begin{bmatrix} 1 & 5/6 \\ 0 & 1/6 \end{bmatrix},$$

(6.6)
$$\sum_{s_1,s_2\geq 0} \frac{q^{3s_1^2+3s_2^2}(-1)^{s_2}(-q^3;q^6)_{s_1}(q^2;q^2)_{3s_1-s_2}}{(q^6;q^6)_{2s_1}(q^6;q^6)_{s_1-s_2}(-q;q)_{2s_2}(q^2;q^2)_{s_2}} = \frac{(q^5,q^6,q^{11};q^{11})_\infty}{(q^{12};q^{12})_\infty(q^3;q^6)_\infty}.$$

Note that the 2nd, 3rd, and 5th identities are distinct even though they correspond to the same group element. Perhaps the easiest version of this is $(S1)(S2) = (S2)(S1)$, see §5.

These six identities, particularly (6.2) and (6.5), are reminiscent of, but not the same as, Andrews' mod 11 identities in [**4**], one of which is

$$\sum_{n,j=0}^{\infty} \frac{(q;q)_{4n+2j}(-1)^j q^{4n^2+12nj+8j^2+j}}{(q^4;q^4)_n(q^2;q^2)_j(q^4;q^4)_{2n+2j}} = \frac{(q^5,q^6,q^{11};q^{11})_\infty}{(q^4;q^4)_\infty}.$$

Perhaps the most exotic double sum which appears corresponds to

$$(T1)(T1)(S1) = \begin{bmatrix} 1 & 13/9 \\ 0 & 1/9 \end{bmatrix},$$

$$\sum_{s_1,s_2\geq 0} \frac{q^{9s_2^2+3s_1^2}(q^3;q^3)_{3s_2-s_1}(q;q)_{3s_1}}{(q^9;q^9)_{2s_2}(q^9;q^9)_{s_2-s_1}(q^3;q^3)_{2s_1}(q^3;q^3)_{s_1}} = \frac{(q^{13},q^{14},q^{27};q^{27})_\infty}{(q^9;q^9)_\infty}.$$

There are 348 words $w = w_1w_2w_3w_4$ of length 4 with corresponding integer values of $11B/(2A+B)$, these could lead to 202 possible triple sum identities modulo multiples of 11.

7. Borwein polynomials. In this section we use the group element

$$(T1) = \begin{bmatrix} 1 & 1/3 \\ 0 & 1/3 \end{bmatrix}$$

to find alternative forms of the Borwein polynomials.

These polynomials were defined by Andrews [**6**] as

$$A_n(q) = \sum_{k=-\infty}^{\infty} \begin{bmatrix} 2n \\ n-3k \end{bmatrix}_q (-1)^k q^{k(9k-1)/2}$$

$$B_n(q) = \sum_{k=-\infty}^{\infty} \begin{bmatrix} 2n \\ n-3k-1 \end{bmatrix}_q (-1)^k q^{k(9k+5)/2}$$

$$C_n(q) = \sum_{k=-\infty}^{\infty} \begin{bmatrix} 2n \\ n-3k+1 \end{bmatrix}_q (-1)^k q^{k(9k-7)/2}.$$

A conjecture of P. Borwein is equivalent (see [**6**], [**9**], [**15**]) to the conjecture that $A_n(q)$, $B_n(q)$, and $C_n(q)$ have non-negative coefficients as polynomials in q. We give in Theorem 2 an alternative form of these polynomials.

First we review some the hook difference polynomials, which may be defined by [**7**]

$$D_{K,i}(N,M;\alpha,\beta)(q) = \sum_{\lambda=-\infty}^{\infty} q^{\lambda(K\lambda+i)(\alpha+\beta)-K\beta\lambda} \begin{bmatrix} N+M \\ N-K\lambda \end{bmatrix}_q$$

$$- \sum_{\lambda=-\infty}^{\infty} q^{\lambda(K\lambda-i)(\alpha+\beta)-K\beta\lambda+\beta i} \begin{bmatrix} N+M \\ N-K\lambda+i \end{bmatrix}_q . \tag{7.1}$$

Note that

$$D_{6,3}(N,M;\alpha,\beta,q) = \sum_{k=-\infty}^{\infty} \begin{bmatrix} N+M \\ N-3k \end{bmatrix}_q (-1)^k q^{3k^2(\alpha+\beta)/2+3k(\alpha-\beta)/2},$$

$$\begin{aligned} D_{6,3}(N,M;\alpha,\beta,q) =& D_{6,3}(M,N;\beta,\alpha,q) \\ D_{6,3}(N,M;\alpha,\beta,q) =& q^{MN} D_{6,3}(M,N;3+M-N-\alpha,3-M+N-\beta,q^{-1}) \end{aligned}$$

so that

$$\begin{aligned} A_n(q) =& D_{6,3}(n,n;4/3,5/3,q) = D_{6,3}(n,n;5/3,4/3,q), \\ B_n(q) =& D_{6,3}(n-1,n+1;7/3,2/3,q) = D_{6,3}(n+1,n-1;2/3,7/3,q), \\ C_n(q) =& D_{6,3}(n+1,n-1;1/3,8/3,q) = q^{n^2-1} D_{6,3}(n+1,n-1;2/3,7/3,q^{-1}). \end{aligned}$$

It is known [**7**] that if α and β are positive integers satisfying

$$\alpha+\beta < K, \quad -i+\beta \le N-M \le K-i-\alpha,$$

then the $D_{K,i}(N,M;\alpha,\beta)(q)$ has non-negative coefficients. The next proposition realizes fractional values as an element of a Bailey pair.

Proposition 1. *For (T1),*
1. *if* $\beta_n = (q;q)_{2n}^{-1} D_{6,3}(n,n;\alpha,\beta,q)$,
   then $\beta'_n = (q^3;q^3)_{2n}^{-1} D_{6,3}(n,n;1+\alpha/3,1+\beta/3,q^3)$,
2. *if* $\beta_n = (q;q)_{2n}^{-1} D_{6,3}(n-1,n+1;\alpha,\beta,q)$,
   then $\beta'_n = q(q^3;q^3)_{2n}^{-1} D_{6,3}(n-1,n+1;(5+\alpha)/3,(1+\beta)/3,q^3)$.

Proof. If $a=1$ in a Bailey pair, we have

$$\beta_n(1,q) = (q;q)_{2n}^{-1} \sum_{k=0}^{n} \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q \alpha_k(1,q).$$

For part (1) choose

$$\alpha_k = \begin{cases} 1 \quad \text{if } k=0, \\ (-1)^K q^{3K^2(\alpha+\beta)/2}(q^{3K(\alpha-\beta)/2} + q^{-3K(\alpha-\beta)/2}), \text{ if } k=3K>0, \\ 0 \quad \text{otherwise.} \end{cases}$$

so that $\beta_n(1,q) = (q;q)_{2n}^{-1} D_{6,3}(n,n;\alpha,\beta,q)$. Applying (T1) we see that

$$\beta'_n = (q^3;q^3)_{2n}^{-1} D_{6,3}(n,n;\alpha',\beta',q^3),$$

where

$$3(\alpha+\beta)/2+9=9(\alpha'+\beta')/2, \quad 3(\alpha-\beta)/2=9(\alpha'-\beta')/2.$$

The solution is $\alpha'=1+\alpha/3$, $\beta'=1+\beta/3$.

For part (2), choose non-zero values

$$\begin{aligned}\alpha_{3k+1}=&(-1)^k q^{3k^2(\alpha+\beta)/2+3k(\alpha-\beta)/2},\\ \alpha_{3k-1}=&(-1)^k q^{3k^2(\alpha+\beta)/2-3k(\alpha-\beta)/2},\end{aligned}$$

and apply (T1). □

Now we use the known values [**13, Proposition 2**]

$$\begin{aligned}&D_{6,3}(n,n;1,2,q)=(1+q^n)\frac{(q^3;q^3)_{n-1}}{(q;q)_{n-1}},\\ &D_{6,3}(n+1,n-1;1,2,q)=D_{6,3}(n-1,n+1;2,1,q)=\frac{(q^3;q^3)_{n-1}}{(q;q)_{n-1}},\end{aligned}$$

to obtain from Proposition 1 the following theorem.

Theorem 2. *We have*

$$\begin{aligned}A_n(q^3)=&\frac{(q;q)_{3n}}{(q^3;q^3)_n}+\sum_{k=1}^{n}\frac{(q;q)_{3n-3k}}{(q^3;q^3)_{n-k}}q^{k^2}\begin{bmatrix}3n-k\\2k\end{bmatrix}_q\frac{(q^3;q^3)_{k-1}}{(q;q)_{k-1}}(1+q^k),\\ qB_n(q^3)=&\sum_{k=1}^{n}\frac{(q;q)_{3n-3k}}{(q^3;q^3)_{n-k}}q^{k^2}\begin{bmatrix}3n-k\\2k\end{bmatrix}_q\frac{(q^3;q^3)_{k-1}}{(q;q)_{k-1}},\\ q^2C_n(q^3)=&\sum_{k=1}^{n}\frac{(q;q)_{3n-3k}}{(q^3;q^3)_{n-k}}q^{k^2+k}\begin{bmatrix}3n-k\\2k\end{bmatrix}_q\frac{(q^3;q^3)_{k-1}}{(q;q)_{k-1}}.\end{aligned}$$

Finding the $n\to\infty$ limits of Theorem 2 gives the mod 27 identities in Slater [**14**]: (93), (91), and (90) respectively.

Unfortunately Theorem 2 does not establish positivity for the polynomials, but new recurrences do follow from Theorem 2 using Axel Riese's q-Zeil package. If $\gamma_n=qB_n(q^3)$ or $q^2C_n(q^3)$ then we have

$$\begin{aligned}\gamma_n=&-q^{-15}(q^8+q^{6n}+q^{4+3n})(q^{10}+q^{6n}+q^{5+3n})\gamma_{n-2}\\ &+q^{-6}(q^6+q^9+q^{6n}+q^{6n+3}+q^{4+3n}+q^{5+3n})\gamma_{n-1}\\ &-q^{6n-7}(1+q+q^2)(q;q)_{3n-6}/(q^3;q^3)_{n-2}, \quad n\ge 2.\end{aligned}$$

8. Remarks. One may ask where the second Rogers-Ramanujan identity appears from the Bailey-Rogers-Ramanujan group. The group may be extended by a simple transformation on Bailey pairs (see [**11, Proposition 4.1**]) which puts

$$\beta_n'(1,q)=q^n\beta_n(1,q).$$

The full Andrews-Gordon identities then appear. However once an element of type (D), (E) or (T) is used in our word the base q has changed, and only certain linear exponents may be inserted. These, in turn, allow special sets of excluded bases on the product sides. An example of these choices is given in [**11, Corollary 4.4**].

Some of the sums involving $\lim_{n\to\infty}$ in §5 have striking finite forms. For example, the finite sum factors to

$$\begin{aligned}
\frac{(q,q^5;q^6)_n}{(q^6;q^6)_{2n}} &\quad \text{for (T1)(D3)},\\
(q;q^2)_{2n-1}(1-q^{4n+1}) &\quad \text{for (E3)(T2)},\\
(q;q^2)_{n-3}(q;q^2)_{n+3} &\quad \text{for (D3)(T2)},\\
\frac{(q;q^2)_{n+2}(q;q^2)_{n-2}}{(q^2;q^2)_{2n}} &\quad \text{for (D3)(D3)},\\
\frac{1-q^{2n}-q^{2n+1}}{(q^2;q^2)_{2n}} &\quad \text{for (D1)(D2)}.
\end{aligned}$$

Perhaps these elements of the group are particularly useful.

Another large set of infinite families of multisum Rogers-Ramanujan identities has been given by Warnaar [**15**].

A combinatorial interpretation of the Rogers-Ramanujan identity which corresponds to a general element of the Bailey-Rogers-Ramanujan group is not known. Some preliminary work on the mod 2^k+2i identities for $i=2$ has been done by Bressoud [**10**].

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