Midterm 1 will consists of 3–4 problems from the following five areas:

1. Linear first-order equations $\frac{dx}{dt} + a(t)x = b(t)$;
2. separable equations $\frac{dx}{dt} = f(x)a(t)$;
3. second order equations $a\frac{d^2}{dt^2}x + b\frac{dx}{dt} + cx = f(t)$;
4. 2×2 systems $\ddot{x} + Ax = 0$ with a symmetric matrix A;
5. first-order 2×2 systems $\dot{x} = Ax$.

Sample problems:

1. Exercise (1), page 19. (You can also do the other 6 exercises on that page, although some of them are computationally more difficult that what will be in the midterm.)

2. Exercise (1), page 25. (You can also do exercises (2) and (3). Exercise (4) was in one of the homework assignments. Exercise (5) concerns homogeneous equations, see p. 23. You can also think about the variant of Exercise (1) when $x(0) = \varepsilon$ is a small number $\varepsilon \neq 0$.)

3. Find the general solution of the equation

$$\ddot{x} + \dot{x} + x = \sin t + \cos 2t\,. \tag{1}$$

Hint: the equation is linear and therefore one can solve it separately for each term on the right-hand side and then add the two solutions.

4. Consider the 2×2 second-order system

$$\ddot{x} = Ax, \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad A = \begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix}. \tag{2}$$

Find a change of variables $x = Py$ (where P is a suitable non-singular 2×2 matrix) in which the system becomes diagonal, i. e.

$$\begin{aligned} \ddot{y}_1 &= \lambda_1 y_1\,, \\ \ddot{y}_2 &= \lambda_2 y_2\,. \end{aligned} \tag{3}$$

Hint: use the eigenvectors of the matrix A. See also lecture 15, pp. 53,54 in the Lecture Log.

5. Consider the 2×2 system of equations

$$\begin{aligned} \dot{x}_1 &= -px_1 + qx_2\,, \\ \dot{x}_2 &= px_1 - qx_2\,, \end{aligned} \tag{4}$$

where $0 \leq p, q < 1$. Consider a solution $x(t)$ of the system with $x_1(0) = \overline{x}_1 > 0$ and $x_1(0) = \overline{x}_2 > 0$. Show that

$$\lim_{t\to\infty} x(t) = \begin{pmatrix} \frac{q}{p+q} \\ \frac{p}{p+q} \end{pmatrix} (\overline{x}_1 + \overline{x}_2)\,. \tag{5}$$

Hint: use the eigenvectors/eigenvalues of the matrix $\begin{pmatrix} -p & q \\ p & -q \end{pmatrix}$. See also the example considered on lecture 14 (Lecture Log, p. 47).