Midterm 2 will consists of 3–4 problems from the following three areas:

1. linear systems with constant coefficients $x' = Ax + f(t)$ (for general matrix A) and the second order systems $x'' + Ax = f(t)$ for a symmetric matrix A;
2. higher-order linear equations with constant coefficients $x^{(m)} + a_1 x^{(m-1)} + \ldots + a_m x = f(t)$;
3. Jordan form for matrices; skew-adjoint matrices (Section 2.11 in the textbook; orthogonal matrices (Section 2.12 in the textbook).

Sample problems:

1. Let $a, b, c \in (0, \infty)$ and consider equation $au'''' - bu'' + cu = 0$ in $\mathbf{R}$ for a complex-valued function $u = u(x)$. Let X be the space of all solutions of the equation and let $X_+ = \{u \in X, \limsup_{x\to\infty} |u(x)| < \infty\}$ and $X_- = \{u \in X, \limsup_{x\to-\infty} |u(x)| < \infty\}$. Determine the dimensions $\dim X_+ , \dim X_-$ and decide whether $X = X_+ \oplus X_-$.

2. Consider the system

$$\begin{aligned} x_1'' + 2x_1 + x_2 &= 0\,, \\ x_2'' + x_1 + 2x_2 &= \cos\omega t\,. \end{aligned}$$

Find all values ω for which all solutions of the system in the interval $(0, \infty)$ remain bounded.

3. Consider the system of equations

$$\begin{aligned} x_1' &= x_2 - x_1\,, \\ x_2' &= x_3 - x_2\,, \\ x_3' &= x_1 - x_3\,. \end{aligned}$$

Show that for each solution $x(t)$ and each $j = 1, 2, 3$ we have $x_j(t) \to \frac{x_1(0)+x_2(0)+x_3(0)}{3}$ as $t \to \infty$.

4. Assume that A is an $n \times n$ complex matrix with characteristic polynomial of the form $\det(A - \lambda I) = (\alpha - \lambda)^n$ for some $\alpha \in \mathbf{C}$. Show that e^{tA} is given by

$$e^{tA} = e^{\alpha t}\left(I + t(A - \alpha I) + \frac{t^2}{2!}(A - \alpha I)^2 + \ldots + \frac{t^{n-1}}{(n-1)!}(A - \alpha I)^{n-1}\right) .$$

(The main point is that the sum is finite.)