1. We recall that $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$ and hence
$\int_0^y \frac{dx}{1+x^2} = \arctan y - \arctan 0 = \arctan y\,.$
Alternatively, one can use the substitution $x = \tan t$. Then $dx = d(\tan t) = (1+\tan^2 t)\,dt\,, \quad \frac{dx}{1+x^2} = dt$ and letting $s = \arctan y$, we have
$\int_0^y \frac{dx}{1+x^2} = \int_0^s dt = s\,.$

2. We recall that $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$ and hence
$\int_0^y \frac{dx}{\sqrt{1-x^2}} = \arcsin y - \arcsin 0 = \arcsin y\,.$
Alternatively, one can use the substitution $x = \sin t\,.$ Then $\frac{dx}{\sqrt{1-x^2}} = dt$ and letting $s = \arcsin y$ we have $\int_0^y \frac{dx}{\sqrt{1-x^2}} = \int_0^s dt = s\,.$
In this problem the value of y is restricted to $|y| \le 1$. [1]

3. We recall that $\frac{d}{dx}\mathrm{arcsinh}\, x = \frac{1}{\sqrt{1+x^2}}$ and hence
$\int_0^y \frac{dx}{\sqrt{1+x^2}} = \mathrm{arcsinh}\, y - \mathrm{arcsinh}\, 0 = \mathrm{arcsinh}\, y = \log(y + \sqrt{y^2+1}).$
Alternatively, one can use the substitution $x = \sinh t$. Then $\frac{dx}{\sqrt{1+x^2}} = dt$ and letting $s = \mathrm{arcsinh}\, y$ we have $\int_0^y \frac{dx}{\sqrt{1+x^2}} = \int_0^s dt = s\,.$

4. (i) $\frac{d}{dt}\log|\sec t| = -\frac{d}{dt}\log|\cos t| = \frac{\sin t}{\cos t} = \tan t$. (Here we have used $\frac{d}{dx}\log|x| = \frac{1}{x}$.) The formula valid in the classical sense only in intervals where $\cos t$ does not vanish.
Alternatively, in the integral $\int \tan t\,dt$ we can set $\cos t = x$ Then $(\sin t)dt = -dx$ and $\int \tan t\,dt = \int -\frac{dx}{x} = -\log x = \log\frac{1}{x} = \log\sec x$, where $t \in (-\frac{\pi}{2}, \frac{\pi}{2}$ and all equalities are considered modulo a constant. (Extension to other intervals is obvious.)
(ii) $\frac{d}{dt}\log|\sec t + \tan t| = \frac{1}{\sec t+\tan t}(\sec' t + \tan' t) = \frac{\cos t}{1+\sin t}(\frac{\sin t}{\cos^2 t} + \frac{1}{\cos^2 t}) = \frac{1}{\cos t}\,.$
If we wish to calculate $\int \sec t\,dt$ "from scratch" there are several substitutions which can be used. For example, the classical substitution $\tan\frac{t}{2} = x$ used for trigonometric integrals gives $\sec t\,dt = \frac{2dx}{1-x^2}$ and hence $\int \sec t\,dt = \int \frac{2dx}{1-x^2} = \int(\frac{1}{1+x} + \frac{1}{1-x})\,dx = \log\left(\frac{1+x}{1-x}\right) = \log(\sec t + \tan t)$, assuming $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and taking all the equalities with the integrals modulo a constant. Another way to do the integral is $\int \sec t\,dt = \int \frac{\cos t\,dt}{\cos^2 t} = \int \frac{\cos t\,dt}{1-\sin^2 t}$ which after the substitution $\sin t = x$ becomes $\int \frac{-dx}{1-x^2} = -\frac{1}{2}\log\frac{1-x}{1+x} = \log\frac{1+x}{\sqrt{1-x^2}} = \log(\sec t + \tan t)$, where we assumed $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$ during the calculation and the equalities are taken modulo constants.

[1] A side remark: one can play with extending the formula beyond this range, but it requires some complex analysis. For example, we have $\arcsin y = \frac{1}{i}\log(\sqrt{1-y^2} + iy)$ and this formula could be used to extend the integral for $|y| > 1$. The extension is not unique, as the function $\arcsin y$ is a multi-valued function when considered in the complex plane, with branch points at $y = \pm 1$.

(iii) $\left[\frac{d}{dt}(\sec t\ \tan t)\right] + \sec t = \tan^2 t\ \sec t + \sec t\,(1+\tan^2 t) + \sec t =$
$= 2\sec t(1+\tan^2 t) = 2\sec^3 t$. If we wish to calculate "from scratch" we can write $\int \sec^3 t\, dt = \int \frac{\cos t\, dt}{(1-\sin^2 t)^2} = \int \frac{-dx}{(1-x^2)^2} = \frac{1}{2}\frac{x}{1-x^2} + \frac{1}{4}\log\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\sec t \tan t +$
$+ \frac{1}{2}\int \sec t\, dt$, where we used (ii). Again, we first work with $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and then extend the formula to the remaining intervals.

5. We can write $\frac{dx}{\sqrt{x^2+1}} = dt$. Recalling the integral $\int \frac{dx}{\sqrt{x^2+1}}$ from Problem 3, integrating between 0 and x on left-hand side and between 0 and t on the right-hand side, we obtain $\operatorname{arcsinh} x = t$ which is the same as $x = \sinh t$. It is easy to verify by direct calculation that this function indeed solves our problem.

6. Letting $\beta = \frac{\alpha}{m}$, we can write $\frac{dv}{g-\beta v} = dt$. Integrating over $(0, v)$ on the left-hand side and over $(0, t)$ on the right-hand side, we obtain $-\frac{1}{\beta}\log(\frac{g-\beta v}{g}) = t$. Solving for v, we obtain $v = \frac{g}{\beta}(1-e^{-\beta t})$. We see that the solution approaches $\frac{g}{\beta}$ (from below) as $t \to \infty$ (and this can be in fact seen without calculation, by looking at the phase diagram as discussed in Lecture 4), and therefore it makes sense to call $\frac{g}{\beta}$ the terminal velocity.

Optional part: we can write $\frac{dv}{g-\sigma v^2} = dt$. We have $\int_0^v \frac{dv}{g-\sigma v^2} =$
$= \frac{1}{2\sqrt{g}}\int_0^v \left(\frac{1}{\sqrt{g}+\sqrt{\sigma}v} + \frac{1}{\sqrt{g}-\sqrt{\sigma}v}\right) dv = \frac{1}{2\sqrt{g\sigma}}\log\frac{\sqrt{\frac{g}{\sigma}}+v}{\sqrt{\frac{g}{\sigma}}-v}$. This expression should be equal to the integral of the right-hand side over $(0, t)$, which is t. An easy calculation now shows $v = \sqrt{\frac{g}{\sigma}}\frac{1-e^{-2\sqrt{g\sigma}t}}{1+e^{-2\sqrt{g\sigma}t}}$. We can see that $v \to \sqrt{\frac{g}{\sigma}}$ (from below) as $t \to \infty$. This can be seen again without calculation, from the phase portrait. The equation in the optional part is sound from the point of view of physics only for $v \geq 0$, although it can be solved also for negative values of v.

7*. (Optional) We can write $\frac{dx}{x^{1-\varepsilon}} = -a dt$ and integrating on both sides we have $\frac{1}{\varepsilon}x^\varepsilon - \frac{1}{\varepsilon}x_0^\varepsilon = -at$. Hence $x(t) = (x_0^\varepsilon - \varepsilon at)^{\frac{1}{\varepsilon}}$ for $t \in [0, \frac{x_0^\varepsilon}{\varepsilon a}]$, with $x(t)$ vanishing at endpoint of this interval, while being strictly positive inside. We can also write $x(t) = x_0(1-\varepsilon\frac{at}{x_0^\varepsilon})^{\frac{1}{\varepsilon}}$. Recalling that $(1-\varepsilon y)^{\frac{1}{\varepsilon}} \to e^{-y}$ as $\varepsilon \to 0_+$ and using that $\varepsilon \to x_0^\varepsilon$ is increasing to 1 as ϵ decreases to 0, we see that for each small $\delta > 0$ we have $e^{-\frac{at}{(1-\delta)}} \leq \liminf_{\varepsilon\to 0_+} x(t) \leq \limsup_{\varepsilon \to 0_+} x(t) \leq e^{-at}$. Taking $\delta \to 0_+$, we obtain the required result. Alternatively, we can calculate $\log x(t) = \log x_0 + \frac{1}{\varepsilon}\log(1-\varepsilon\frac{at}{x_0^\varepsilon})$ and use $\log(1-y) = -y + O(y^2)$ for $y \to 0$, or calculate the limit of the expression $\frac{1}{\varepsilon}\log(1-\varepsilon\frac{at}{x_0^\varepsilon})$ from l'Hôpital's rule.

8*. (Optional) Writing $\frac{dx}{x} = -a(t)dt$ and integrating on both sides we have $x(t) = x_0 e^{-A(t)}$, with $A(t) = \int_0^t a(s)ds$ and the conclusion is clear from this formula. Alternatively, we could argue as follows: clearly $x(t) \to 0$ for $t \to T_+$ if and only if $\log x(t) \to -\infty$ for $t \to T_+$. The intergartion of $\frac{dx}{x} = -a(t)dt$ gives $\log x(t) = \log x_0 - A(t)$, the statement follows.