1. The characteristic polynomial of the homogeneous equation $u''+u'+u=0$ is $\lambda^2+\lambda+1=0$. The roots are $\lambda_{1,2}=\frac{-1\pm i\sqrt{3}}{2}=e^{\pm\frac{2\pi i}{3}}$. The general solution of the homogeneous equation is $C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}$. (There are other equivalent expressions, such as $\left[c_1e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right)+c_2e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$ or $Ce^{-\frac{t}{2}}\cos(\frac{\sqrt{3}}{2}(t-t_0))$.) We need to find a particular solution for the inhomogeneous equation. As $3\sin(\sigma t)=3\,\mathrm{Im}\,e^{i\sigma t}$, we can first solve $u''+u'+u=3e^{i\sigma t}$ and then take the imaginary part. As we did in class, we seek the solution of the last equation as $Ae^{i\sigma t}$. This gives $A=\frac{3}{1-\sigma^2+i\sigma}$ and hence a particular solution of the inhomogeneous equation is $v(t)=3\,\mathrm{Im}\,\frac{e^{i\sigma t}}{1-\sigma^2+i\sigma}=\frac{-3\sigma}{(1-\sigma^2)^2+\sigma^2}\cos\sigma t+\frac{3(1-\sigma^2)}{(1-\sigma^2)^2+\sigma^2}\sin\sigma t$. The general solution of the inhomogeneous equation then is $u(t)=v(t)+C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}$. (This expression can again be written in several ways.) One can also find the solution of the inhomogeneous equation by starting from $a\cos\sigma t+b\sin\sigma t$. When we substitute this expression into the equation, we get a system of two equations for the two unknowns a,b, which we can solve and arrive at $a=\frac{-3\sigma}{(1-\sigma^2)^2+\sigma^2}$, $b=\frac{3(1-\sigma^2)}{(1-\sigma^2)^2+\sigma^2}$, confirming the previous calculation.

2. We need to maximize $|A|$ from the previous problem. This is the same as minimizing $(1-\sigma^2)^2+\sigma^2$. Setting $\sigma^2=\tau$, we need to minimize $g(\tau)=(1-\tau)^2+\tau$ over $\tau\geq 0$. We can write $g(\tau)=(\frac{1}{2}-\tau)^2+\frac{3}{4}$ from which we see that the minimum is attained at $\tau=\frac{1}{2}$. (Instead of completing the square, we can work with the equation $g'(\tau)=0$.) Going back to σ we obtain $\sigma=\pm\frac{\sqrt{2}}{2}$. If we work in the real setting, writing the solution in the form $a\cos\sigma t+b\sin\sigma t$, we need to use the fact the the amplitude of the function given by the last expression is $\sqrt{a^2+b^2}$. (This can be seen several ways, for example by writing $a\cos\sigma t+b\sin\sigma t=\mathrm{Re}\,(a-ib)e^{i\sigma t}$, or $a\cos\sigma t+b\sin\sigma t=\sqrt{a^2+b^2}\cos\sigma(t+s)$ for a suitable s.)

3. We will solve $x''+x=e^{it}$ and take the imaginary part. The general solution of the homogeneous equation is $x(t)=C_1e^{it}+C_2e^{-it}$. To calculate a solution of the inhomogeneous equation, we can use the variation of constant, see lecture 10 in the lecture log. In the last expression we consider C_1 and C_2 as functions of t and set $C_1'e^{it}+C_2'e^{-it}=0$. The inhomogeneous equation then gives $iC_1'e^{it}-iC_2'e^{-it}=e^{it}$. Solving for C_1',C_2' (by using Cramer's rule, for example), we obtain $C_1'=-\frac{i}{2}$, $C_2'=\frac{i}{2}e^{2it}$. Hence we can take $C_1=-\frac{it}{2}$, $C_2=\frac{1}{4}e^{2it}$. Then $C_1e^{it}+C_2e^{-it}=e^{it}(-\frac{it}{2}+\frac{1}{4})$. Noticing that e^{it} is a solution of the homogeneous equation, we can take for our particular solution the function $-\frac{it}{2}e^{it}$. To obtain a particular solution of $x''+x=\sin t$, we take the imaginary part of $-\frac{it}{2}e^{it}$, obtaining $-\frac{1}{2}t\cos t$. One can check directly that this is a particular solution of our equation. The general solution then is $x(t)=-\frac{1}{2}t\cos t+C_1e^{it}+C_2e^{-it}$ here C_j are now constants, or, alternatively, $x(t)=-\frac{1}{2}t\cos t+c_1\cos t+c_2\sin t$, where c_1,c_2 are again constants. One can also do the variation of constants starting from $c_1 \cos t + c_2 \sin t$, considering c_1, c_2 as functions of t. If you do it this way, you may obtain expressions such as, for example, $x(t) = -\frac{1}{2} t \cos t + \frac{1}{4} \sin 2t \cos t + \frac{1}{2} \sin^3 t$.[1] This may at first look different than the expression obtained above, but it describes the same solutions: we note that $\frac{1}{4} \sin 2t \cos t + \frac{1}{2} \sin^3 t = \frac{1}{2} \sin t \cos^2 t + \frac{1}{2} \sin t \sin^2 t = \frac{1}{2} \sin t$ and the last function solves the homogeneous equation.

4. We have $(t^r)' = rt^{r-1}$ and $(t^r)'' = r(r-1)t^{r-2}$. Substituting these expression into the equation, we get $ar(r-1) + br + c = 0$. Alternatively, we can use the substitution $t = e^s$. Our equation then changes to $ax'' + (b-a)x' + cx = 0$ and the function t^r changes to e^{rs}. The characteristic equation for r will now be $ar^2 + (b-a)r + c = 0$, which is the same as $ar(r-1) + br + c = 0$.

5. The linear space of the solutions of the homogeneous equation has dimension 2 in this case. Hence we only have to show that the functions t^{r_1} and t^{r_2} are linearly independent over $\mathbf{C}$ in $(0, \infty)$. Let us consider the equation $C_1 t^{r_1} + C_2 t^{r_2} = 0$ for some constants C_1, C_2. Assuming the equation is satisfied at $t = t_1 > 0$ and at $t = t_2 > 0$, $t_2 \neq t_1$, we see that the constants C_1, C_2 must vanish when $\det \begin{pmatrix} t_1^{r_1} & t_1^{r_2} \\ t_2^{r_1} & t_2^{r_2} \end{pmatrix} = t_1^{r_1} t_2^{r_2} - t_2^{r_1} t_1^{r_2} \neq 0$. Letting $\frac{t_1}{t_2} = s$, we see that the determinant will not vanish when $s^{r_1} \neq s^{r_2}$, which is the case as long as $s \neq 1$ and $r_1 \neq r_2$. Hence when $r_1 \neq r_2$ the the expression $C_1 t^{r_1} + C_2 t^{r_2}$ is a general solution. Alternatively, we can use the change of variables $t = e^s$ to reduce our example to the case of the equation with the constant coefficients.

6. We have $\frac{d}{dt} E(t) = m\dot{x}\ddot{x} + V'(x)\dot{x} = \dot{x}(m\ddot{x} + V'(x)) = -\alpha \dot{x}^2 \leq 0$.

7*. (Optional) Substituting $p(z) = C\rho(z)$ into the equation $\frac{dp}{dz} = -g(z)\rho(z)$, we obtain $\frac{d\rho}{dz} = -g(z)\rho(z)\frac{1}{C}$, which is the same as $\frac{d\rho}{\rho} = -\frac{g(z)dz}{C}$. Integrating between ρ_0 and ρ on the left-hand side and between 0 and z on the right-hand side, we obtain $\log \frac{\rho}{\rho_0} = -\frac{1}{C}(V(z) - V(0))$, where $V(z) = -\frac{\kappa M}{(R+z)}$. This gives $\rho = \rho_0 e^{-\frac{V(z)-V(0)}{C}}$. Then $\lim_{z \to \infty} \rho(z) = \rho_0 e^{\frac{V(0)}{C}} > 0$, and hence the mass of the atmosphere cannot be finite (assuming the atmosphere is at equilibrium). When g is constant, a similar (an, in fact, easier) calculation gives $\rho = \rho_0 e^{-\frac{gz}{C}}$, which is equivalent to replacing $V(z) - V(0)$ by $V'(0)z$ in the formula for variable g.

8*. (Optional) We have $x'(t) = p(x(t))$. Hence $x'' = \frac{dp}{dx} x' = p\frac{dp}{dx}$. Hence $x'' = f(x, x')$ gives $p\frac{dp}{dx} = f(x, p)$.

[1] Other forms are possible, depending on how we choose the constants of integration.