1. (i) This is a matter of straightforward calculation. A nice way to write the calculation is to use the identity $w = \det(x, x')$, where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$ and then systematically use that $\det(y, y) = 0$ for any vector y. We have $w' = \det(x', x') + \det(x, x'') = \det(x, -px' - qx) = -p \det(x, x') - q \det(x, x) = -p \det(x, x') = -p\, w$.
(ii) The general solution of $w' + p(t)w = 0$ is $w = Ce^{P(t)}$, where P is a primitive of p. This function can vanish at t_0 only when $C = 0$ in which case we clearly have $w(t) = 0$ for each t.

2. The homogeneous equation is $x'' + \frac{x'}{t} = 0$. Letting $x' = y$, we can write $y' = -\frac{y}{t}$ which gives $y = \frac{C_1}{t}$. The integration of $x' = y$ then gives $x = C_1 \log t + C_2$. We now have to find a particular solution of the inhomogeneous equation. This can either be done by (educated) guessing, or by variations of constants. To make a good guess, we note that when applied to $x(t) = t^m$, both terms x'' and $\frac{x'}{t}$ lower the degree of the polynomial by 2. In particular, the quadratic polynomial t^2 is taken into a constant, so ct^2 will be a particular solution of equation (3) for a suitable constant c. One now checks easily that we must take $c = \frac{1}{4}$.
If we do the variation of constant instead, we seek the solution of the inhomogeneous solution as $x(t) = C_1 \log t + C_2$, where C_j are now considered as functions of t and, moreover, one has $x'(t) = C_1(\log t)' + C_2(1)' = \frac{C_1}{t}$. This means that $C_1' \log t + C_2' = 0$ and – after using the equation – that $\frac{C_1'}{t} = 1$. Hence we can take $C_1 = \frac{1}{2}t^2$ and $C_2 = -\int t \log t = -\frac{1}{2}t^2 \log t + \frac{1}{4}t^2$ and $x(t) = \frac{1}{4}t^2$.
Yet another way to find the solution is to write the equation as $(tx')' = t$ which implies $tx' = \frac{1}{2}t^2 + c_1$, hence $x(t) = \int \left(\frac{1}{2}t + \frac{c_1}{t}\right) dt + c_2$.
No matter which method we use, the general solution we obtain will be
$x(t) = \frac{1}{4}t^2 + C_1 \log t + C_2$.

3. Substituting $x = \frac{y}{t}$ into our equation we obtain $\left(\frac{y}{t}\right)'' + \frac{2}{t}\left(\frac{y}{t}\right)' + \frac{y}{t} = 0$. Using Leibnitz rule, we can write the expression on the left-hand side as
$\frac{y''}{t} - \frac{2y'}{t^2} + \frac{2y}{t^3} + \frac{2}{t}\frac{y'}{t} - \frac{2}{t}\frac{y}{t^2} + \frac{y}{t} = \frac{y''}{t} + \frac{y}{t}$ and the equation simplifies to $y'' + y = 0$, with the general solution $y(t) = C_1 \cos t + C_2 \sin t$. The general solution of the original equation then is $x(t) = C_1 \frac{\cos t}{t} + C_2 \frac{\sin t}{t}$.

4*. 1. Let $b_{ij} = \delta_{ij} + s a_{ij}$, with the usual definition $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. We have $\det(I + sA) = \det B = \sum \operatorname{sign}(i_1 \dots i_n)\, b_{1i_1} \dots b_{ni_n}$, where the sum is taken over all permutations $i_1 \dots i_n$. This expression is clearly a polynomial in s, of the form $1 + p_1 s + p_s s^2 + \dots p_n s^n$. We need to show that $p_1 = \operatorname{Tr} A$. For any permutation $i_1 \dots i_n$ which is different from the trivial permutation $1, 2, \dots, n$ the expression $b_{1i_1} \dots b_{ni_n}$ contains at least two off-diagonal elements, and hence it will depend on s with the power s^m for $m \geq 2$. Therefore the contribution to p_1 can come only from $b_{11}b_{22}\dots b_{nn} = (1+sa_{11})(1+sa_{22})\dots(1+sa_{nn})$. This expression can be written as a sum of 2^n terms of the form $1, sa_{kk}, s^2a_{kk}a_{ll}, s^3a_{kk}a_{ll}a_{mm}$, etc. Only the terms sa_{kk} can contribute to $p_1 s$, and their total contribution is easily seen to be $sa_{11} + \dots sa_{nn} = s\mathrm{Tr}\, A$.

2. In the notation above, we have $\frac{d}{ds}B(s)|_{s=0} = p_1 = \mathrm{TrA}$.

3. $\frac{d}{ds}|_{s=0} \det C(s) = \frac{d}{ds}|_{s=0} \det\left(C(s)C^{-1}(0)\right)\det C(0) = \mathrm{Tr}\,\left(C'(0)C^{-1}(0)\right)\det C(0)$.

5*. Letting $y = x'$ and $z = \begin{pmatrix} x \\ y \end{pmatrix}$, so that $z_1 = x_1, z_2 = x_2, z_3 = y_1, z_4 = y_2$, we can write our system as $z' = Az$, where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{pmatrix} . \tag{1}$$

A direct calculation of $\det(A-\lambda I)$ (e. g. by "developing" the determinant by the first column) gives $\det(A-\lambda I) = \lambda^4+\lambda^2+1$. To solve the equation $(A-\lambda I)z = 0$ for the eigenvalues, one can add the $-\lambda$-multiple of the third row of the matrix $A-\lambda I$ to its first row and add the $-\lambda$-multiple of the the fourth row of $A-\lambda I$ to its second row. This gives

$$\begin{pmatrix} 0 & 0 & \lambda^2+1 & \lambda \\ 0 & 0 & \lambda & \lambda^2+1 \\ -1 & 0 & -\lambda & -1 \\ 0 & -1 & -1 & -\lambda \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = 0 . \tag{2}$$

Note that this works for any λ, we did not need to calculate $\det(A-\lambda I)$ to get this equivalent form of the equation $(A-\lambda I)z = 0$. In fact, we can see from (2) that $\det(A-\lambda I) = \det\begin{pmatrix} \lambda^2+1 & \lambda \\ \lambda & \lambda^2+1 \end{pmatrix} = \lambda^4+\lambda^2+1$, confirming our previous calculation of $\det(A-\lambda I)$. Let $\omega = e^{\frac{2\pi}{3}i}$, $\zeta = e^{\frac{\pi}{3}i}$. The roots of the characteristic polynomial $\det(A-\lambda I) = \lambda^4+\lambda^2+1$ are

$$\lambda_1 = \omega, \quad \lambda_2 = \overline{\omega}, \quad \lambda_3 = \zeta, \quad \lambda_4 = \overline{\zeta}. \tag{3}$$

Note that

$$\lambda_1^2+1 = -\lambda_1, \quad \lambda_1^2+1 = -\lambda_2, \quad \lambda_3^2+1 = \lambda_3, \quad \lambda_4^2+1 = \lambda_4 . \tag{4}$$

The corresponding eigenvectors are now easily seen to be

$$z^{(1)} = \begin{pmatrix} -\lambda_1-1 \\ -\lambda_1-1 \\ 1 \\ 1 \end{pmatrix}, \; z^{(2)} = \begin{pmatrix} -\lambda_2-1 \\ -\lambda_2-1 \\ 1 \\ 1 \end{pmatrix}, \; z^{(3)} = \begin{pmatrix} -\lambda_3+1 \\ \lambda_3-1 \\ 1 \\ -1 \end{pmatrix}, \; z^{(4)} = \begin{pmatrix} -\lambda_4+1 \\ \lambda_4-1 \\ 1 \\ -1 \end{pmatrix} . \tag{5}$$

The general solution of our original second-order system is

$$x(t) = C_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{\lambda_2 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_3 e^{\lambda_3 t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_4 e^{\lambda_4 t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} . \quad (6)$$

We note that $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 < 0$ and $\operatorname{Re}\lambda_3 = \operatorname{Re}\lambda_4 > 0$. Hence the solution of our system which are bounded in $(0, \infty)$ are exactly the solutions

$$x(t) = C_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{\lambda_2 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} . \quad (7)$$

Remark: We can also solve the system in this problem directly, without rewriting it as a first order system. In this approach we seek the solutions as $x(t) = e^{\lambda t} b$, where $b \in \mathbf{C}^2$ is a fixed vector. Substituting this expression into the equation, we obtain $\lambda^2 b + \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} b + b = 0$, which is the same as $\begin{pmatrix} \lambda^2 + 1 & \lambda \\ \lambda & \lambda^2 + 1 \end{pmatrix} b = 0$. This equation can have non-trivial solutions b only when $\det \begin{pmatrix} \lambda^2 + 1 & \lambda \\ \lambda & \lambda^2 + 1 \end{pmatrix} = 0$, which gives the four roots in (4). Calculating the corresponding vectors b, we again obtain the general solution (6).