1. On the final exam there will be 7 problems, any 5 of them will count as 100%. You can use any books or notes you like, as well as a calculator. However, no electronic devices with wireless communication capabilities will be allowed.

2. The problems on the exam will be similar to the problems which appeared during the semester in the two midterms, the two posted practice tests, the three homework assignments, and the three problems below.

Problem 1

Let $\kappa_1, \kappa_2 > 0$ be two given frequencies and let ρ and T be two positive constants. For $x \in (0, L)$ and $t \in \mathbf{R}$ consider the wave equation with a forcing term

$$\rho \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2} + \left[\sin \frac{\pi x}{L} + 0.1 \sin \frac{2\pi x}{L} \right] \sin \kappa_1 t + 0.001 \sin \frac{3\pi x}{L} \ \sin \kappa_2 t \ , \tag{1}$$

with the boundary conditions $u(0,t) = 0\,,\ u(L,t) = 0$, and the initial conditions $u(x,0) = 0\,,\ u_t(x,0) = 0$. (Here u_t denotes the time derivative.) Determine the conditions on the pair on the frequencies κ_1, κ_2 so that the solution of the problem stays bounded for all time.

Problem 2

Let $c > 0, K_0 > 0$ and $a, b \in \mathbf{R}$ be parameters. Consider the heat equation (with a source term) on the real line

$$c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + b e^{-at} \,, \tag{2}$$

and assume the initial condition $u_0(x) = u(x,0)$ is given by $u_0 = \chi_{(\alpha,\beta)}$, the characteristic function of an interval (α, β). Determine $\lim_{t\to\infty} u(x,t)$.

Comments: This is a slighly more complicated version of the example we did in class. There will be several cases, depending on the sign of a. Also, in class we only worked with the fundamental solution of the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, let us call it $\Gamma(x,t)$. You can check that the fundametal solution of the equation $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$, where $\nu > 0$ is $\Gamma(x, \nu t)$.

Problem 3

For $\varepsilon > 0$ we define a function $u_0^\varepsilon(x)$ on the real line as follows:

$$u_0^\varepsilon = \begin{cases} 0\,, & x \notin [1-\varepsilon, 1] \\ \frac{1}{\varepsilon}\,, & x \in [1-\varepsilon, 1]\,. \end{cases} \tag{3}$$

Let u^ε be the solution of the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ for $(x,t) \in (-\infty, \infty) \times (0, \infty)$ with the initial condition $u(x,0) = u_0^\varepsilon$. Determine $v(x,t) = \lim_{\varepsilon \to 0_+} u^\varepsilon(x,t)$ for $(x,t) \in (-\infty,\infty) \times (0,\infty)$.

Hint: Use the fact that for ny continuous function $g(x)$ on the real line we have $\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} g(x) u_0^\varepsilon(x)\, dx = g(1)$.