Math 5587 **Homework Assignment 1** Fall 2017

due Sept 26, 2017, at the beginning of class, 2:30pm

Do at least four of the following six problems.[1]

1. Find the steady-state solution of the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + f\,, \tag{1}$$

in interval $(0, L)$ when we assume that $k > 0$ and $f > 0$ are constants, and the boundary conditions are

$$\frac{\partial u}{\partial x}(0,t) = Hu(0,t)\,, \quad \frac{\partial u}{\partial x}(L,t) = -Hu(L,t)\,, \tag{2}$$

where H is a positive constant.

2. Determine the behavior of the solutions in Problem 1 when $H \to 0_+$ and also when $H \to +\infty$.

3. In the derivation of the heat equation in Section 1.2 of the textbook, three quantities appear characterizing the material: c, ρ, and K_0. There is also the quantity $k = K_0/c\rho$, which appears in the heat equation (1). If we are choosing a (homogeneous) insulating material for a house to minimize heat loss in the winter, of the four parameters c, ρ, K_0, k of the material, what is the most important parameter to consider and why?

4. Assume that if we measure the temperature in the degrees of Kelvin, the length in meters, and the time in seconds, the coefficient k in the heat equation (1) of a homogeneous material is given by some specific number, which we will denote by k_{SI} (after the SI unit system). How does the number change if we instead start measuring the length in miles, the time in months, and the temperature in the degrees of Fahrenheit?

5. Assume that a smooth function $u(x_1, x_2, t)$ defined for $(x_1, x_2) \in \mathbf{R}^2$ and $t > 0$ satisfies the 2-dimensional hear equation

$$\frac{\partial u}{\partial t} = k\Delta u\,, \qquad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\,. \tag{3}$$

Let $\tilde{u}$ be a function obtained from u by a rotation about the origin by angle θ:

$$\tilde{u}(x_1, x_2, t) = u((\cos\theta)x_1 + (\sin\theta)x_2, -(\sin\theta)x_1 + (\cos\theta)x_2, t)\,. \tag{4}$$

Show that $\tilde{u}$ again satisfies the same heat equation (3).

6. Let us consider the following simplified model of the heat equation. We measure the temperature of a rod only at 6 equally distributed points including the two ends, where the temperature is kept at 0. The non-zero temperatures can only be at the four points which are not at the end of the rod, and let us denote these temperatures (which think of as time-dependent) $u_1(t), u_2(t), u_3(t), u_4(t)$. Assume the (column) vector u of these four temperatures satisfies the equation

$$\dot{u} = kAu\,, \qquad A = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}, \tag{5}$$

where $k > 0$ is a parameter.
(a) Show that any solution $u(t)$ of (5) converges to 0 as $t \to +\infty$.
(b) Let $U(t) = \max_j |u_j(t)|$. What is the shortest possible time (in terms of the parameter k) one has to wait to be sure that, no matter what the initial condition is, $U(t)$ drops at least by 99%? (The answer can be approximate, with an error below 10%, and you can use a computer.)

[1] For grading purposes, any 4 problems correspond to 100%. You can get extra credit if you do more.