Math 5587 **Homework Assignment 2** Fall 2017

due October 31

Please submit via Moodle by midnight, Oct 31

Do at least four of the following six problems.[1]

1. (a) Determine the Fourier coefficients c_k in the Fourier series[2]

$$f(x) = \sum_{k=-\infty}^{k=\infty} c_k e^{\pi i k x} \tag{1}$$

of the function $f(x) = 1 - x^2$ on the interval $(-1, 1)$.
What identity do we get from the general formula $\int_{-1}^{1} |f(x)|^2\, dx = 2\sum_k |c_k|^2$ in this example?
(b) Decide whether the formula

$$f'(x) = \sum_{k=-\infty}^{\infty} \pi i k c_k e^{\pi i k x} \tag{2}$$

which we obtain from (1) by formally differentiating the series term-by-term leads to a convergent series and expresses the derivative f' correctly for $x \in (-1, 1)$
(c) Explain why the series (1) defines a 2-periodic function on $(-\infty, \infty)$ which is not differentiable at $x = 1$. Verify that the series (2) still converges at $x = 1$ and calculate the sum for $x = 1$. Explain why the value of the sum differs from $\frac{d}{dx}(1 - x^2)|_{x=1}$.

2. (a) On the interval $(0, \pi)$ calculate[3] the sine-Fourier series[4] of the function $\sin^2 x$ and the cosine-Fourier series[5] of the function $\cos^2 x$. Explain why the latter is finite while the former is not finite.
(b) If we write $\sin^2 x = \sum_{n=1}^{\infty} B_n \sin nx,\ x \in (0, \pi)$, how many times can we differentiate term by term before the series on the right-hand side fails to converge (in the traditional point-wise sense)?

3. Assume you have at your disposal a machine which can do the following:
(i) For any 2π-periodic piece-wise smooth function f it can calculate the coefficients c_k of the Fourier series

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}\,. \tag{3}$$

(ii) For any given coefficients $\{c_k\}_{k=-\infty}^{k=\infty}$ for which the sum (3) is convergent (in some suitable sense), it can recover the function f.
Write an algorithm which can freely use the machine and solves the following task:
Given $L > 0, c > 0, \gamma \geq 0$ and two smooth functions $u_0(x)$ and $u_1(x)$ vanishing at the endpoints of the inteval $(0, L)$, find the value $u(x, t)$ (where $x \in (0, L)$ and $t \in (-\infty, \infty)$) of the solution of the following problem:

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} - \gamma u\,, \qquad x \in (0, L)\,,\ t \in (-\infty, \infty)\,,\\
u(0, t) &= 0\,,\\
u(L, t) &= 0\,,\\
u(x, 0) &= u_0(x)\,,\\
\frac{\partial u}{\partial t}(x, 0) &= u_1(x)\,.
\end{aligned}$$

[1] For grading purposes, any 4 problems correspond to 100%. You can get extra credit if you do more.
[2] You can use Wolfram Alpha, it can handle calculations of the relevant definite integrals with e^{ikx} where k is an undetermined parameter, but it is not enough just to copy the results the computer gives you - the expressions need to be simplified.
[3] As in Problem 1, you can use Wolfram Alpha to calculate the series, although in this case one needs to be even more careful in interpreting the results and simplifying the expressions.
[4] see Section 3.3.1 of the textbook
[5] see Section 3.3.2 of the textbook

4. Suppose a standard guitar string has density[6] ρ and produces (for a given guitar) a given tone at tension T.
(a) How do we have to change the density of the string if we wish to produce the same tone (on the same guitar) with tension $T/2$.
(b) If we wish to use the original string on a another guitar which is much bigger, so that the effective length of the string would double, what tension in the string would be needed to produce the same tone as on the original guitar?
5. Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{4}$$

in the whole $1+1$–dimensional space-time, which we will identify with $\mathbf{R} \times \mathbf{R}$.
(a) Let us consider a Galilean transformation of the coordinates (x,t) to coordinates $(\tilde{x},\tilde{t})$ of an observer moving at a constant velocity v:

$$t = \tilde{t}\,, \qquad x = \tilde{x} + v\tilde{t}\,. \tag{5}$$

Assuming a quantity u satisfies (4) in the coordinates (x,t), derive the equation satisfied by u in the coordinates $(\tilde{x},\tilde{t})$ of the moving observer. Can the moving observer determine (in principle) the velocity v from observing the behavior of the quantity u?
(b) Assume now that we transform the coordinates via a different transformation, namely

$$\begin{array}{rcl} ct & = & c\tilde{t}\cosh\theta + \tilde{x}\sinh\theta \\ x & = & c\tilde{t}\sinh\theta + \tilde{x}\cosh\theta\,, \end{array} \tag{6}$$

where $\theta \in \mathbf{R}$ is a parameter. This change of coordinates is called the Lorentz transformation.[7]
(i) Explain why θ can be related to the velocity v of observer associated with the coordinates $(\tilde{x},\tilde{t})$ as observed in coordinates (x,t) via the relation $v = c\tanh\theta$.
(ii) Show that if a quantity u satisfies (4) in the coordinates (x,t), then u also satisfies the same equation in the coordinates $(\tilde{x},\tilde{t})$.
(iii) Express the transformation (6) in terms of the velocity v (rather than the parameter θ).
6. Let n be a positive integer. In Lecture 7 we considered the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w & w^2 & \dots & w^{n-1} \\ 1 & w^2 & w^4 & \dots & w^{2(n-1)} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & w^{n-1} & w^{2(n-1)} & \dots & w^{(n-1)(n-1)} \end{pmatrix}, \tag{7}$$

where

$$w = e^{\frac{2\pi i}{n}}\,. \tag{8}$$

See pages 10 and 11 of the Lecture Log for the connection of A to the Discrete Fourier Transformation.
(a) Prove that A is invertible and its inverse is given by

$$A^{-1} = \frac{1}{n} A^*\,, \tag{9}$$

where we use the usual notation A^* for the (Hermitian) adjoint matrix given by $(A^*)_{kl} = \overline{A}_{lk}$, with the bar denoting the complex conjugation.
(b) Show that if $c_1, \dots, c_n$ is a vector of complex numbers and

$$f_k = \sum_{l=1}^{n} A_{kl} c_l\,, \qquad k = 1, 2, \dots, n\,, \tag{10}$$

then

$$\frac{1}{n}\sum_{k=1}^{n} |f_k|^2 = \sum_{k=1}^{n} |c_k|^2\,. \tag{11}$$

Remark: One of the many remarkable properties of A is that $A^4 = n^2 I$, where I is the identity matrix. This of course also shows that A is invertible, and, moreover, it implies that the spectrum of A is contained in the set $\{1, i, -1, -i\}$. It is a good exercise to try to prove this, although it is not a part of this assignment.

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[6] mass per unit legth
[7] Usually it is written in a form in which the values of $\cosh\theta$ and $\sinh\theta$ are expressed in terms of $v = c\tanh\theta$, as in part (iii) of the problem.