1. *Let $U = U(x)$ be a steady-state solution of the heat equation*

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q\,, \tag{1}$$

in interval $(0, L)$ where we assume that $k > 0$ and Q are constant, and the boundary conditions are

$$u(0,t) = 0\,, \quad \frac{\partial u}{\partial x}(L,t) = 0\,. \tag{2}$$

Let us denote

$$\alpha = \frac{\partial U}{\partial x}(0)\,. \tag{3}$$

Assuming the values of L and α are known, what is the value of $U(L)$?

Solution: The general solution of $ku''(x) + Q = 0$ is $u(x) = -\frac{Q}{2k}x^2 + c_1 x + c_2$. Determining c_1, c_2 from the boundary conditions gives $U(x) = \frac{Q}{2k}x(2L - x)$. Using $U'(0) = \alpha$ we see that $\frac{QL}{k} = \alpha$, or $\frac{Q}{k} = \frac{\alpha}{L}$. Using this in the formula for U, we get $U(L) = \frac{1}{2}\alpha L$.

There are many variations of the calculation. Based on dimensional considerations, the only way to express U in terms of α and L which has the right dimension is $U(L) = c\alpha L$, where c is some constant. The value $c = \frac{1}{2}$ can be seen for example from the fact that the profile of the solution is a parabola with its vertex at $x = L$.

2. *Let L, l, K, k be strictly positive numbers. Assume $U = U(x,t)$ satisfies the heat equation*

$$\frac{\partial U}{\partial t} = K\frac{\partial^2 U}{\partial x^2} \tag{4}$$

in $(-L, L)$ (for $t > 0$) with the boundary conditions

$$\frac{\partial U}{\partial x}(-L,t) = 0\,, \qquad \frac{\partial U}{\partial x}(L,t) = 0 \tag{5}$$

and the initial condition

$$U(x,0) = 100\,\mathrm{sign}\,(x)\,. \tag{6}$$

Let T_1 be the minimal time $t > 0$ for which $U(L,t) \le 1$.
Similarly, assume $u = u(x,t)$ satisfies the the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{7}$$

in $(-l, l)$ (for $t > 0$) with the boundary conditions

$$\frac{\partial u}{\partial x}(-l,t) = 0\,, \qquad \frac{\partial u}{\partial x}(l,t) = 0 \tag{8}$$

and the initial condition

$$u(x,0) = \mathrm{sign}\,(x)\,. \tag{9}$$

Let t_1 be the minimal time $t > 0$ for which $u(l,t) \le 0.01$. Assuming the values of $T_1, t_1, L, l,$ and K are known from measurements, what is the value of k?

Solution: The solutions u, U are "similar", in the sense that they are related by a transformation $U(x,t) = Au(\alpha x, \beta t)$ for some positive constants A, α, β. Using the information about the solutions we have, we conclude $U(x,t) = 100\,u(\frac{l}{L}x, \frac{t_1}{T_1}t)$. Substituting this into the equation for U and using the equation for u, we obtain $k = K\frac{l^2}{L^2}\frac{T_1}{t_1}$.

One can also solve the problem by dimensional considerations. The dimension of k is length^2/time, so the quantities kt_1/l^2 and KT_1/L^2 are dimension-less. Given that the two solutions can be thought of as describing the same situation, except in different units, the two dimension-less quantities must coincide, i. e., $kt_1/l^2 = KT_1/L^2$, which again gives the formula for k above.

3. *Find a formula for the solution of the following problem (where we assume $k > 0$):*

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}\,, & x \in (0,\pi)\,,\ t > 0\,, \quad (10)\\ u(x,0) &= -0.2\sin 50x\,, & x \in (0,\pi)\,, \quad (11)\\ u(0,t) = u(\pi,t) &= 0\,, & t > 0. \quad (12) \end{aligned}$$

Solution: Here we can follow the textbook, see section 2.3.5 and just insert our particular values into formula 2.3.26 (for example), to obtain $u(x,t) = -0.2e^{-k50^2 t}\sin 50x = -0.2e^{-2500kt}\sin 50x$.

4. *Let u be the solution of the following problem (where we assume $k > 0$):*

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}\,, & x \in (0,\pi)\,,\ t > 0\,, \quad (13)\\ u(x,0) &= \cos^2 x\,, & x \in (0,\pi)\,, \quad (14)\\ \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) &= 0\,, & t > 0. \quad (15) \end{aligned}$$

Determine

$$u_\infty(x) = \lim_{t\to\infty} u(x,t)\,, \qquad x \in (0,\pi)\,. \quad (16)$$

Solution: We can calculate the solution explicitly, following the textbook, section 2.4.1, and using $\cos^2 x = (1 + 2\cos x)/2$ (which gives the cosine Fourier series for $\cos^2 x$, so we do not need to calculate the coefficients by integration). We get $u(x,t) = \frac{1}{2} + \frac{1}{2}e^{-4kt}\cos 2x$, which converges to $\frac{1}{2}$ for each x as $t \to \infty$, hence $u_\infty(x) = \frac{1}{2}$.
Alternatively, from the representation of the general solution by formula (2.4.19) in the textbook we see that $u_\infty(x) = A_0$, with A_0 given by formula (2.4.23).

5. *Find a formula for the solution of the Laplace equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (17)$$

in the unit disc $B_1 = \{(x,y)\,;\ x^2 + y^2 < 1\}$, with the boundary condition

$$u(x,y) = y^2 \text{ at the boundary of the disc.} \quad (18)$$

Solution: We can use formula (2.5.25) in the textbook, only have to determine the coefficients A_n, B_n. In the notation of the textbook, we have $f(\theta) = \sin^2\theta = (1 - \cos 2\theta)/2$, so we see that $A_0 = \frac{1}{2}$, $A_2 = -\frac{1}{2}$ and all the other coefficients vanish. Hence $u(r,\theta) = \frac{1}{2} - \frac{1}{2}r^2\cos 2\theta$. Going back to the (x,y) coordinates, we have $u(x,y) = \frac{1}{2} - \frac{1}{2}x^2 + \frac{1}{2}y^2$.
Alternatively, one does not have to use the method in the textbook and try instead to seek the solution as a quadratic polynomial, and a short calculation gives again $u(x,y) = \frac{1}{2} - \frac{1}{2}x^2 + \frac{1}{2}y^2$. (This may not look like a systematic method, but rather like an *ad hoc* idea, but it is more general than it seems at first. It is mentioned here only for completeness.)

6. *Consider the planar domain $\Omega = \{(x,y) \in \mathbf{R}^2\,, 0 < x < 1\,,\ y > 0\}$ and a function f on the boundary of Ω defined by*

$$f(0,y) = 0\,,\ f(1,y) = 0\,, \quad y > 0\,, \qquad f(x,0) = (\sin\pi x)(\cos 2\pi x)\,, \quad x \in (0,1). \quad (19)$$

Find a harmonic function in Ω which is bounded and coincides with f at the boundary of Ω. (Recall we call a function harmonic if it satisfies the Laplace equation (17).)

Solution: By separation of variables we see that functions of the form $\sin(\pi nx)(A_n e^{\pi ny} + B_n e^{-\pi ny})$ solve the equation and satisfy the boundary condition in the finite region. The required behavior for $y \to \infty$ implies that we should only consider the solutions with $A = 0$. We can then write our solution as a superposition of these solutions, the only remaining issue is find the (sine) Fourier series for $f(x,0)$. Using $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$ and $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$, we obtain $\sin\theta\cos 2\theta = \frac{1}{2}\sin 3\theta - \frac{1}{2}\sin\theta$. Our solution then is $u(x,y) = -\frac{1}{2}(\sin\pi x)e^{-\pi y} + \frac{1}{2}(\sin 3\pi x)e^{-3\pi y}$.