Do at least four of the following six problems.[1]

1. *Find the steady-state solution of the heat equation*

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q\,, \tag{1}$$

in interval $(0, L)$ *when we assume that* $k > 0$ *and* $Q > 0$ *are constants, and the boundary conditions are*

$$\frac{\partial u}{\partial x}(0,t) = Hu(0,t)\,, \quad \frac{\partial u}{\partial x}(L,t) = 0\,, \tag{2}$$

where H *is a positive constant.*

A variant of the problem: assume that for a steady-state solution of (2) we know H, k, L and $u(0)$ (which is assumed to be independent of time). Calculate Q (which is assumed to be independent of x and t.)
Further problems of this type can be found on page 18 of the textbook.

2. *Let* $k > 0$ *and* $L > 0$*. Consider the heat equation*

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{3}$$

in the interval $(0, L)$*, with the boundary conditions* $u(0,t) = u(L,t) = 0$ *(for* $t > 0$*) and the initial condition* $u(x, 0) = U$*, where* U *is a constant. Let us denote the first time the solutions satisfies* $|u(x,t)| \le 0.01\,U$ *for all* $x \in (0, L)$ *by* $T(k, L, U)$*. Assume that from a measurement we know that* $T(1, 1, 1) = \tau$*. Find the formula for* $T(k, L, U)$ *in terms of* k, L, U, τ*.*

Hint: If $u_1(x,t)$ is the solution in the special case $k = 1$, $L = 1$, $U = 1$, seek the solution of the general in the form $u(x,t) = A\,u_1(\alpha t, \beta x)$, for suitable A, α, β.

3. *Find a formula for the solution of the following problem (where we assume* $k > 0$*):*

$$\begin{aligned}
\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}\,, && x \in (0,\pi)\,,\ t > 0\,, && (4)\\
u(x,0) &= 7\sin 2x + 3\sin 15x\,, && x \in (0,\pi)\,, && (5)\\
u(0,t) = u(\pi,t) &= 0\,, && t > 0. && (6)
\end{aligned}$$

For further problems in this direction, see for example exercise 2.3.3 on page 51 of the textbook. Also, you can try to solve the problem with the initial condition $u(x, 0) = 1$, which is somewhat more difficult, because the relevant Fourier series is infinite and its coefficients need to be calculated.

[1] For grading purposes, any 4 problems correspond to 100%. You can get extra credit if you do more. You can use the textbook, any notes, and a calculator, as long as it does not have wireless capabilities. Devices with wireless communication capabilities are not allowed. Hints for solutions such as the ones above might not be included on the real exam.

4. *Let u be the solution of the following problem (where we assume $k>0$):*

$$\begin{aligned}
\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}\,, \qquad , \qquad x\in(0,\pi)\,,\ t>0\,, &(7)\\
u(x,0) &= 1+2\cos^3 x\,, \qquad x\in(0,\pi)\,, &(8)\\
\frac{\partial u}{\partial x}(0,t)=\frac{\partial u}{\partial x}(\pi,t) &= 0\,, \qquad t>0. &(9)
\end{aligned}$$

Determine the function $u_\infty(x)$ defined by

$$u_\infty(x)=\lim_{t\to\infty}u(x,t)\,. \tag{10}$$

The method for solving this problem is discussed in Section 2.4.1 of the textbook.

5. *Find the formula for the solution of the Laplace equation*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \tag{11}$$

in the unit disc $B_1=\{(x,y)\,;\ x^2+y^2<1\}$, with the boundary condition

$$u(x,y)=-1+x^2-y^3 \text{ at the boundary of the disc.} \tag{12}$$

Hint: One way to solve this is to use the method of Section 2.5.2 in the textbook, in particular formula 2.5.45. To apply this method, one needs to use the polar coordinates $x=r\cos\theta, y=r\sin\theta$. It may also be useful (although not necessary) to remember the observation we made in class that one can calculate the Fourier series of expressions such as $\cos^3\theta$ in the following way $\cos^3(\theta)=\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^3=\frac{1}{8}\left(e^{3i\theta}+3e^{i\theta}+3e^{-i\theta}+e^{-3i\theta}\right)=\frac{1}{4}\cos 3\theta+\frac{3}{4}\cos\theta$. You can also work with the analogue of the formula 2.5.45 using the complex representation:

$$u(r,\theta)=\sum_{n=-\infty}^{\infty}c_n r^{|n|}e^{in\theta}\,. \tag{13}$$

6. *Consider the planar domain in $\Omega=\{(x,y)\in\mathbf{R}^2\,,0<x<1\,,\ y>0\}$ and a function f on the boundary of Ω defined by*

$$f(0,y)=0\,,\ f(1,y)=0\,,\quad y>0\,,\qquad f(x,0)=2\sin\pi x+\sin 3\pi x\,. \tag{14}$$

Find a harmonic function $u(x,y)$ in Ω which is bounded and coincides with f at the boundary of Ω. (Recall we call a function harmonic if it satisfies the Laplace equation (11).)

Hint: Use separation of variables. In the textbook similar problems are discussed in case the domain is a rectangle in Section 2.5.1. When the rectangle is replaced by an infinite strip and we demand that the solution be bounded, the calculation is in fact simpler.