Do at least four of the following six problems.[1]

**1.** *Let $L, c, \gamma$ be strictly positive numbers. Find a formula for the solution $u(x,t)$ of the following problem:*

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} - \gamma u\,, \qquad x \in (0,L) \\ u(0,t) &= 0\,, \\ u(L,t) &= 0\,, \\ u(x,0) &= \sin\frac{\pi x}{L}\,, \\ \frac{\partial u(x,0)}{\partial t} &= \sin\frac{\pi x}{L}\cos\frac{\pi x}{L}\,. \end{aligned}$$

Comments: In principle one could solve this problem "from scratch", by using the separation of variables, seeking first special solutions in the form $h(t)\phi(x)$, and then using their linear combinations. However, we have done enough of these problems to know that we should seek the solutions as $u(x,t) = \sum_k B_k(t)\sin(\frac{\pi k x}{L})$. Substituting this to the PDE, we get an ODE for $B_k(t)$ for each $k$, which we then solve using the information from the initial condition at time $t = 0$. Note that $u(x,0)$ already is written as a (very simple) sine-Fourier series. This is not the case for $\frac{\partial u(x,0)}{\partial t}$, but we can use the identity $\sin\alpha\cos\alpha = \frac{1}{2}\sin 2\alpha$ to obtain the corresponding (very simple) Fourier series without much work.

**2.** *Consider the wave equation for an inhomogeneous string*

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u}{\partial x^2}\,, \qquad x \in (0,L)\,, \tag{1}$$

*with the boundary conditions at the endpoints given by*

$$u(0,t) = u(L,t) = 0\,. \tag{2}$$

*We assume that $\rho(x)$ is a smooth, strictly positive function on the closed interval $[0,L]$ and that $T > 0$. Let $\phi_1(x), \phi_2(x), \dots$ be the eigenvalues of the Sturm-Liouville problem $-T\phi''(x) = \lambda\rho(x)\phi(x)\,,\ \phi(0) = \phi(L) = 0\,,$ with the corresponding eigenvalues $0 < \lambda_1 < \lambda_2 < \dots$. Find a formula for the solution of (1) with the boundary conditions (2) satisfying the initial conditions*

$$\begin{aligned} u(x,0) &= a_1\phi_1(x) + a_2\phi_2(x) + \dots + a_m\phi_m(x)\,, \\ \frac{\partial u(x,0)}{\partial t} &= b_1\phi_1(x) + b_2\phi_2(x) + \dots + b_m\phi_m(x)\,, \end{aligned}$$

*where $m > 0$ is an integer and $a_1, a_2, \dots, a_m$ and $b_1, b_2, \dots, b_m$ are given real numbers.*

Hint: Search the solution as $u(x,t) = c_1(t)\phi_1(x) + c_2(t)\phi_2(x) + \dots + c_m(t)\phi_m(x)$.

**3.** *Let us consider two inhomogeneous strings, both fixed at the ends and under tension. The first string has length $L_1$, a part from one end to its middle has a constant density[2] $\rho_1$ while its other part has a constant density $2\rho_1$. The tension in this string is $T_1$ and we know from a measurement (or a calculation) that the lowest possible frequency of its oscillations (under tension $T_1$) is $\omega_1$. The second string is similar, except that the value of the corresponding quantities is $L_2, \rho_2, T_2$ and we have not measured/calculated $\omega_2$. Express $\omega_2$ in terms of $\rho_1, \rho_2, L_1, L_2, T_1, T_2, \omega_1$.*

Hint: The equation of motion for the strings number $i$ is $\tilde\rho_i(x)\frac{\partial^2 u_i}{\partial t^2} = T_i\frac{\partial^2 u_i}{\partial x_i^2}$ where the functions are considered on $(0, L_i)$ and $\tilde\rho_i(x) = \rho_i$ for $x \in (0, \frac{L_i}{2})$ and $\tilde\rho_i(x) = 2\rho_i$ for $x \in (\frac{L_i}{2}, L_i)$. We can now use the change of variables $u_1(x,t) = \alpha u(\beta x, \gamma t)$

[1] For grading purposes, any 4 problems correspond to 100%. You can get extra credit if you do more. You can use the textbook, any notes, and a calculator, as long as it does not have wireless capabilities. Devices with wireless communication capabilities are not allowed. Hints for solutions such as the ones above might not be included on the real exam.

[2] here we interpret density as mass per unit length

for suitable $\alpha, \beta, \gamma > 0$ to map the two situations onto each other.
If you are familiar with dimensional analysis, one can say that the non-dimensionalized equations are the same in both cases, and hence the non-dimensionalized frequencies must be the same. Expressing this, we obtain the desired formula, which is of course the same as the one obtained from the change of variables above.
If you are not familiar with the dimensional analysis, one can explain it in this case as follows. Note that by a suitable choice of units of mass, length, and time (different for each string) we can assume without loss of generality that $\rho_i = 1, L_i = 1$ and $T_i = 1$. Let us call these units the basic units of the corresponding string. The main point now is that the two equations expressed in the basis units are the same and are defined on the same interval. This means that the frequencies must be the same (as expressed in terms of the basic units of the string). Going back to the original units, we can obtain the desired expression.

**4.** *Let us write the second order equation differential equation $\ddot{x} = -\omega^2 x$ for a scalar function $x = x(t)$ (where $\ddot{x} = \frac{d^2x}{dt^2}$) as a first-order system*

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= -\omega^2 x\,, \end{aligned} \tag{3}$$

*and consider the following numerical scheme (forward Euler method) for the system:*

$$\begin{aligned} \frac{x(t+\tau)-x(t)}{\tau} &= y(t)\,, \\ \frac{y(t+\tau)-y(t)}{\tau} &= -\omega^2 x(t)\,. \end{aligned} \tag{4}$$

*Here $\tau > 0$ is the time-step (and we think of it as a small number), and the scheme is used to start from $t = 0$ and successively evaluate the solution at the times $t = \tau, 2\tau, 3\tau, \dots$.*
*(a) Show that for the original equation the solution $x(t)$ is always a periodic function of time. In particular it is bounded for all time and does not approach 0, unless it vanishes identically.*
*(b) Show that for any $\tau > 0$ the sequence of vectors with coordinates $x(k\tau), y(k\tau)$, $k = 1, 2, 3, \dots,$ which we get from the numerical approximation will never be bounded as $k \to \infty$, unless it is identically zero. (This does not mean the scheme is not convergent, it just means that it is not suitable for investigating the long-time behavior of the solution.)*
Hint: Let us denote $X$ the vector with components $x, y$ and let us write (4) as $X(t + \tau) = (I + \tau A)X(t)$ for a suitable $2 \times 2$ matrix $A$, where $I$ is the $2 \times 2$ identity matrix. The key is to look at the eigenvalues of the matrix $I + \tau A$.
Comments: (i) You can also check (as an optional exercise) that the scheme is convergent on any finite time-interval, in the following sense: for any $t > 0$ and any natural number $n$ we set $\tau_n = \frac{t}{n}$. Then the value $x(n\tau_n), y(n\tau_n)$ of the solution obtained from scheme with some initial conditions $(x(0), y(0))$ converges to the correct value as $n \to \infty$.
(ii) It is also a good exercise to ask the same questions for the backward Euler scheme

$$\begin{aligned} \frac{x(t+\tau)-x(t)}{\tau} &= y(t+\tau)\,, \\ \frac{y(t+\tau)-y(t)}{\tau} &= -\omega^2 x(t+\tau)\,. \end{aligned} \tag{5}$$

or the Crank-Nicolson scheme

$$\begin{aligned} \frac{x(t+\tau)-x(t)}{\tau} &= \tfrac{1}{2}(y(t+\tau)+y(t))\,, \\ \frac{y(t+\tau)-y(t)}{\tau} &= -\omega^2 \tfrac{1}{2}(x(t+\tau)+x(t))\,. \end{aligned} \tag{6}$$

These schemes are all convergent, but their long-time behavior for a given $\tau > 0$ is different. The key is again to write the formulae in a suitable matrix form and look at the eigenvalues of the relevant matrices.

**5.** *Let us consider the equation*

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{7}$$

*for functions $u(x, t)$ which are $2\pi-$periodic, i. e. $u(x+2\pi, t) = u(x, t)$. Consider the following approximation of the equation*

$$\frac{u(x, t+\tau) - u(x, t)}{\tau} + a\frac{u(x+h, t) - u(x, t)}{h} = 0\,, \tag{8}$$

*Assume we implement (8) as a numerical method for calculating (approximate) solutions for functions which are $2\pi-$peridic in $x$: for a positive integer $n$ we set $h = \frac{2\pi}{n}$ and consider a grid $x_0 = 0, x_1 = h, x_2 =$*

$2h, \ldots, x_{n-1} = (n-1)h$. *As our functions are* $2\pi-$*periodic, we can identify* $x_n = nh$ *with* $x_0$. *Representing a function on the grid by vectors with coordinates* $u_0, u_1, u_2, \ldots u_{n-1}$, *we can write the scheme also as*

$$\frac{u(t+\tau) - u(t)}{\tau} + \frac{a}{h} A u = 0\,, \tag{9}$$

*with*

$$A = \begin{pmatrix} -1 & 1 & 0 & \ldots & 0 \\ 0 & -1 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & -1 & 1 \\ 1 & 0 & \ldots & 0 & -1 \end{pmatrix}. \tag{10}$$

*(a) Using vectors with coordinates of the form* $v_l = e^{ikx_l}$, *show that the eigenvalues of* $A$ *are of the form* $\lambda_k = e^{ikh} - 1\,,\ k = 0, 1, 2, \ldots, n-1$.
*(b) Show that if* $a > 0$ *and we use the scheme with* $\tau$ *comparable to* $h$, *such as, for example,* $\tau = h$ *when trying to calculate the solution, the calculation will be unstable, and typically will "blow-up", not leading to a reasonable result.*
*(c) Explain why for* $a < 0$ *the same calculation can be expected to be stable.*

Hint: Look at what happens during the iteration when the initial condition is an eigenvector corresponding to an eigenvalue $\lambda_k$ with $\lambda_k < 0$.

**6.** *Let* $A$ *be an* $n \times n$ *symmetric matrix with positive eigenvalues* $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. *Let* $M = \mathrm{diag}(m_1, m_2, \ldots, m_n)$ *be a diagonal matrix with the entries on the diagonal given by the numbers* $m_j$, *which are assumed to be strictly positive. Assume* $\underline{m} = \min\{m_1, \ldots, m_n\}$ *and* $\overline{m} = \max\{m_1, \ldots, m_n\}$. *Consider the system*

$$M\ddot{x} + Ax = 0\,, \tag{11}$$

*which may describe small oscillation of some mechanical system with* $n$ *degrees of freedom. We know that there are* $n$ *linearly independent vectors* $\mathbf{b}^{(1)}, \mathbf{b}^{(2)}, \ldots, \mathbf{b}^{(n)}$ *in* $\mathbf{R}^n$ *and frequncies* $0 \le \omega_1 \le \omega_2 \le \cdots \le \omega_n$ *such any solution of (11) can be written as*

$$x(t) = A_1 \mathbf{b}^{(1)} \sin \omega_1 (t - t_1)) + A_2 \mathbf{b}^{(2)} \sin \omega_2 (t - t_2) + \cdots + A_n \mathbf{b}^{(n)} \sin \omega_n (t - t_n) \tag{12}$$

*for some real numbers* $A_1, \ldots, A_n$ *and* $t_1, \ldots, t_n$. *Explain why the fequencies* $\omega_j$ *will satisfy*

$$\frac{\lambda_1}{\overline{m}} \le \omega_j^2 \le \frac{\lambda_n}{\underline{m}}\,, \qquad j = 1, 2, \ldots n. \tag{13}$$

Hint: Use the characterization of the lowest and highest (generalized) eigenvalues (given by $Ax = \mu Mx$) through the Rayleigh quotient: $\mu_{\min} = \min_{x \ne 0} \frac{(Ax,x)}{(Mx,x)}$ and $\mu_{\max} = \max_{x \ne 0} \frac{(Ax,x)}{(Mx,x)}$.
Comment: We note for completeness that the vectors $\mathbf{b}^{(j)}$ are mutually orthogonal with respect to the (generalized) scalar product $(x,y)_M = (Mx, y)$, but not necessarily with respect to the standard scalar product $(x,y) = x_1 y_2 + x_2 y_2 + \cdots + x_n y_n$.