1. Let $G_{B_R}(x,y) = -\frac{1}{4\pi|x-y|} + \frac{R}{4\pi|y||x-y^*|}$ (where $y^* = y\frac{R^2}{|y|^2}$) be the Green's function for B_R. Let us denote $y' = (y_1, y_2, -y_3)$. For $y \in B_R^+$ we set $G(x,y) = G_{B_R}(x,y) - G_{B_R}(x,y')$. Then clearly $G(x,y)$ vanishes when $|x| = R$. When $x_3 = 0$, then $G_{B_R}(x,y) = G_{B_R}(x,y')$, because in that case we have $|x-y| = |x-y'|$ and $|x-y^*| = |x-y'^*|$, due to the reflection symmetry about the x_1x_2 plane. Hence $G(x,y)$ vanishes when $x_3 = 0$ and we see that the function $x \to G(x,y)$ vanishes at the boundary of B_R^+. Clearly $\Delta_x G(x,y) = \delta(x-y)$ in B_R^+ (as y is the only point of the set y, y^*, y', y'^* which lies in B_R^+), and therefore $G(x,y)$ is the desired Green's function of B_R^+.

2. Extend f first to the square $[0,1]\times[0,1]$ by $f(x_1,x_2) = -f(x_2,x_1)$. Then to the square $[0,2]\times[0,2]$ by $f(x_1,x_2) = -f(2-x_1,x_2)$; $(x_1,x_2) \in [1,2]\times[0,1]$, $f(x_1,x_2) = -f(x_1,2-x_2)$, $(x_1,x_2) \in [0,1]\times[1,2]$; $f(x,y) = -f(2-x,y) = -f(x,2-y)$, $f(x_1,x_2) = f(2-x_1,2-x_2)$, $(x_1,x_2) \in [1,2]\times[1,2]$. And finally to a function on $\mathbf{R}^2$ (still denoted by f) which is 2-periodic in x_1 and $2-$periodic in x_2. Set $F(x) = f(2x)$. The function F is $1-$periodic in x_1 and x_2, and we use the machine to calculate its Fourier coefficients $\hat{F}(k)$. We set $\hat{U}(k) = -\frac{\hat{f}(k)}{\pi^2|k|^2}$ (note that there is no 4 in front of π^2) when $k \neq 0$ and $\hat{U}(0,0) = 0$, and use the machine to obtain $U(x) = \sum_k \hat{U}(k)e^{2\pi i(k_1x_1+k_2x_2)}$. The function U solves $\frac{1}{4}\Delta U = F$, with the factor $\frac{1}{4}$ coming from skipping the 4 in front of π^2 as noted above. We set $u(x) = U(\frac{x}{2})$ and note that $\Delta u(x) = \frac{1}{4}(\Delta U)(\frac{x}{2}) = F(\frac{x}{2}) = f(x)$. The function u has the same symmetries as f, and hence vanishes at the boundary of our triangle, and is the (unique) solution to our problem.

3. Searching $u(x) = \frac{v(r)}{r}$ (with $r = |x|$), we obtain the equation $v'' = -\lambda v$ for v. We are looking for solutions which vanish at $r = 1$. In addition, the solutions also have to vanish at $r = 0$, so that $u(x) = \frac{v(r)}{r}$ is not singular at $r = 0$. We have seen this problem before. The solutions are of the form $v(r) = \sin(\pi k r)$, $\lambda = (\pi k)^2$, $k = 1, 2, \ldots$ Hence the radial eigenfunctions are $\frac{\sin \pi k r}{r}$ with the corresponding eigenvalues $\pi^2 k^2$.

4. It is enough to prove the statement for $P(z) = z^m$. We have $\frac{\partial}{\partial x}(x+iy)^m = m(x+iy)^{m-1}$, $\frac{\partial^2}{\partial x^2}(x+iy)^m = m(m-1)(x+iy)^{m-2}$. This can also be written as $\frac{\partial^2}{\partial x^2}z^m = m(m-1)z^{m-2}$. Similarly, $\frac{\partial}{\partial y}(x+iy)^m = im(x+iy)^{m-1}$, $\frac{\partial^2}{\partial y^2}(x+iy)^m = -m(m-1)(x+iy)^{m-2}$. This amounts to $\frac{\partial^2}{\partial y^2}z^m = -m(m-1)z^{m-2}$. Hence $\Delta z^m = 0$. There are many other ways to arrive at the same conclusion. For example, one can use the polar coordinates to write $z^m = r^m e^{im\theta}$ and use the expression $\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{r^2\partial r^2}$ to check that $r^m e^{im\theta}$ is harmonic.
Remark: The above problem asks to verify "by hand" a statement which is arises from some basic considerations of complex analysis. There one works with $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$, with $i = \sqrt{-1}$. One can check by an easier version of the above calculations that $\frac{\partial}{\partial \bar{z}}P(z) = 0$. In addition, $\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = \frac{1}{4}\Delta$.

5. Option 1: The direction perpendicular to H is given by the vector $a = (1,1,1)$. Hence for $x \in \mathbf{R}^3$, its projection $x' = Px$ will be given by the conditions $x' = x - ta$ and $x_1' + x_2' + x_3' = 0$. The last condition is the same as $x_1 + x_2 + x_3 - 3t = 0$, which gives $t = \frac{x_1+x_2+x_3}{3}$. Substituting this into the expression $x' = x - ta$, we obtain $P = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$.
Option 2: We need to find two mutually perpendicular vectors $a, b \in H$. For our particular H it is not hard to find such vectors without much calculation. For example, $a = (1,1,-1,-1)$ and $b = (1,-1,-1,1)$ have the desired properties.The length of both a and b is $\sqrt{1+1+1+1} = 2$, so the desired projection is $P = \frac{1}{4}a\otimes a + \frac{1}{4}b \otimes b = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$.

6. For $u(x) = (R^2 - x_1^2 - x_2^2 - x_3^2)(b_1x_1 + b_2x_2 + b_3x_3 + b_0)$, we obtain $\Delta u = -10b_1x_1 - 10b_2x_2 - 10b_3x_3 - 6b_0$. Hence the solution is obtained by taking $b_j = -\frac{1}{10}$, $j = 1,2,3$ and $b_0 = -\frac{1}{6}$.
In general, if P is a polynomial of degree at most m in x_1, x_2, x_3, then $Lu = \Delta[(R^2 - x_1^2 - x_2^2 - x_3^2)\,P(x_1,x_2,x_3)]$ is again a polynomial of degree ot most m. Denoting by $\mathcal{P}_m$ the linear space of all polynomials of degree $\leq m$ in $\mathbf{R}^3$, we see that L maps $\mathcal{P}_m$ to $\mathcal{P}_m$. Clearly L is a linear mapping (and hence it can be represented by a matrix, if we choose a bases in $\mathcal{P}_m$. We claim that the equation $LP = 0$ for $P \in \mathcal{P}_m$ only has the trivial solution $P = 0$. To see that, we note that $LP = 0$ implies that $u = (R^2 - |x|^2)P$ is harmonic. At the same time, u vanishes at the boundary of B_R. These tow facts imply that $u = 0$ and hence also $P = 0$. (One can use the maximum principle, for example, or other methods used to prove uniqueness of solutions for $\Delta u = 0$ in B_R and $u|_{\partial B_R} = 0$.) We see that the equation $LP = 0$ for $P \in \mathcal{P}_m$ has only the trivial solution $P = 0$. Hence the equation $LP = Q$ has a unique solution $P \in \mathcal{P}_m$ for every $Q \in \mathcal{P}_m$.