due March 22

Please submit via Moodle by midnight, March 22

Do at least four of the following six problems.[1]

1. Consider the half-ball B_R^+ of radius R in $\mathbf{R}^3$ given by

$$B_R^+ = \{x = (x_1, x_2, x_3)\,,\; x_1^2 + x_2^2 + x_3^3 < R^2\,,\; x_3 > 0\}\,. \tag{1}$$

Find a formula for the Green's function of B_R^+ using the method of images.

Hint: Use the Green's function of the ball B_R (see, for example, formula (121) in the Lecture Log) as a starting point.

2. Let $\Omega \subset \mathbf{R}^2$ be the interior of the triangle given by the points $(0,0), (0,1)$ and $(1,1)$. For functions $f\colon \Omega \to \mathbf{R}$ we wish to solve the problem

$$\begin{aligned} \Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{at } \partial\Omega. \end{aligned} \tag{2}$$

Assume that you have a machine which can do the following two tasks:
(i) For a function $f\colon \mathbf{R}^2 \to \mathbf{R}$ which is 1−periodic in both x_1 and x_2, i. e. $f(x_1+1, x_2) = f(x_1, x_2) = f(x_1, x_2+1)$ for each $x \in \mathbf{R}^2$, the machine can calculate the Fourier coefficients $\hat{f}(k) = \int_0^1 \int_0^1 f(x_1, x_2) e^{-2\pi i (k_1 x_1 + k_2 x_2)}\, dx_1\, dx_2\,,\; k = (k_1, k_2) \in \mathbf{Z}^2$.
(ii) For each set of Fourier coefficients $\hat{f}(k)\,, k \in \mathbf{Z}^2$, the machine can sum the corresponding Fourier series and obtain the function $f(x) = \sum_{k \in \mathbf{Z}^2} \hat{f}(k) e^{2\pi i (k_1 x_1 + k_2 x_2)}$.
Given $x \in \Omega$, how can one use the machine to calculate solution of the problem (2)?

Hint: You can extend f from the triangle to a 2-periodic function F in x_1 and x_2, so that the 2−periodic solution of $-\Delta U = F$ (which can be calculated using Fourier series) will satisfy $U = 0$ at $\partial\Omega$. Note that the function $x \to F(2x)$ will be 1-periodic.

3. Let $B_1 = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3\,,\; x_1^2 + x_2^2 + x_3^2 < 1\}$ be the unit ball in $\mathbf{R}^4$ centered at the origin. Describe all *radial* solutions of the eigenvalue problem

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } B_1, \\ u &= 0 && \text{at } \partial B_1\,, \end{aligned} \tag{3}$$

and the corresponding eigenvalues λ. Recall that a function $u\colon B_1$ is radial if it only depends on $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

Hint: Search the solutions in the form $u(x) = \frac{v(r)}{r}$ and note that from the first equation of (3) we get a simple equations for $v(r)$.

4. Let $z = x_1 + ix_2$, and let $P(z)$ be a polynomial in z, i. e. an expression of the form $a_0 z^m + a_1 z^{m-1} + \dots + a_{m-1} z + a_m$, where $a_0, a_1, \dots, a_m$ are complex numbers. We can consider P as a complex-valued function of two variables x_1 and x_2 and write it as

$$P(z) = P(x_1 + ix_2) = u(x_1, x_2) + iv(x_1, x_2)\,, \tag{4}$$

where u, v are real-valued functions of x_1 and x_2. Show that both u and v are harmonic, in the sense that $\Delta u = 0$ and $\Delta v = 0$.

Hint: Note that it is enough to show the statement for the special cases $P(z) = z^m$, with $m = 0, 1, 2, \dots$.

5. This problem has two options. If your background in linear algebra is not very strong, choose Option 1. If you have a good background in linear algebra, choose Option 2.
Option 1: Let $H \subset \mathbf{R}^3$ be the linear subspace given by $H = \{x \in \mathbf{R}^3\,,\; x_1 + x_2 + x_3 = 0\}$ and let $P\colon \mathbf{R}^3 \to H$ be the orthogonal projection. Recall that P is a linear map from $\mathbf{R}^3$ into $H \subset \mathbf{R}^3$, and as such is given by a 3×3 matrix. Find this matrix.
Option 2: Let $H \subset \mathbf{R}^4$ be the linear subspace given by $H = \{x \in \mathbf{R}^4\,,\; x_1 + x_2 + x_3 + x_4 = 0\,,\; x_1 - x_2 + x_3 - x_4 = 0\,\}$. Let $P\colon \mathbf{R}^4 \to H$ be the orthogonal projection. Find the 4×4 matrix representing P.

Hint: One way which works for both options is the following. Find two unit vectors $e, f \in H$ which are perpendicular to each other. Then $P = e \otimes e + f \otimes f$. The notation is the same as the one we used in class: for $b = (b_1, b_2, \dots, b_n) \in \mathbf{R}^n$ we define $b \otimes b$ as the matrix with entries $b_i b_j$. For Option 1 there is a simpler way: take $x \in \mathbf{R}^3$ and find $t \in \mathbf{R}$ such that $x - t(1,1,1) \in H$. Then $Px = x - t(1,1,1)$.

Remark: The relevance of this problem for PDE is the following: if we have some set of functions $\phi_1, \phi_2, \dots, \phi_n$ and denote H the linear space of functions of the form $c_1\phi_1 + c_2\phi_2 + \dots + c_n\phi_n$, it is often useful to have an "orthogonal projection" P of a suitable class of function on the space H. For example, in the theory of Fourier series, the projection P is associated with the Dirichlet kernel.

6. Let $B_R = \{x \in \mathbf{R}^3\,,\; |x|^2 < R^2\}$ be the ball of radius R in $\mathbf{R}^3$ centered at the origin. Let a_0, a_1, a_2, a_3 be real numbers. Show that the solutions of the problem

$$\begin{aligned} \Delta u &= a_1 x_1 + a_2 x_2 + a_3 x_3 + a_0 && \text{in } B_R \\ u &= 0 && \text{at } \partial B_R \end{aligned} \tag{5}$$

is a cubic polynomial in the variables x_1, x_2, x_3.

Hint: Search the solution as $u(x) = (R^2 - x_1^2 - x_2^2 - x_3^2)(b_1 x_1 + b_2 x_2 + b_3 x_3 + b_0)$. You can also try to show a more general fact: if f is a polynomial of degree at most m in x_1, x_2, x_3, then the solution of $\Delta u = f$ in B_R and $u|_{\partial B_R} = 0$ is of the form $(R^2 - x_1^2 - x_2^2 - x_3^2)\, P(x_1, x_2, x_3)$, where $P(x_1, x_2, x_3)$ is a polynomial of degree at most m.

[1] For grading purposes, any 4 problems correspond to 100%. You can get extra credit if you do more.