due April 26

Please submit via Moodle by midnight, April 26

Do at least four of the following six problems.[1]

1. Let $n \in \{1,2,3\}$ and let $\delta_{\mathbf{R}^n}$ be the Dirac function in $\mathbf{R}^n$. Let A be an $n \times n$ matrix with $\det(A) \neq 0$. Show that

$$\delta_{\mathbf{R}^n}(Ax) = \frac{1}{|\det(A)|}\delta_{\mathbf{R}^n}(x)\,. \tag{1}$$

Hint: In the formula $\int_{\mathbf{R}^n} \delta_{\mathbf{R}^n}(Ax)\varphi(x)\,dx$ make the change of variables $y = Ax$.

2. Let a_1, a_2, a_3 be non-zero real numbers. Find the solution of the equation

$$a_1^2\frac{\partial^2 u}{\partial x_1^2} + a_2^2\frac{\partial^2 u}{\partial x_2^2} + a_3^2\frac{\partial^2 u}{\partial x_3^2} = \delta_{\mathbf{R}^3}(x) \tag{2}$$

which satisfies $u(x) \to 0$ as $x \to \infty$.

Hint: Make a change of variables after which our equation becomes $-\Delta u = c\delta_{\mathbf{R}^3}$ for some $c \in R$.

Remark: After calculating u, you can see what happens if we take $a_1 = 1, a_2 = \sqrt{-1}, a_3 = \sqrt{-1}$ in the resulting formula (although we calculated the solution only for real a_j). For such a choice of a_j our equation becomes a wave equation. The function u you will see (with $a_1 = 1, a_2 = \sqrt{-1}, a_3 = \sqrt{-1}$) is related to the fundamental solution of the wave equation in 2+1 dimensions which we calculated in class. It is not equal to the solution everywhere, but this is in some sense not surprising, as the fundamental solution of the wave equation is not unique.

3. Find a solution of the

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} = \delta(x_1, x_2, x_3, t) \tag{3}$$

in $\mathbf{R}^3 \times \mathbf{R}$ which vanishes for $t > 0$.

4. Calculate the solution of the problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= \delta(x) \qquad && x \in \mathbf{R}, \qquad t \geq 0 \\ u(x,0) &= 0\,, && x \in \mathbf{R} \\ \frac{\partial u(x,0)}{\partial t} &= 0\,, && x \in \mathbf{R}. \end{aligned} \tag{4}$$

Hint: Use the fundamental solution of the wave equation in one spatial dimension.

5. Let a, b, c, d be real numbers, with $a, c > 0$. Find the solution of the problem

$$\begin{aligned} a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} - c\frac{\partial^2 u}{\partial^2 x} + d\,u = 0 \qquad x \in \mathbf{R}\,, t > 0\,, \\ u(x,0) = \delta(x)\,. \end{aligned} \tag{5}$$

6. For smooth functions $u\colon \mathbf{R}^3 \times (t_1, t_2) \to \mathbf{R}$ consider the functional

$$J(u) = \int_{t_1}^{t_2}\int_{\mathbf{R}^3} \left(\frac{1}{2}(u_t)^2 - \frac{1}{2}|\nabla u|^2 - \frac{c}{2}u^2 - fu\right)\,dx\,dt\,, \tag{6}$$

where c is a real number, $f = f(x,t)$ is a given function, and ∇u denotes the spatial gradient of u, i. e. the 3-vector with coordinates $\frac{\partial u}{\partial x_j}$, $j = 1, 2, 3$. To make sure that the integral is well-defined, we can assume that u and f vanish outside a bounded region. Let X be the class of smooth functions on $\mathbf{R}^3 \times [t_1, t_2]$ which vanish outside a bounded set and also vanish for all x whenever $t = t_1$ or $t = t_2$. Calculate the equation which we obtain from the requirement that for each $\varphi \in X$ the derivative of the function $\varepsilon \to J(u + \varepsilon\varphi)$ vanishes at $\varepsilon = 0$.

Remark: One might be tempted to say that at the point where the derivative of the function $\varepsilon \to J(u + \varepsilon\varphi)$ vanishes at $\varepsilon = 0$ for each $\varphi \in X$ the functional J attains it minimum or maximum, but that is not the case. As an optional part of the problem you can show that on the class of functions we consider, the functional J is not bounded from above and not bounded from below.

[1]For grading purposes, any 4 problems correspond to 100%. You can get extra credit if you do more.