

Lecture 1, 9/8/2010

The Laplace operator in \mathbb{R}^n is

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \quad (1)$$

It was introduced by P.-S.Laplace in a 1784 paper in connection with gravitational potentials.

Recall that if we have masses M_1, \dots, M_l at points a_1, \dots, a_n of \mathbb{R}^3 , then the *gravitational potential* of the configuration is given by

$$u(x) = -\varkappa \frac{M_1}{|x - a_1|} \cdots - \varkappa \frac{M_l}{|x - a_l|}, \quad (2)$$

where \varkappa is the gravitational constant.

The Newton law of gravity can be reformulated in terms of the potential u :

The force on a point particle of mass m located at x is

$$\text{force} = -m\nabla u(x).^1 \quad (3)$$

The value of the potential $u(x)$ at a point $x \in \mathbb{R}^3$ can be interpreted as the potential energy due to gravity of a particle of unit mass located at x : if we would like to move this particle to the spatial infinity, we need to act against the gravitational force, and the total work we do while moving the particle to the spatial infinity is exactly $-u(x)$.²

Assume now that instead of point masses we are dealing with a continuous distribution of mass described by a density $\rho(x)$: the mass an an arbitrary domain $\mathcal{O} \subset \mathbb{R}^3$ is given by $\int_{\mathcal{O}} \rho(x) dx$. Often we think of ρ being non-zero only in some domain Ω (representing a planet or a star, for example). The gravitational potential due to the mass described by the density ρ is

$$u(x) = \int_{\mathbb{R}^3} -\varkappa \frac{\rho(y)}{|x - y|} dy \quad (4)$$

If we wish to emphasize that ρ is only non-zero in a domain Ω , we can write

$$u(x) = \int_{\Omega} -\varkappa \frac{\rho(y)}{|x - y|} dy. \quad (5)$$

Now Laplace's observation is that at a point x in a neighborhood of which we have no masses (i. e. ρ vanishes close to x) the potential u satisfies

$$\Delta u = 0. \quad (6)$$

This can be easily seen from the following two facts:

1. $\Delta \left(\frac{1}{|x-a|} \right) = 0$ away from $x = a$, and

¹Excercise: Check by direct calculation that (3) indeed gives Newton's law.

²Excercise: Verify this statement

2. equation (6) is linear, i. e. a linear combination of solutions is again a solution.³

One could perhaps think that since we have the explicit representation (5), we can answer any questions about the gravitational field by calculating the corresponding integrals. So what is the point of bringing in equation (6)?

Let us illustrate the usefulness of equation (6) on a very simple example. Consider the situation when the total mass M is uniformly distributed over a solid sphere B_R of radius $R > 0$. So we have $\rho = M/(4\pi R^3)$ in B_R and $\rho = 0$ outside B_R . What is the gravitational potential (5) outside $\Omega = B_R$?

First, try to calculate the potential by direct integration. It may look difficult, but with some effort one can still evaluate the integral explicitly⁴.

If we use equation (6), we can avoid the explicit calculation of the integrals, and still get the exact u . We first note that the potential u is rotationally symmetric: $u(x)$ depends only on $r = |x|$.⁵ With some abuse of notation we will write $u(x) = u(r)$. For such symmetric functions u in \mathbb{R}^3 we have

$$\Delta u = u'' + \frac{2u'}{r}, \quad (7)$$

where u' denotes the derivative of u with respect to r .⁶

As a simple exercise in solving ordinary differential equations, we can find the general solution of

$$u'' + \frac{2u'}{r} = 0. \quad (8)$$

It is given by

$$u(r) = \frac{A}{r} + B. \quad (9)$$

We now determine the value of the constants A and B by looking at the potential for large x .

We have

$$\begin{aligned} u(x) &= \int_{B_R} -\varkappa \frac{\rho(y)}{|x-y|} dy \\ &= \int_{B_R} -\varkappa \frac{\rho(y)}{|x|} dy \\ &\quad + \int_{B_R} \varkappa \rho(y) \left(\frac{-1}{|x-y|} + \frac{1}{|x|} \right) dy \\ &= -\varkappa \frac{M}{|x|} + O\left(\frac{1}{|x|^2}\right) \quad \text{as } x \rightarrow \infty, \end{aligned}$$

³Excercise: Verify all these statements

⁴Excercise: Do this calculation

⁵Excercise: Verify that this is indeed the case

⁶Excercise: Verify the formula

where we have used the usual “ O -notation”, which has the following meaning: $O(1/|x|^2)$ denotes any function whose absolute value for large x is below $C/|x|^2$, for some (fixed) $C > 0$.

Combining the last equation with (9), we see that we must have

$$A = -\varkappa M, \quad B = 0.$$

We see that

$$u(x) = -\varkappa \frac{M}{|x|} \quad \text{for } |x| > R.$$

We have shown the classical result (probably going back to Newton) that the gravitational potential outside the sphere is exactly the same as the potential of a point charge at the origin, with the same mass as the total mass of the sphere.

Let us replace in the above example the solid sphere by a spherical shell $\{R_1 < |x| < R_2\}$, with the total mass M distributed uniformly over the shell. The same argument as above shows that outside the shell we again have $u(x) = -\varkappa \frac{M}{|x|}$.

As an exercise, you can show that inside the shell (i. e. for $|x| < R_1$) the potential u is constant, i. e. $u(x) = B$ for some constant B . This means that the gravitational force inside the shell vanishes.

All this can be done again by calculating directly the integral (5), but the above argument is computationally much simpler.⁷

In more complicated situations, e. g. in electrostatics of conducting surfaces of general shape, the direct calculation of the integrals in many cases is no longer feasible, whereas arguments using the equation $\Delta u = 0$ still work very well⁸ although we have to study the properties of general solutions (not necessarily symmetric) in more detail. This will be our program in the next few lectures.

⁷Exercise: Do the direct calculation of the integral in this case

⁸In fact, during the 19th century there has been an important shift in thinking about the basic laws of physics. One can probably say that in the times of Newton, the law of force exemplified by explicit representation formulae, such as (2) and (3), were considered as fundamental. However, the importance of partial differential equations satisfied by the “fields of force” gradually grew, and eventually the equations themselves became viewed as fundamental. The theory of the electromagnetic field by Faraday and Maxwell (completed in 1865) played an important role in this transformation.

Lecture 2, 9/10/2010

Last time we looked at the gravitational potential

$$u(x) = \int_{\mathbb{R}^3} -\kappa \frac{\rho(y)}{|x-y|} dy \quad (10)$$

in the regions where ρ vanishes, and we saw that in these regions we have $\Delta u = 0$. What happens in the regions where ρ does not vanish? Around 1812, S. D. Poisson discovered the following fundamental fact:

If ρ is sufficiently regular⁹, then $\Delta u = 4\pi\kappa\rho$.

This is why the equation $\Delta u = f$ is usually called the Poisson equation. Before going to the proof, we will adjust our notation and recall some definitions.

We set

$$G(x) = -\frac{1}{4\pi|x|} \quad (11)$$

and for a sufficiently regular function f (say, continuous, compactly supported) we set

$$\mathcal{G}f(x) = \int_{\mathbb{R}^3} G(x-y)f(y) dy, \quad (12)$$

which is one of the standard forms of the gravitational potential used in PDE textbooks¹⁰. We can think of \mathcal{G} as a linear operator taking functions to functions. Recalling that the convolution of two functions f, g on \mathbb{R}^n is defined¹¹ by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y) dy = \int_{\mathbb{R}^n} g(x-y)f(y) dy, \quad (13)$$

we can also write

$$\mathcal{G}f = G * f. \quad (14)$$

The above notation is somewhat “heavy”. You will soon see that it makes sense to use G for both the kernel (11) and the linear operator \mathcal{G} . If we write $Gf = G * f$ instead of $\mathcal{G} = G * f$, it may look quite ambiguous at first, as Gf can in principle mean both the pointwise product of the kernel G with the function f and the result of applying the operator G to the function f . However, in practice such confusion does not arise, as the intended meaning is usually clear from the context.

With the above notation, we can now write the Poisson’s result as follows:

$$\Delta(\mathcal{G}f) = f. \quad (15)$$

We will see that the proof also gives¹²

$$\mathcal{G}(\Delta u) = u. \quad (16)$$

⁹We will specify what this exactly means later.

¹⁰The other standard form is the one one gets by changing the sign of the kernel.

¹¹under some assumptions on f, g , e. g. f bounded measurable and compactly supported and g locally integrable

¹²under appropriate assumptions

Therefore we can write, somewhat loosely,

$$\mathcal{G} = (\Delta)^{-1} \quad \text{and} \quad \Delta = (\mathcal{G})^{-1}, \quad (17)$$

i. e. the integral operator \mathcal{G} is the inverse of the differential operator Δ , and vice versa.

This illustrates an important general point that, roughly speaking, the inverses of differential operators are closely related to integral operators.¹³ The oldest example of this is of course the Fundamental Theorem of Calculus: the inverse of taking derivatives is integration.¹⁴

We recall the following rule for differentiating convolutions:

$$\frac{\partial}{\partial x_i}(f * g) = \left(\frac{\partial}{\partial x_i}f\right) * g = f * \left(\frac{\partial}{\partial x_i}g\right). \quad (18)$$

Of course, one needs some assumptions. For example, if f is locally integrable and g is continuously differentiable and compactly supported, you can check as an exercise that with the “classical interpretation” of all the expressions the part

$$\frac{\partial}{\partial x_i}(f * g) = f * \left(\frac{\partial}{\partial x_i}g\right), \quad (19)$$

is fine, while the expression $\left(\frac{\partial}{\partial x_i}f\right) * g$ needs some interpretation, as the term $\frac{\partial}{\partial x_i}f$ may not be well-defined by the classical point-wise definitions. We will deal with issues such as this later, for now we will only use the expressions which are defined classically.

By repeated application of (18) we see that, under appropriate assumptions, we have for any differential operator L with constant coefficients

$$L(f * g) = (Lf) * g = f * (Lg). \quad (20)$$

Let $\phi: \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth function compactly supported in the unit ball with $\int_{\mathbb{R}^n} \phi = 1$. For $\varepsilon > 0$ we set

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon^n} \phi\left(\frac{x}{\varepsilon}\right). \quad (21)$$

If f is a locally integrable function, the function $f * \phi_\varepsilon$ gives a good approximation of f by smooth functions. The operation $f \rightarrow f * \phi_\varepsilon$ is called mollification

¹³The converse of this statement, which would be that the inverses of integral operators are closely related to differential operators, is not really true unless we substantially generalize the notion of a differential operator. You can see this already by looking at integral operators with kernels $1/|x|^\alpha$.

¹⁴As an exercise, you can check the following formulation of this theorem: Define $G: \mathbb{R} \rightarrow \mathbb{R}$ by $G(x) = 0$ for $x < 0$ and $G(x) = 1$ for $x > 0$. Let $\mathcal{G}f = G * f$. Let Du denote the first derivative of u . Then, under natural assumptions on f and u we have $D(\mathcal{G}f) = f$ and $\mathcal{G}(Du) = u$. We also have $D^k(\mathcal{G}^k)f = f$ for $k = 1, 2, \dots$. You can calculate the kernel of the operator \mathcal{G}^k as an exercise. You can also check what happens if we change G to, say, $G(x) = \text{sign}(x)/2$.

and the function ϕ is called a mollifier. (Often the term mollifier is also used for the linear operator $f \rightarrow f * \phi_\varepsilon$.)

Heuristically, for small $\varepsilon > 0$ the function $f * \phi_\varepsilon$ is a convex combination of small “shifts” of a function f .¹⁵ We expect that for a small y the shifted function $x \rightarrow f(x - y)$ will be close to f , and hence $f * \phi_\varepsilon$ should be close to f for small $\varepsilon > 0$. As an exercise, you can show that for a compactly supported continuous f the functions $f * \phi_\varepsilon$ converge uniformly to f as $\varepsilon \rightarrow 0$. Also, you can show that if $p \in [1, \infty)$ and $f \in L^p(\mathbb{R}^n)$, the functions $f * \phi_\varepsilon$ converge to f in $L^p(\mathbb{R}^n)$.¹⁶

The last tool we need for the proof of Poisson’s result is integration by parts: if $\Omega \subset \mathbb{R}^n$ is a smooth domain and ν and $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$ is the outer unit normal at $x \in \partial\Omega$ (the boundary of Ω), then for any sufficiently regular function v in Ω we have

$$\int_{\Omega} \frac{\partial}{\partial x_i} v \, dx = \int_{\partial\Omega} v \nu_i \, dx. \quad (22)$$

We can now proceed with the proof of $\Delta(\mathcal{G}f) = f$.

Let us choose any smooth function $K: \mathbb{R}^3 \rightarrow \mathbb{R}$ satisfying $K(x) = G(x) = -1/(4\pi|x|)$ for $|x| > 1/2$. We note that

1. ΔK is compactly supported in the unit ball B_1 , and
2. $\int_{\mathbb{R}^3} \Delta K = 1$. (Hint: Use (22).)

For $\varepsilon > 0$ we set

$$K_\varepsilon(x) = \frac{1}{\varepsilon} K\left(\frac{x}{\varepsilon}\right). \quad (23)$$

Note that $K_\varepsilon(x) = G(x)$ for $|x| > \varepsilon/2$ and that $\int_{\mathbb{R}^3} |G - K_\varepsilon|^p \, dx \rightarrow 0$ as $\varepsilon \rightarrow 0$ for $1 \leq p < 3$.

In addition, we have

$$\Delta K_\varepsilon = \frac{1}{\varepsilon^3} \Delta K\left(\frac{x}{\varepsilon}\right). \quad (24)$$

In other words, the function ΔK_ε can be considered as mollifier, similar to (21). Assume now that f is compactly supported and has continuous second derivatives.¹⁷

Set $u = G * f$ and $u_\varepsilon = K_\varepsilon * f$. Let L be any derivative of order ≤ 2 . Then $Lu_\varepsilon = K_\varepsilon * (Lf)$, and this clearly converges uniformly to $G * (Lf) = Lu$ as $\varepsilon \rightarrow 0$. Therefore u_ε converges to u uniformly, together with derivatives up to order two as $\varepsilon \rightarrow 0$. On the other hand, we have $\Delta u_\varepsilon = (\Delta K_\varepsilon) * f$, showing that Δu_ε is just a mollification of f by the mollifier ΔK_ε . Therefore $\Delta u_\varepsilon \rightarrow f$ uniformly as $\varepsilon \rightarrow 0$. We see that $\Delta u = f$, as claimed.

¹⁵Recall that a shift of a function f by a vector y is a function $x \rightarrow f(x - y)$, whose graph is a copy of the graph of f shifted in by the vector y in the plane of the coordinates x_1, \dots, x_n .

¹⁶The proof of the first statement is elementary. The precise formulation and the proof of the second statement requires the theory of the Lebesgue integration.

¹⁷We are not aiming for the most general assumptions here. Our goal is to illustrate the main point.

Remark: Some suitably defined convergence of $\Delta K_\varepsilon * f$ to f as $\varepsilon \rightarrow 0$ is to be expected for quite general functions f , so we expect that the conclusion $\Delta u = f$ should be true in a much more general situation. The main issue to clarify in the above proof when f has less regularity is the question of convergence of the second derivatives of u_ε .¹⁸ As an exercise, you can try to relax the assumptions on f in the above proof. For example, it is not hard to see that it is enough to work with one derivative of f only (as one can put the other derivative on K_ε).

¹⁸Depending on of what type of convergence we look for, the question can be subtle. For example, for a general continuous and compactly supported f , it may not be the case that the second derivatives of u_ε will be uniformly bounded as $\varepsilon \rightarrow 0$. On the other hand, if f is Hölder continuous, the convergence of the second derivatives will hold. We will eventually see that, in some sense, many of these subtleties can be bypassed if one works with good definitions.

We start with a few remarks on the proof of the identity $\Delta(G * f) = f$ we did last time.

The expression $u = G * f$ is well-defined for quite general functions f . For example, $f \in L^1(\mathbb{R}^3)$ is enough.¹⁹ There is no problem with the convergence of the approximations $u_\varepsilon = K_\varepsilon * f$ to $G * f$, and the convergence of the Laplacians $\Delta u_\varepsilon = (\Delta K_\varepsilon) * f$ to f as $\varepsilon \rightarrow 0$. So the identity $\Delta u = f$ should be true for quite general functions f .

However, we have to deal with some technical issues here. First of all, we have to say how we define Δu . If we use the classical point-wise definitions of the derivatives, we have to worry about the existence of the limits which appear in these definitions. What do we have to know about f so that we can conclude that u is twice differentiable in the classical sense? If we say “twice differentiable”, do we really mean “twice continuously differentiable”, or is it enough if the limits defining the derivatives exist, without requiring that $\nabla^2 u(x)$ is continuous as a function of x ? You can see that one can get quite quickly into all kinds of difficult questions. Over time people realized that things become much easier if the point-wise definition of derivatives are replaced by a more flexible definition.²⁰ We will get into this topic later. For now, I will mention only the following:

- In general, if f continuous, compactly supported, then $G * f$ may not be twice continuously differentiable.²¹
- If f compactly supported and α -Hölder continuous²², then the second derivatives of $G * f$ exist classically, and are also α -Hölder continuous.

Let us now turn to the proof of the identity

$$G * (\Delta u) = u \tag{25}$$

mentioned earlier in (16). For now we work with the point-wise definitions of the derivatives, so we assume that u is twice continuously differentiable. We

¹⁹To see this, consider a compactly supported smooth function φ with $\varphi = 1$ in a neighborhood of 0, write $G_1 = \varphi G$ and $G_2 = G - G_1$. Then $G * f = G_1 * f + G_2 * f$. Note that $G_2 * f$ is well-defined point-wise, i. e. the integral defining $(G_2 * f)(x)$ is well-defined for each x . (In fact, the function $G_2 * f$ is continuous, as you can easily check.) The function $G_1 * f$ is not defined point-wise in general. (Give an example!) However, since G_1 is in L^p for each $p < 3$ and $f \in L^1$, the function $G_1 * f$ is defined as an L^p function, so $G * f$ is a sum of an L^p function and a continuous function. Note that this argument works also when we replace the function f by a Radon measure.

²⁰There is some analogy here with the convergence of the Fourier series $f(x) = \sum_k c_k e^{ikx}$. Studying the point-wise convergence of the series to f can be difficult. On the other hand, if we replace the point-wise convergence by the convergence in L^2 , things become much easier, at least from the modern point of view, in which we take the Lebesgue integration for granted.

²¹The more flexible definition of derivatives we introduce later will render this seemingly unpleasant fact quite harmless.

²²Recall that this means that $|f(x) - f(y)| \leq C|x - y|^\alpha$ for some $C > 0$.

also assume it is compactly supported. The proof is very simple:

$$G * (\Delta u) = \lim_{\varepsilon \rightarrow 0} K_\varepsilon * \Delta u = \lim_{\varepsilon \rightarrow 0} (\Delta K_\varepsilon) * u = u, \quad (26)$$

where in the last equality we used that ΔK_ε is a mollifier.

Identity (25) can be used to study the solutions of general solutions of the Laplace equation $\Delta u = 0$.

A function u defined in an open domain $\Omega \subset \mathbb{R}^n$ is called *harmonic* if $\Delta u = 0$.

At the moment this definition assumes that the derivatives in the definition are point-wise, so we need to assume that u has two continuous derivatives, say. This technical condition will be removed later when we introduce a more flexible notion of derivative.

For a domain $\Omega \subset \mathbb{R}^3$ and a point $a \in \Omega$, the function $G_a(x) = -(4\pi|x - a|^{-1})$ is clearly harmonic in Ω . By taking linear combinations of functions of this form we can generate many harmonic functions. Our aim is to show that, roughly speaking, every harmonic function in Ω arises in this way.

Let $\Omega_1 \subset \Omega$ be open with the closure $\overline{\Omega}_1 \subset \Omega$. Consider a compactly supported function $\varphi: \Omega \rightarrow \mathbb{R}$ such that $\varphi = 1$ in a neighborhood of the closure $\overline{\Omega}_1$.

By (25), when $x \in \Omega_1$, we have

$$u(x) = \varphi(x)u(x) = \int_{\Omega} G(x - y)\Delta(\varphi u)(y) dy. \quad (27)$$

We compute

$$\Delta(\varphi u) = \Delta\varphi u + 2\nabla\varphi\nabla u + \varphi\Delta u = \Delta\varphi u + 2\nabla\varphi\nabla u, \quad (28)$$

where we used $\Delta u = 0$. Letting $f = \Delta\varphi u + 2\nabla\varphi\nabla u$, we note that the support of f is contained in $\Omega \setminus \overline{\Omega}_1$. Hence we can write, for $x \in \Omega_1$,

$$u(x) = \varphi(x)u(x) = \int_{\Omega} G(x - y)f(y) dy. \quad (29)$$

Formula (29) says that in Ω_1 the function u is a “linear combination” (if we can use this term for integrals) of functions $x \rightarrow G(x - y)$.

Although in many ways (29) is not optimal, we can already draw an important conclusion from it: when $x \in \Omega_1$, the integral on the right-hand side can be differentiated in x as many times as we wish, and therefore u is infinitely differentiable in Ω_1 , although we initially only assumed that it was twice differentiable. In fact, one can conclude from (29) that u is analytic in Ω_1 .²³

²³Exercise: verify this statement.

Lecture 4, 9/15/2010

Let us return to the situation we considered last time and formula (27). Our aim is to improve formula (29) by moving the derivatives of u on the left-hand side to G and φ . This can be done by integration by parts. Let us replace (28) by the following formula

$$\Delta(\varphi u) = 2 \operatorname{div}(\nabla \varphi u) - \Delta \varphi u. \quad (30)$$

We can write

$$\varphi u = G * \Delta(\varphi u) = G * (2 \operatorname{div}(\nabla \varphi u)) - G * (\Delta \varphi u) = 2G_{,i} * (\varphi_{,i} u) - G * (\Delta \varphi u), \quad (31)$$

where $G_{,i}, \varphi_{,j}$ denote partial derivatives, and we use the summation convention of summing over repeated indices. Writing out the integrals explicitly, we have for $x \in \Omega_1$

$$u(x) = \int_{\operatorname{support} \varphi} [2G_{,i}(x-y)\varphi_{,i}(y)u(y) - G(x-y)\Delta\varphi(y)u(y)] dy, \quad x \in \Omega_1. \quad (32)$$

This formula implies some classical results about harmonic functions.

First, we prove the following a-priori estimate:

Let u be a harmonic function in the ball $B_{2R} \subset \mathbb{R}^3$ of radius $2R$. Then, for $k = 0, 1, 2, \dots$, we have the following point-wise estimate for the derivatives of u in the ball B_R :

$$|\nabla^k u(x)| \leq \frac{C_k}{R^{3+k}} \int_{B_{2R} \setminus B_R} |u(y)| dy, \quad x \in B_R, \quad (33)$$

where C_k are constants independent of u and R .

We will see later that this estimate can still be improved in various ways. Nevertheless, it is sufficient for many purposes.

The proof of (33) follows easily from (32): differentiate (32) to obtain

$$\nabla^k u(x) = \int_{\operatorname{support} \varphi} [2\nabla_x^k G_{,i}(x-y)\varphi_{,i}(y)u(y) - \nabla_x^k G(x-y)\Delta\varphi(y)u(y)] dy, \quad x \in \Omega_1. \quad (34)$$

Consider a smooth function ψ compactly supported in the ball B_2 of radius 2 with $\psi = 1$ in $B_{\frac{3}{2}}$ and set $\varphi_R(x) = \psi(x/R)$. Use (34) with $\varphi = \varphi_R$ and note that for $x \in B_R$ and y is in the support of $\nabla\varphi_R$ we have

$$|\nabla_x^k G_{,i}(x-y)| \leq \frac{c_k}{R^{2+k}}, \quad |\nabla_x^k G(x-y)| \leq \frac{c_k}{R^{1+k}} \quad (35)$$

and

$$|\nabla\varphi_R(y)| \leq \frac{c'_1}{R}, \quad \Delta\varphi_R(y) \leq \frac{c'_2}{R^2}. \quad (36)$$

This immediately gives (33).

A simple consequence of (33) is the following estimate
For a harmonic function u in B_R we have

$$\sup_{B_R} |\nabla^k u| \leq \frac{C_k}{R^k} \sup_{B_{2R}} |u|, \quad (37)$$

where the constants C_k are independent of R and u .

We can now prove a classical result:

Liouville Theorem.

A bounded harmonic function in R^3 is constant.

Proof: We use (37) for $k = 1$ and $R \rightarrow \infty$.

One can use the same idea to prove a more general statement:

Liouville Theorem, version 2.

Assume that u is harmonic in R^3 and that $|u(x)| \leq C(|x|^{m-\varepsilon} + 1)$ for some positive integer m , with $C > 0$ and $\varepsilon > 0$. Then u is a polynomial of order at most $m - 1$.

Proof: Use (37) with $k = m$ and $R \rightarrow \infty$.

Note that the last theorem is more or less optimal, as for each integer m there are non-trivial harmonic polynomials of degree m .

As there are no derivatives in the right-hand side (32), one expects that the assumption that u has two derivatives we used in the definition of harmonic functions is of technical nature, and is not really essential. We will address general issues related to this later in a systematic way, but perhaps it is worth making some remarks on this even now.

Let us consider the mollifier $\phi_\varepsilon(x) = \frac{1}{\varepsilon^n} \phi(\frac{x}{\varepsilon})$ we introduced earlier. For a locally integrable function u defined in a domain $\Omega \subset \mathbb{R}^n$, the function $u * \phi_\varepsilon$ is unambiguously well defined in $\Omega_\varepsilon = \{x \in \Omega, \text{dist}(x, \partial\Omega) < \varepsilon\}$, as the support of ϕ_ε is contained in the ball B_ε . Assume now that u is a locally integrable function in Ω and that $u_\varepsilon = u * \phi_\varepsilon$ is harmonic in Ω_ε for small ε . Write (32) for u_ε :

$$u_\varepsilon(x) = \int_{\text{support } \varphi} [2G_{,i}(x-y)\varphi_{,i}(y)u_\varepsilon(y) - G(x-y)\Delta\varphi(y)u_\varepsilon(y)] dy, \quad x \in \Omega_1. \quad (38)$$

Note that the support of φ is contained in Ω_ε for small ε . Now as $\varepsilon \rightarrow 0$, the functions u_ε converge to u in $L^1(\text{support of } \varphi)$. The right-hand side of (38) converges to the same expression with u_ε replaced by u . Therefore $u_\varepsilon(x)$ must

converge to a limit for each $x \in \Omega_1$ as $\varepsilon \rightarrow 0$. (As an exercise, you can check that the convergence of u_ε is uniform in Ω_1 .) At the same time, u_ε converge to u in $L^1(\Omega_1)$, by general properties of integrable functions. We conclude that the representation formula holds true for the function u , i. e. we can remove the epsilons in (38). Therefore u is smooth in Ω_1 and satisfies all the estimates we obtained above. Since we can take for Ω_1 any ball whose closure is in Ω , we see that u is in fact smooth in Ω .

We could call a locally integrable function u defined in Ω weakly harmonic if $u * \phi_\varepsilon$ is harmonic in Ω_ε for each $\varepsilon > 0$. However, such a definition does not look appealing. For example, is it not immediately clear that it is independent of the choice the mollifier. A much more elegant definition²⁴ which is in fact equivalent,²⁵ is the following:

Definition: A locally integrable function in $\Omega \subset \mathbb{R}^3$ is *weakly harmonic* if $\int_\Omega u \Delta \phi = 0$ for each smooth, compactly supported function $\phi: \Omega \rightarrow \mathbb{R}$.

If u is weakly harmonic in Ω and ϕ_ε is the same mollifier as above, it is immediate that $\Delta(u * \phi_\varepsilon) = 0$. Hence the above arguments prove the following statement.

Weyl's Lemma.

A weakly harmonic function u is smooth and harmonic.

Proof: See above.

²⁴a special case of defining derivatives as distributions, as introduced by L. Schwartz

²⁵Exercise: prove the equivalence

Lecture 5, 9/17/2010

The definition of a weakly harmonic function can be immediately generalized to Poisson's equation $\Delta u = f$.

Definition.

Let u and f be locally integrable functions in a domain $\Omega \subset \mathbb{R}^n$. We say that the equation $\Delta u = f$ is satisfied weakly in $\int_{\Omega} u \Delta \varphi = \int_{\Omega} f \varphi$ for each smooth, compactly supported $\varphi: \Omega \rightarrow \mathbb{R}$.

This is again a special case of defining derivatives in the theory of distributions, due to L. Schwartz in 1940s. An equivalent definition is that, $\Delta(u * \varphi_{\varepsilon}) = f * \varphi_{\varepsilon}$ in Ω_{ε} for every (small) $\varepsilon > 0$, similar to the situation with weakly harmonic functions mentioned in the last lecture.

The idea of these definitions is that we do not really try to say what point-wise values of functions such as u or f are. We only consider point-wise values of the smooth compactly supported functions ϕ , called the test functions.²⁶

With the definition above, the difficulties we had to deal with when we considered the point-wise definitions of derivatives are not present. For example, we have

Theorem.

Assume $f \in L^1(\mathbb{R}^3)$ and let $u = G * f$.²⁷ Then $\Delta u = f$ weakly.

Proof: Using the approximation K_{ε} of G defined in lecture 2, we can write²⁸

$$\begin{aligned} \int (G * f) \Delta \varphi &= \lim_{\varepsilon \rightarrow 0} \int (K_{\varepsilon} * f) \Delta \varphi = \lim_{\varepsilon \rightarrow 0} \int \Delta (K_{\varepsilon} * f) \varphi \\ &= \lim_{\varepsilon \rightarrow 0} \int [(\Delta K_{\varepsilon}) * f] \varphi = \int f \varphi, \end{aligned}$$

where we have used that ΔK_{ε} is a mollifier.

Note how the problems which we had to deal with when we used point-wise definitions of derivatives for Δu seemingly disappeared. In reality, the problems about the regularity of u are still there, but the point is that we need not to

²⁶If we think about it, such approach makes a lot of sense from the point of view of measuring physical quantities such as u (the gravitational potential) and f (the density of matter): in practice, we can never measure u or f at a point. Any measuring device really measures some kind of average of the measured quantity around the point where we are making the measurement. So instead of measuring $u(x)$, in reality we measure something like $\int u(x-y)\varphi(y)dy$, where $\int \varphi = 1$. The support of the function φ becomes smaller as the measurement becomes more accurate, but it is never zero. In this picture the test function φ represent some kind of measuring device.

²⁷We have seen in the beginning of lecture 3 that $G * f$ is well defined, although not point-wise, in general.

²⁸Exercise: justify this precisely using standard Lebesgue integration theorems.

answer them and still can proceed. We simply give the equation $\Delta u = f$ a new meaning.

One issue we have to deal with when we weaken the definition of what an equation says is the issue of uniqueness. We can imagine that we could go too far with the weakening of the definitions, and that for the weak version of the equation some unexpected solutions will appear. However, in the example above this does not happen, as we can see from the following:

Theorem.

If u_1, u_2 and f are locally integrable in \mathbb{R}^3 , and the equations $\Delta u_1 = f$ and $\Delta u_2 = f$ are satisfied weakly, then $h = u_2 - u_1$ is a smooth harmonic function.

Proof: Check that $u_2 - u_1$ is weakly harmonic and apply Weyl's lemma from the previous lecture.

We see that the notion of the weak solution does not bring any new non-uniqueness. The non-uniqueness is the same as in the case when all quantities are considered smooth and the equation is satisfied point-wise. It is of course clear that the equation $\Delta u = f$ can determine u at best only up to a smooth harmonic function. To get uniqueness, we have to impose an additional restriction on u to eliminate the harmonic function. For example, in the case when f is compactly supported it is natural to require that $u(x) \rightarrow 0$ as $x \rightarrow \infty$.²⁹ With this or another similar condition at ∞ , the equation $\Delta u = f$ determines u uniquely.

The mean value property of harmonic functions.

We will now continue our study of harmonic functions by proving the following classical result³⁰: If u is harmonic in the ball $B_{x,R} = \{y \in \mathbb{R}^3, |y - x| < R\}$ and continuous in the closure of the ball, then

$$u(x) = \frac{1}{|\partial B_{x,R}|} \int_{\partial B_{x,R}} u(y) dy. \quad (39)$$

Here we denote by $\partial B_{x,R}$ the boundary of the ball of radius R centered at x and $|\partial B_{x,R}|$ denotes the area of the boundary.

Another form of the mean value property is

$$u(x) = \frac{1}{|B_{x,R}|} \int_{B_{x,R}} u(y) dy, \quad (40)$$

where this time we take average over the solid ball $B_{x,R}$. As an exercise, you can show that (39) for all radii $R \in (0, R_0)$ implies (40) for all radii $R \in (0, R_0)$ and vice versa.³¹

²⁹Recall that we are \mathbb{R}^3 .

³⁰going back to Gauss

³¹Hint: for the first implication, integrate in R ; for the second, differentiate in R .

One way to verify these formulae is to prove them for the special case $u(x) = G(x-a)$ by direct calculation and then use the fact that every harmonic function is in some sense a linear combination of function of the form $x \rightarrow G(x-a)$. For the special case of $x \rightarrow G(x-a)$, we have to verify

$$\frac{1}{|\partial B_{x,R}|} \int_{\partial B_{x,R}} G(y-a) dy = G(x-a). \quad (41)$$

However, this is exactly the statement we discussed in lecture 1: the gravitational potential at point a of a homogeneous shell of unit mass and radius R centered at x is exactly the same as the gravitational potential at a of a unit mass located at x . Thus our statement is proved.

Another way to do the proof is for example as follows. Let us assume without loss of generality that $x = 0$. Let f_1 and f_2 be two radial functions (i. e. $f_1 = f_1(|x|), f_2 = f_2(|x|)$) supported in the ball of radius R , with $\int f_1 = \int f_2 = 1$. From lecture 1 we know that $G * f_1 = G * f_2$ outside the support of f_1 and f_2 . This means that $v = G * f_1 - G * f_2$ is compactly supported in B_R . Hence we have

$$\int u(f_1 - f_2) = \int u \Delta v = \int \Delta u v = 0. \quad (42)$$

We can now choose functions $f_1 = f_{1\varepsilon}$ and $f_2 = f_{2\varepsilon}$ so that $\int u f_{1\varepsilon} \rightarrow u(x)$ and $\int u f_{2\varepsilon} \rightarrow \frac{1}{|\partial B_{x,R}|} \int_{\partial B_{x,R}} u$ as $\varepsilon \rightarrow 0$.³²

³²Exercise: do this in detail.

Lecture 6, 9/20/2010

Today we will talk about the maximum principle, strong maximum principle, and the Harnack inequality for harmonic functions. These properties are, in increasing order of “depth”, important hallmarks of the behavior of harmonic functions, and are in fact shared by solutions of much more general second order scalar elliptic and parabolic equations. With the exception of some easy alternative proofs of the maximum principle, at this stage our proofs will be based on the mean-value property of the harmonic functions. Such proofs are not easily adapted to more general classes of equations, for which different approaches have to be found. Nevertheless, it is useful to see what can be proved by elementary methods, before starting to use some of the more advanced techniques.

In what follows we denote by Ω a bounded domain (= connected open set) in \mathbb{R}^3 .

A simple consequence of the mean value property is the following:

Lemma.

Assume that u is harmonic in a ball $B_{x,R}$. If $u(x) = \sup_{B_{x,R}} u$, then u is constant in $B_{x,R}$.

Proof: We know that $u(x) = \frac{1}{|B_{x,R'}|} \int_{B_{x,R'}} u$ for every $R' < R$. Now it is enough to note that if an averaged quantity is never above its average, it has to be equal to the average.

Corollary: (The strong maximum principle)

If $u: \Omega \rightarrow \mathbb{R}$ is harmonic and $u(x) = \sup_{\Omega} u$ for some $x \in \Omega$, then u is constant in Ω .

Proof: By the lemma above, $u = u(x)$ in any ball $B_{x,R} \subset \Omega$. Take the maximal R with this property. Now we can replace x by $x' \in B_{x,R}$ and repeat the argument. We can get a still larger set where $u = u(x)$. Since Ω is connected, it is easy to see that the property $u = u(x)$ will “propagate” to all Ω . (To do this argument in a more formal fashion, you can check that the set $\{y \in \Omega, u(y) = u(x)\}$ is both closed and open in Ω and hence has to coincide with Ω , due to the assumption that Ω is connected.)³³

³³If one uses the fact that the harmonic functions are analytic, one can see that one has the following stronger statement:

If $u: \Omega \rightarrow \mathbb{R}$ is harmonic and for some ball $B_{x,R} \subset \Omega$ we have $\sup_{B_{x,R}} u = u(x)$, then u is constant in Ω .

For the proof we only have to note that u is constant in $B_{x,R}$ by the lemma above and use analyticity.

We see from the proof that the validity of the statement is a result of two effects: the strong maximum principle and analyticity. As we mentioned, the strong maximum principle generalizes to quite general classes (scalar, second order) equations. The generalization of analyticity in this context is known as “unique continuation”. It does hold for quite general equations,

Alternative proofs of the maximum principle.

By the maximum principle we usually mean the statements that a harmonic function $u: \Omega \rightarrow \mathbb{R}$ cannot attain its *strict* maximum over Ω inside Ω . If u is continuous up to the boundary of Ω , this means that we cannot have $u(x) > \sup_{\partial\Omega} u$ for some $x \in \Omega$. This statement can be proved quite easily in a number of ways, without using the mean-value property.

We will illustrate this by two different proofs.³⁴

Proof 1: This proof is based on the simple observation that at an interior point of a local maximum of u the second derivatives of u must be negative. We did not say “strictly negative”, what we really meant is “not above zero”. This looks close to having a contradiction with the equation $\Delta u = 0$, but it is still not a contradiction. However, since the maximum over Ω is strictly bigger than the maximum over $\partial\Omega$, one has some room to perturb the situation to a real contradiction: we note that for a sufficiently small $\varepsilon > 0$ the function $u_\varepsilon(y) = u(y) + \varepsilon|y|^2$ still attains its maximum at an interior point of Ω . We use the same argument with the second derivatives, and note that $\Delta u_\varepsilon > 0$ (strict inequality!), so this time we do get a contradiction. This proof can be used for quite general equations, assuming we have sufficiently regular coefficients to be able to work with the point-wise values of the second derivatives.

Proof 2: This proof is based on a very important observation, due to Dirichlet or perhaps Gauss, that a harmonic function minimizes u the integral $\int_\Omega |\nabla u|^2$ among all (sufficiently regular) functions having the same boundary value.³⁵ (We assume that all functions are sufficiently regular up to the boundary.) The proof is quite easy: Let u be harmonic in Ω and smooth up to the boundary, and v be smooth up to the boundary, with $v|_{\partial\Omega} = 0$. Then, trivially

$$\int_\Omega |\nabla(u+v)|^2 = \int_\Omega |\nabla u|^2 + 2\nabla u \nabla v + |\nabla v|^2. \quad (43)$$

The main point now is that the cross term vanishes, as can be seen from integration by parts:

$$\int_\Omega 2\nabla u \nabla v = \int_\Omega -2(\Delta u)v = 0. \quad (44)$$

Therefore we have

$$\int_\Omega |\nabla(u+v)|^2 = \int_\Omega |\nabla u|^2 + \int_\Omega |\nabla v|^2 \quad (45)$$

but not quite as general as the strong maximum principle. For example, for equations of the form $a_{ij}(x)u_{x_i x_j} = 0$ in dimensions higher than 2 one needs that the coefficients $a_{ij}(x)$ are Lipschitz continuous.

³⁴Neither of these proofs can be easily adapted to give the *strong* maximum principle, but they are still quite instructive.

³⁵We will be exploring some of the far-reaching implication of this observation in some detail later.

and the claim follows.

Now the maximum principle can be seen easily: if u has a peak inside Ω , it cannot be minimizing, as we could cut off the peak and save the energy. Note that this argument does not require much regularity. It can work well in more general situation, if the equation satisfies an analogue of the Dirichlet principle.

The Harnack inequality

We have seen that if u is harmonic and $u(x) = \sup_{\Omega} u$ for some $x \in \Omega$, then u must be constant. In some sense, the Harnack inequality addresses the question what happens if we slightly relax the assumption $u(x) = \sup_{\Omega} u$ to $u(x)$ being close to $\sup_{\Omega} u$. Will then $u(x)$ be close to $\sup_{\Omega} u$, at least on a large part of Ω ? In other words, we are asking if the behavior described by the strong maximum principle is in some sense stable. As we will see, the answer is positive.

The Harnack inequality is traditionally defined in terms of non-negative functions. To relate this formulation to the maximum principle, we should look at $M - u$, where $M = \sup_{\Omega} u$.

Harnack Inequality

For each compact subset $K \subset \Omega$ there exists a constant $C = C(K, \Omega)$ such that for every non-negative harmonic function u in Ω we have

$$\sup_K u \leq C \inf_K u. \quad (46)$$

Proof: Assuming the statement fails, we can obtain a sequence of harmonic functions $u_n \geq 0$ in Ω , and two sequences of points $x_n, y_n \in K$ such that:

- $u_n \leq 1$ in K , and $u_n(y_n) = 1$,
- $u_n(x_n) \rightarrow 0$,
- $x_n \rightarrow \bar{x} \in K$, $y_n \rightarrow \bar{y} \in K$.

Let us now consider an arbitrary point $a \in K$. Choose r so that the closure of the ball $B_{a,r}$ is contained in Ω . We have

$$\frac{1}{|B_{a,r}|} \int_{B_{a,r}} u_n = u_n(a) \leq 1. \quad (47)$$

By estimate (33) we know that u_n together with their derivatives up to any given order are bounded point-wise in $B_{a,r/2}$. The bounds may possibly depend on a , but since K is compact, we can cover it by finitely many balls $B_{a,r/2}$ and get estimates which are uniform in K . In other words, the functions u_n , together

with their derivatives up to a given order are bounded in a neighborhood of \mathcal{O} of K . Note that we can assume without loss of generality that \mathcal{O} is connected.³⁶ Since we have the uniform bounds on the derivatives up to any given order in \mathcal{O} , we can use the Arzela-Ascoli theorem to choose a suitable subsequence of the sequence $u_n, n = 1, 2, \dots$ which converges uniformly with its derivatives up to a given order to a function \bar{u} . This subsequence will still be denoted as $u_n, n = 1, 2, \dots$. The function \bar{u} is harmonic in \mathcal{O} , non-negative, with $\bar{u}(\bar{x}) = 0$ and $\bar{u}(\bar{y}) = 1$. This contradicts the strong maximum principle, and the proof is finished.³⁷

³⁶Exercise: verify this point.

³⁷Many steps of the proof above work for more general equations. The main difficulty we have to deal with then is to obtain an estimate in the direction of (47). Note that in this estimate we obtain information about the behavior of u_n in a *neighborhood* of K from its behavior only on K , and the assumption $u_n \geq 0$.

Lecture 7/22/2010

Today we start talking about boundary value problems. Historically, these became important in connection with electrostatics.³⁸ The basic objects in electrostatics are electric charges. They satisfy Coulomb's law, force = $-kqQ/r^2$, which gives the force between charges q and Q at distance r , where k is a constant depending on the choice of units. We can introduce the electric potential u , similar to the gravitational potential, in that the force on point-charge q located at x is given by $-q\nabla u(x)$. The difference with gravity is that the charge can have both positive and negative sign, and that charges of the same sign repel each other. Therefore we have to change some signs in some formulae we used for gravity. There is more than one way to do it, but the convention is that the potential of a positive charge is positive. So the potential of a point-charge Q located at the origin is $kQ/|x|$, which is of the opposite sign than our convention for the gravitational potential. These issues with signs are not important for the PDE analysis, but it is good to be aware of them when looking at textbooks of the electromagnetism.³⁹

Let us now consider an example of a situation where a boundary value problem naturally arises. Consider a conducting ball B_1 centered at a_1 with charge Q_1 . The electrical potential outside B_1 is $u_1 = kQ_1/|x - a_1|$. Inside the ball the potential u_1 is constant, with the constant determined by the requirement that u_1 is continuous across the boundary of the ball. Imagine we now move into this field from very far away another charged ball B_2 with charge Q_2 , and position it so that its center is at a_2 (sufficiently far away from a_1 so that the balls do not touch). If B_1 were absent, the electric potential of the ball B_2 would be $u_2 = kQ_2/|x - a_2|$ outside of the ball and constant inside B_2 .

What is the electric potential when both balls are present? If we were dealing with gravity and B_1, B_2 would be solid bodies, the resulting potential would be $u = u_1 + u_2$. The same would be the case if the charges in the balls were each "attached" to certain points so that they could not move. However, charges in conductors can move. Ideal conductors are characterized by the property that the electric potential has to be constant in it. Therefore the resulting potential will not be $u = u_1 + u_2$, even in the case when $Q_2 = 0$.⁴⁰ So how do we determine the resulting potential? We can try to figure out what is going on with the charges and how they should distribute themselves in the balls and get some equations for the distributions, but it is easier to work the directly with the potentials. The distribution of charges can be obtained once the potential is known.

The potential u of the two balls has to satisfy the following conditions:

³⁸In Newtonian gravity the boundary value problems are not really as important, since there is no known way of manipulating the gravitational potential in the same way the electrical potential can be manipulated by introducing conductors or dielectrics into the electrical field.

³⁹We also note that there are several systems of units which are used, so the value of k can be different in different textbooks. You can check http://en.wikipedia.org/wiki/Centimetre_gram_second_system_of_units for more details.

⁴⁰As an exercise, try to answer the following question: if $Q_2 = 0$ and $Q_1 \neq 0$, i. e. we move an uncharged ball into the field of a charged ball, will the balls repel each other, attract one another, or will there be no force between them?

- u has to be constant in B_1 , say, $u = c_1$ in B_1 ,
- u has to be constant in B_2 , say, $u = c_2$ in B_2 ,
- u has to be continuous across the boundaries of the balls,
- $u(x) \rightarrow 0$ as $x \rightarrow \infty$,
- u has to be harmonic outside the balls, i. e. $\Delta u = 0$ in $\mathbb{R}^3 \setminus (B_1 \cup B_2)$,
- the total charge of B_1 is Q_1 . This translates to $\int_{\partial B_1} \frac{\partial u}{\partial \nu} = -kQ_1$, where $\frac{\partial u}{\partial \nu}$ is the derivative in the direction of outer unit normal to B_1 .
- the total charge of B_2 is Q_2 , so $\int_{\partial B_2} \frac{\partial u}{\partial \nu} = -kQ_2$ similarly to the previous point.

The problems of finding a function u satisfying the above conditions is an example of a boundary value problem. We seek a function satisfying some equation, and we know some conditions at the boundaries of the domains where the equation is satisfied. Our task is to find the function based on these conditions.

As the first example of a boundary-value problem we will try to solve, we choose a simpler example than the above. Roughly speaking, we will deal with a situation that we have some given distribution charges which cannot move, and our goal is to determine its electric potential in a presence of a conductor connected to ∞ . This leads to the following problem. Consider an open set $\Omega \subset \mathbb{R}^3$. Let $f: \Omega \rightarrow \mathbb{R}$ be a given function (“distribution of charge”). The function is considered as fixed. We wish to find a function u (corresponding to the electric potential), such that

$$\Delta u = f \quad \text{in } \Omega, \text{ and} \tag{48}$$

$$u = 0 \quad \text{at the boundary } \partial\Omega. \tag{49}$$

Note that we have reverted to our old notation, and do not write $-\Delta u = f$ as one perhaps should in the context of electrostatics.

We will first solve (48)–(49) in a domain for which the problem is quite easy, the half-space $\Omega = \mathbb{R}_+^3 = \{x \in \mathbb{R}^3, x_3 > 0\}$. In this case we can write the solutions explicitly: extend f to a function $\tilde{f}: \mathbb{R}^3 \rightarrow \mathbb{R}$ which is odd in x_3 , and for $x \in \Omega$ set $u(x) = G * \tilde{f}$. We know that u satisfies $\Delta u = f$ in Ω and it is easy to see that u is also odd in x_3 , and hence $u(x_1, x_2, 0) = 0$.

We can think about this solution in the following way: to a charge at $y \in \Omega$ we associate a fictitious charge of the opposite sign at $y^* = (y_1, y_2, -y_3)$. The potential of those two charges is a function of x . As we are using our old sign convention, it is natural to take the function as

$$G_\Omega(x, y) = -\frac{1}{4\pi|x-y|} + \frac{1}{4\pi|x-y^*|}. \tag{50}$$

Modulo the right sign conventions, this is the electric potential we would get by placing a single charge at y in the “empty” domain Ω whose boundary is made out of a conductor. The function G_Ω is known as Green’s function of the domain Ω , and plays the same role which the kernel $-1/(4\pi|x - y|)$ has in the full space: it inverts the Laplace operator Δ for the zero boundary conditions.⁴¹ Now the solution u can be written as

$$u(x) = \int_{\Omega} G_\Omega(x, y) f(y) dy. \tag{51}$$

⁴¹For $\Omega = \mathbb{R}^3$ we also have a “boundary condition” at ∞ : we need to assume $u(x) \rightarrow 0$ (perhaps in some weak sense), so that our solution is unique.

Lecture 8, 9/24/2010

We first have a look at our notation. So far we have mostly used G for the function

$$G(x) = -\frac{1}{4\pi|x|}. \quad (52)$$

We have also used K_ε for a certain smooth approximation of G , see (23). We have already seen that it makes sense to write G for Green's function of \mathbb{R}^3 . In that case can write

$$G(x, y) = -\frac{1}{4\pi|x-y|}. \quad (53)$$

As $K_\varepsilon(x)$ converges to $G(x)$ as $\varepsilon \rightarrow 0$, it also makes sense to write

$$K_0 = -\frac{1}{4\pi|x|}, \quad (54)$$

so that we have, for example,

$$G(x, y) = K_0(x - y). \quad (55)$$

This looks better than

$$G(x, y) = G(x - y) \quad (56)$$

which might not be considered as good notation.

We will use G_Ω to denote Green's function of the domain Ω , although later we will often not insist on writing the index Ω , if the context is clear.

Although the above notation is slightly ambiguous, you will see that in practice it does not lead to confusion.

We will now continue with exploring simple properties of Green's functions, assuming the functions exist. Based on what we have covered so far, their existence is, strictly speaking, not proven, except in cases where we have constructed them explicitly. The full existence proof will be given later.

You will also see later that there are approaches to PDEs which do not emphasize Green's functions.

Recall that in the context of Green's function G_Ω of the Laplacian in Ω , we can think about the following analogy: Ω represents "empty space" surrounded by a conductor, which is connected to ∞ , so that the value of the electric potential on $\partial\Omega$ is fixed to 0. We put a unit charge⁴² at location $y \in \Omega$ and the electric potential of that charge in Ω , is the function $x \rightarrow G_\Omega(x, y)$. In comparison with the situation when no conductor is present, the potential in Ω will be modified by charges in the boundary $\partial\Omega$ which will appear due to the presence of the charge at y . Let us denote the potential produced by those charges by $H^{(y)}$.

⁴²Be aware of our sign conventions.

Clearly $x \rightarrow H^{(y)}(x)$ is a harmonic function in Ω (as the charges do not enter Ω), and we expect

$$G_\Omega(x, y) = K_0(x - y) + H^{(y)}(x), \quad G(x, y) = 0 \text{ for } x \in \partial\Omega, y \in \Omega. \quad (57)$$

Strictly speaking, one should be more precise in what sense the “boundary condition” $G_\Omega(x, y) = 0$ is attained. For now we will require that $x \rightarrow G_\Omega(x, y)$ is continuous in the closure of Ω except at y , and - as above - that $G_\Omega(x, y) = 0$ when $x \in \partial\Omega$ and $y \in \Omega$.

Note that we have not proved the existence of $H^{(y)}$, at this point we only gave a heuristic argument for its existence. In fact, if the boundary of Ω is sufficiently irregular, the issue in what sense the boundary condition $G_\Omega(x, y) = 0$ should be satisfied can become subtle. However, under relatively mild regularity assumptions on $\partial\Omega$ the boundary condition is satisfied in the point-wise sense. The last condition determines the boundary condition for $H^{(y)}$ and from the maximum principle we see the following:

Lemma.

G_Ω is uniquely determined by the above requirements.

Proof: Apply the maximum principle as indicated above.

The next statement is also a direct consequence of the definition, at least at the formal level. Assume that f is a “sufficiently regular” function in Ω and let

$$u(x) = \int_\Omega G_\Omega(x, y) f(y) dy. \quad (58)$$

Lemma: In the notation above, $\Delta u = f$ in Ω . Under relatively mild regularity assumptions on $\partial\Omega$ and f the function u is also continuous up to the boundary $\partial\Omega$ and $u(x) = 0$ for $x \in \partial\Omega$.

“Proof”: Let us extend f by 0 to \mathbb{R}^3 . We can write $u(x) = (K_0 * f)(x) + \int_\Omega H^{(y)}(x) f(y) dy$. We know that $\Delta(K_0 * f) = f$ in Ω . The expression

$$\int_\Omega H^{(y)}(x) f(y) dy \quad (59)$$

can be thought of as a linear combination of functions $H^{(y)}$, each of which is harmonic in Ω , and therefore it should also be harmonic in Ω . Finally, since $G(x, y) = 0$ for each $x \in \partial\Omega$ and each $y \in \Omega$, we clearly have $u(x) = 0$ for $x \in \partial\Omega$.

As you may have noticed, the above argument is not really a rigorous proof. It shows why the statements should be true, but to make it into a real proof one would need to fill in some technical details. For example, we have to justify that the integral (59) is well-defined and that we can differentiate it in x . Another problem is that while it is clear that $\int_\Omega G_\Omega(x, y) f(y) dy = 0$ when $x \in \partial\Omega$, our statement claims something stronger: that the same expression is small when x is close to $\partial\Omega$. That may not look so obvious at first, as $G_\Omega(x, y)$ is unbounded

as a function of y near x . It takes some work to clarify these and other issues. It is not hard, but for now we prefer to proceed with further calculations.

Lemma:

Green's function G_Ω is symmetric in x, y . In other words,

$$G_\Omega(x, y) = G_\Omega(y, x). \quad (60)$$

This statement is perhaps more surprising than the previous two lemmas. The definition of G_Ω is not transparently symmetric. The reason why G_Ω is symmetric is, roughly speaking, that the Laplace operator (augmented with the zero boundary conditions) is symmetric, and G_Ω is the inverse of Δ (with zero boundary conditions). The inverse of a symmetric operator is symmetric and there are many ways to see it. For example, one can use the following argument. Assume A is a self-adjoint symmetric operator with respect to a given scalar product $x, y \rightarrow (x, y)$, which means that $(Ax, y) = (x, Ay)$. Now one can replace x with $A^{-1}x$ and y with $A^{-1}y$ and we see that A^{-1} is also symmetric.

Our proof that $G_\Omega(x, y) = G_\Omega(y, x)$ is similar. The relevant symmetry of the Laplace operator is $\int_\Omega (\Delta v_1) v_2 dx = \int_\Omega v_1 \Delta v_2 dx$ for all (sufficiently regular) functions v_1, v_2 vanishing at $\partial\Omega$.

Let us define $G_{\varepsilon, \Omega}$ by

$$G_{\varepsilon, \Omega}(x, y) = K_\varepsilon(x - y) + H^{(y)}(x), \quad (61)$$

where $H^{(y)}$ has the same meaning as above. Clearly $G_{\varepsilon, \Omega}(x, y) = G_\Omega(x, y)$ when $|x - y| \geq \varepsilon$.

Let us now prove the symmetry of G_Ω . Take y_1, y_2 in Ω , $y_1 \neq y_2$. For sufficiently small ε we have

$$\int_\Omega \Delta_x G_{\varepsilon, \Omega}(x, y_1) G_{\varepsilon, \Omega}(x, y_2) dx = \int_\Omega G_{\varepsilon, \Omega}(x, y_1) \Delta_x G_{\varepsilon, \Omega}(x, y_2) dx \quad (62)$$

by the symmetry of Δ with the zero boundary-condition. We note that $\Delta_x G_{\varepsilon, \Omega}(x, y_1)$ is the same as $\Delta_x (K_\varepsilon(x - y_1))$ and can be considered as a mollifier located at y_1 . Hence the left-hand side of (62) converges to $G_\Omega(y_1, y_2)$ as $\varepsilon \rightarrow 0$. Similarly, the right-hand side converges to $G_\Omega(y_2, y_1)$ and the symmetry of G_Ω follows.

Green's functions can be used to obtain a better representation formula for harmonic functions than (32). We recall the following notation. For $x \in \partial\Omega$

we denote by $\nu(x)$ the outer unit normal to Ω at x . (We assume that $\partial\Omega$ is sufficiently regular.) For a function u defined in the closure of Ω , we denote by

$$\frac{\partial u}{\partial \nu} \quad (63)$$

the normal derivative of u , i.e. the derivative of u in the direction of ν .⁴³

⁴³As an exercise, formulate precisely how $\frac{\partial u}{\partial \nu}$ is defined and what regularity one needs for u to be able to define it.

Lemma:

Let Ω be bounded and sufficiently regular. Let $h: \Omega \rightarrow \mathbb{R}$ be harmonic with a sufficiently regular extension to the closure of Ω . Then, for each $x \in \Omega$ we have

$$h(x) = \int_{\partial\Omega} \frac{\partial G_{\Omega}(y, x)}{\partial \nu_y} h(y) dy. \quad (64)$$

Proof:

$$\begin{aligned} h(x) &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \Delta_y G_{\varepsilon, \Omega}(y, x) h(y) dy \\ &= \int_{\partial\Omega} \frac{\partial G_{\Omega}(y, x)}{\partial \nu_y} h(y) dy + \lim_{\varepsilon \rightarrow 0} \int_{\partial\Omega} \nabla_y G_{\varepsilon, \Omega}(y, x) \nabla h(y) dy. \end{aligned}$$

Now the first integral in the last line coincides with the integral in (64), while the second integral transparently vanishes after one more integration by parts.

The function $\frac{\partial G_{\Omega}(y, x)}{\partial \nu_y}$ is called the Poisson kernel and is often denoted by $P_{\Omega}(x, y)$, or simply $P(x, y)$, if the dependence on Ω does not need to be emphasized. Rewriting the above formula with this notation, we have

$$h(x) = \int_{\partial\Omega} P_{\Omega}(x, y) h(y) dy. \quad (65)$$

It should be mentioned again that the above calculations are formal. We have not proved at this stage that the normal derivatives of G_{Ω} are well-defined, for example. (Such questions have been studied in some detail, of course.) When G_{Ω} is known explicitly, we can verify such details directly. For example, last time we calculated G_{Ω} for the half-space. We can use it to calculate the Poisson kernel $P(x, y)$ for the half-space. We get

$$P(x, y) = \frac{1}{2\pi} \frac{x_3}{|x - y|^3}. \quad (66)$$

When we derived formula (65), we assumed that h was harmonic. We can now look at it slightly differently: for any (integrable) function h at the boundary $\partial\Omega$, the formula gives a harmonic extension of h into Ω . We will look at this in more detail next time.

Lecture 9, 9/27/2010

Let us look in more detail at the Poisson kernel

$$P(x, y) = \frac{1}{2\pi} \frac{x_3}{|x - y|^3} \quad (67)$$

we introduced last time.

Recall that we thought about it as a function on $\mathbb{R}_+^3 \times \partial\mathbb{R}_+^3$, with $x \in \mathbb{R}_+^3$ and $y \in \partial\mathbb{R}_+^3$. We will identify $\partial\mathbb{R}_+^3$ with \mathbb{R}^2 in the obvious way: $y = (y_1, y_2, 0)$ is identified with (y_1, y_2) , which may also be denoted by y .

Let us first look at $P(x, y)$ as a function of x for a fixed y . To understand this function, we can assume $y = 0$, as the function $P(x, y)$ is invariant with respect to shifts along $\partial\mathbb{R}_+^3$: if $b \in \partial\mathbb{R}_+^3$, then $P(x - b, y - b) = P(x, y)$. We have

$$P(x, 0) = \frac{1}{2\pi} \frac{x_3}{|x|^3}. \quad (68)$$

We note that this is exactly $2\frac{\partial}{\partial x_3}K_0$. Sometimes this is called the potential of a dipole (located at the origin and oriented along the x_3 -axis). A dipole potential can be thought of as the potential of two charges of opposite sign which are close to each other. If we locate a unit charge⁴⁴ at $(0, 0, h)$ and an opposite charge at $(0, 0, -h)$, the potential will be

$$x \rightarrow K_0(x_1, x_2, x_3 - h) - K_0(x_1, x_2, x_3 + h). \quad (69)$$

If we want a non-zero limit for this expression as $h \rightarrow 0$, we have to make the magnitude of the charges proportional to $1/h$, otherwise the charges will cancel in the limit $h \rightarrow 0$. So we should look at

$$\frac{K_0(x_1, x_2, x_3 - h) - K_0(x_1, x_2, x_3 + h)}{h}, \quad (70)$$

which in the limit $h \rightarrow 0$ gives the expression $-2\frac{\partial}{\partial x_3}K_0$. (We get the same sign as above if we take charges of the opposite sign.) If you go carefully through our derivation of $P(x, y)$ through Green's function of \mathbb{R}_+^3 , you will see that all of the above is in some sense already present its definition.

A dipole has an orientation - it is given by the line on which the two opposite charges lie. In the case of (68) the orientation is along the x_3 axis, so we can say that the dipole is perpendicular to the boundary $\partial\mathbb{R}_+^3$. For $x \rightarrow P(x, y)$ the dipole is located at y .

As $x \rightarrow \partial\mathbb{R}_+^3$, then $P(x, y)$ approaches 0 except at the point y , where there is a singularity. This of course makes a lot of sense - the potential of a dipole should of course vanish on its plane of symmetry.

Now we look at $P(x, y)$ as a function of y for a fixed x . The function $y \rightarrow P(x, y)$ can be thought of as a function on \mathbb{R}^2 . To see how it looks, we can only consider

⁴⁴Keep in mind our sing conventions.

$x = (0, 0, x_3)$, because of the invariance under shifts parallel to $\partial\mathbb{R}_+^3$. Let us set $x_3 = \varepsilon > 0$ and look at

$$y \rightarrow \phi_\varepsilon(y) = P((0, 0, \varepsilon), y). \quad (71)$$

We have

$$\phi_\varepsilon(y) = \frac{1}{2\pi} \frac{\varepsilon}{(y_1^2 + y_2^2 + \varepsilon^2)^{3/2}} = \frac{1}{\varepsilon^2} \frac{1}{2\pi} \frac{1}{(|\frac{y}{\varepsilon}|^2 + 1)^{3/2}} = \frac{1}{\varepsilon^2} \phi\left(\frac{y}{\varepsilon}\right), \quad (72)$$

where

$$\phi(y) = \frac{1}{2\pi} \frac{1}{(y_1^2 + y_2^2 + 1)^{3/2}}. \quad (73)$$

Note that this is a nice radially symmetric smooth function in \mathbb{R}^2 , which is integrable, as the decay for $y \rightarrow \infty$ is $\sim |y|^{-3}$.

As an exercise, you can check that

$$\int_{\mathbb{R}^2} \phi(y) dy = 1. \quad (74)$$

So we see that ϕ_ε can be thought of as mollifiers generated by ϕ . The mollifiers we have encountered before were compactly supported, but for many purposes that condition is not essential. What is important for us at the moment is that $\phi \geq 0$ and $\int \phi = 1$. With those conditions it is clear that $(\phi_\varepsilon * f)(y) \rightarrow f(y)$ for each $y \in \mathbb{R}^2$ as $\varepsilon \rightarrow 0$ for any bounded continuous function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$. In fact, the convergence will be uniform if f is uniformly continuous, and will be uniform on compact subsets of \mathbb{R}^2 for any bounded continuous f . As an exercise, you should verify these statements in detail. It is easy and it is important to understand how it works.

Let us not consider a bounded, continuous $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ and let us set

$$u(x) = \int_{\mathbb{R}^2} P(x, y)g(y) dy. \quad (75)$$

Clearly u is harmonic in \mathbb{R}_+^3 and we wish to investigate its behavior near $\partial\mathbb{R}_+^3$. Let us write $x = (x', x_3)$, with $x' = (x_1, x_2)$. We will also write x' for $(x_1, x_2, 0)$. We note that, using the notation with ϕ introduced above, we can write

$$u(x) = u(x', x_3) = \int_{\mathbb{R}^2} P(x, y)g(y) dy = \int_{\mathbb{R}^2} P(x-x', y-x')g(y) dy = (\phi_{x_3} * g)(x'), \quad (76)$$

where $*$ denotes convolution in \mathbb{R}^2 . Formula (76) together with the above discussion make it clear that the functions $x' \rightarrow u(x', x_3)$ converge locally uniformly to g as $x_3 \rightarrow 0$. Another way of stating this is:

⁴⁵Hint: use polar coordinates. Also, the integral should be equal to one from the definition of $P(x, y)$. We have $\int_{\partial\Omega} P(x, y) dy = \int_{\partial\Omega} \frac{\partial G_\Omega(x, y)}{\partial \nu_y} dy = \int_{\partial\Omega} \frac{\partial G_{\varepsilon, \Omega}(x, y)}{\partial \nu_y} dy = \int_\Omega \Delta_y G_{\varepsilon, \Omega}(x, y) dy = 1$. This is a complete proof for bounded domains, but for unbounded domains we have to verify that we have the right contribution "from infinity"

Theorem:

With the assumptions above, the function u defined by the Poisson integral (75) is continuous up to the boundary⁴⁶ and $u = g$ at $\partial\mathbb{R}_+^3$.

A suitable form of convergence $\phi_\varepsilon * g \rightarrow g$ as $\varepsilon \rightarrow 0$ holds for more general classes of functions and the theorem can be generalized to those classes, if we replace the continuity up to the boundary of u by suitable other definitions.

We note that if g is bounded, then u is also bounded, with $\inf g \leq u \leq \sup g$.

Is u determined uniquely by the boundary conditions and the equation? If u_1, u_2 are two harmonic functions continuous up to the boundary and $u_1 = u_2$ at the boundary, then $v = u_2 - u_1$ is a harmonic function, continuous up to boundary, which vanishes at the boundary. In general, such function do not have to vanish, as the example $v = x_3$ shows. However, one can show the following

Liouville Theorem for half-space:

Let v be harmonic in \mathbb{R}_+^3 , continuous up to the boundary⁴⁷, with $v = 0$ at the boundary. Assume that $v(x)/|x| \rightarrow 0$ as $x \rightarrow \infty$, $x \in \mathbb{R}_+^3$. Then $v = 0$ in \mathbb{R}_+^3 .

The proof of the theorem can be done as a non-trivial exercise on some of the material which we have covered so far.⁴⁸ I recommend that you try to think about it, to appreciate the theorem. The proof is not straightforward if you have not seen something similar before.

The Poisson formula can be used to prove a number of other results. For example, we note that for $g \in L^\infty(\mathbb{R}^2)$ it gives a bounded harmonic function u in \mathbb{R}_+^3 , with $u(x', x_3) \rightarrow g(x')$ as $x_3 \rightarrow 0$ for almost every $x' \in \mathbb{R}^2$. Does every bounded harmonic function in $\partial\mathbb{R}_+^3$ arise this way? As an exercise, you can try to show that this is indeed the case⁴⁹.

The Poisson extension $g \rightarrow u$ plays an important role in Harmonic Analysis, as it provides a very good way to have all the various approximations $\phi_\varepsilon * g$ encoded in a single and elegant object - the harmonic function u .

⁴⁶We should really say that u can be continuously extended up to the boundary, as the formula transparently defines it only in the interior of \mathbb{R}_+^3 .

⁴⁷This condition can be relaxed, if necessary.

⁴⁸I will omit it at this point, as it might be - with some hints - good material for our first homework assignment.

⁴⁹This needs some knowledge of things from the theory of Lebesgue spaces which are simple once you know them, but may be not so easy to come up with if you work from scratch.

Lecture 10, 9/29/2010

Last time we were looking at boundary value problems in $\Omega = \mathbb{R}_+^3$, and we have formulae which define the solution of

$$\left. \begin{aligned} \Delta u &= f && \text{in } \Omega, \\ u|_{\partial\Omega} &= g. \end{aligned} \right\} \quad (77)$$

To get uniqueness we also need to impose some decay condition for large x . For example, we can search for a solution in the class of bounded u , or, more generally, u with some bounded averages over balls at large distances. The right condition can be determined by remembering what the obstacle to uniqueness is – smooth harmonic functions vanishing at $\partial\Omega$, such as $u(x) = x_3$. Our condition must rule out such functions.

We recall that we can write

$$u(x) = \int_{\partial\Omega} G_\Omega(x, y) f(y) dy + \int_{\partial\Omega} P_\Omega(x, y) g(y) dy. \quad (78)$$

We will now briefly discuss the question in what sense to the functions defined by (78) satisfy the equations (77). The situation with the equation $\Delta u = f$ inside Ω is the same as in \mathbb{R}^3 , which we discussed in lecture 5. If we want u to be twice continuously differentiable (say), the assumption that f be continuous is not sufficient. A good class of functions to work with are for example the Hölder continuous functions. If we work with more general functions f , we can again look at the weak form of the equation:

$$\int_{\Omega} u \Delta \varphi = \int_{\Omega} f \varphi \quad (79)$$

for each smooth, compactly supported $\varphi: \Omega \rightarrow \mathbb{R}$, exactly as in lecture 5.

What about the boundary condition? We can look at the point-wise definitions first. A safe point-wise definition is that u be continuous up to the boundary and that $u|_{\partial\Omega} = g$ point-wise, so clearly g must be continuous, if we wish to use this definition. We have seen in the last lecture that if g is continuous, then the second integral in (78) defines a function u_2 continuous up to the boundary, with $u_2|_{\partial\Omega} = g$. So we need also that the first integral define a function u_1 continuous up to the boundary, with $u_1|_{\partial\Omega} = 0$. The situation here is quite more favorable for the point-wise interpretations, as we do not have to take derivatives. You can prove as an exercise that, for example, when f is bounded and compactly supported, then u_1 satisfies the conditions above. In fact, any compactly supported $f \in L^p$ with $p > 3/2$ is good.⁵⁰ For more singular f the point-wise approach is no longer adequate. Can the notion of the weak solution be generalized to the boundary value problem, including the boundary condition?

To incorporate the boundary conditions into the definition of weak solutions, we have to work with test functions with support extending up to the boundary.

⁵⁰This is still quite elementary - one only needs to use Hölder inequality.

The right weak formulation is obtained again by integrating by parts, this time with non-vanishing boundary terms. If we take a function $\varphi: \Omega \rightarrow \mathbb{R}$ which is smooth up to the boundary and compactly supported in the closure $\bar{\Omega}$ of Ω , we obtain from (77) by formal integration by parts

$$\int_{\Omega} u \Delta \varphi = \int_{\Omega} f \varphi + \int_{\partial \Omega} \left(\frac{\partial \varphi}{\partial \nu} u - \frac{\partial u}{\partial \nu} \varphi \right). \quad (80)$$

If we assume that $u = g$ at $\partial \Omega$ and $\varphi = 0$ at $\partial \Omega$, we have

$$\int_{\Omega} u \Delta \varphi = \int_{\Omega} f \varphi + \int_{\partial \Omega} \frac{\partial \varphi}{\partial \nu} g \quad (81)$$

for each smooth φ compactly supported in the closure of Ω , and vanishing at $\partial \Omega$. It is important that the normal derivative $\frac{\partial \varphi}{\partial \nu}$ at $\partial \Omega$ does not have to vanish. We can now take this as a definition of a weak solution of (77). We start with u, f, g for which the integrals in (81) are well-defined, e. g. u, f locally integrable up to the boundary in Ω ⁵¹ and g locally integrable in $\partial \Omega$.

Definition

With u, f, g as above, we say that u is a (very) weak solution of (77) if (81) is satisfied for each smooth φ compactly supported in the closure of Ω , and vanishing at $\partial \Omega$.

For now we can say just “weak solution” instead “very weak solution”, but later we will introduce another notion of weak solution, and we will have to be more careful with the terminology.

The definition above makes sense for any domain where the normal derivative of smooth functions can be defined at the boundary. In particular, the normal to the boundary has to exist, in some sense (which also can be weak). We will use this definition only for “sufficiently regular” domains.⁵²

One can now check that for $\Omega = \mathbb{R}_+^3$, under fairly general assumptions, formula (78) defines a weak solution, and – importantly – the weak solution is unique.⁵³ We will not go into the proofs right now, I just wanted to illustrate one way in which one can bypass point-wise interpretations of functions also for boundary-value problems. As is usually the case with weak solutions, verifying that some function is a solution becomes easier with this definition, but verifying uniqueness becomes harder.

Let us now turn to determining Green’s functions for some other domains. First, we note that the same idea which works for \mathbb{R}_+^3 works also for perpendicular intersections of half-spaces, such as $\{x_2 > 0, x_3 > 0\}$ or the first octant $\{x_1 > 0, x_2 > 0, x_3 > 0\}$. Let us illustrate the method on $\Omega = \{x_2 > 0, x_3 > 0\}$. Let T_3 be the map $(y_1, y_2, y_3) \rightarrow (y_1, y_2, -y_3)$ and let T_2 be the map $(y_1, y_2, y_3) \rightarrow$

⁵¹This means that u, f are integrable over any compact subset of the closure of Ω .

⁵²Questions about what happens in domains with lower regularity lead very quickly to advanced topics Geometric Measure Theory.

⁵³Of course, one has to impose some conditions which rule out smooth harmonic functions.

$(y_1, -y_2, y_3)$. Given a function $f: \Omega \rightarrow \mathbb{R}$, we can extend it to $\tilde{f}: \mathbb{R}^3 \rightarrow \mathbb{R}$, with $\tilde{f}(T_j x) = -\tilde{f}(x)$, $j = 2, 3$. It is easy to see that the inversion of the laplacian $\tilde{u} = K_0 * \tilde{f}$ in \mathbb{R}^3 has the same symmetries and hence vanishes on $\partial\Omega$. Green's function can be re-constructed from this procedure. It is

$$G_\Omega(x, y) = -\frac{1}{4\pi} \left(\frac{1}{|x-y|} - \frac{1}{|x-y^*|} + \frac{1}{|x-y^{**}|} - \frac{1}{|x-y^{***}|} \right), \quad (82)$$

with $y^* = T_3 y$, $y^{**} = T_2 T_3 y$ and $y^{***} = T_2 y$.

The case of the first octant is similar, with eight terms in the formula.

Let us now look at Green's function of a ball B_R of radius R . At first it is not obvious that we can use the some similar trick as for the domains above, but we can make the following observation: Take $A, B \in (0, \infty)$, $a, b \in \mathbb{R}^3$, consider the potential

$$u(x) = \frac{A}{|x-a|} - \frac{B}{|x-b|}, \quad (83)$$

and consider the set $Z = \{x, u(x) = 0\}$ where it vanishes. Note that for Z is a plane for $A = B$, and for $A \neq B$ the set Z is bounded. We can write the equation for Z as

$$A^2|x-b|^2 - B^2|x-a|^2 = 0. \quad (84)$$

which shows that for $A \neq B$ the set Z must be a sphere. We can calculate the center of the sphere in terms of A, a, B, b . Then we can use the calculation for constructing Green's function for $B_R = \{x, |x| < R\}$ in the following way: for $y \in B_R$ find y^* and c so that the function

$$x \rightarrow -\frac{1}{4\pi|x-y|} + c \frac{1}{4\pi|x-y^*|} \quad (85)$$

vanishes at ∂B_R . We get

$$y^* = \frac{R^2}{|y|^2} y, \quad c = \frac{R}{|y|}. \quad (86)$$

Therefore we conclude that for $\Omega = B_R$ the Green function is

$$G(x, y) = G_\Omega(x, y) = -\frac{1}{4\pi|x-y|} + \frac{R}{|y|} \frac{1}{4\pi|x-y^*|}. \quad (87)$$

Remarks:

1. We know that $G(x, y) = G(y, x)$. The expression we obtained is not transparently symmetric. As an exercise, you can try to write it in a way which makes the symmetry transparent.

2. The map $y \rightarrow \frac{R^2}{|y|^2} y$ which appears in this calculation is quite interesting in itself. It is called the inversion of B_R and it has the remarkable property that

it is conformal: if two curves intersect at a certain angle, their images by the inversion intersect at the same angle. In dimensions $n \geq 3$ there are not many such mappings. In fact, modulo compositions with the obvious conformal maps, the inversion is the only “non-obvious” one. You can also check that it maps balls to balls or half-spaces.

The local behavior of G near the boundary is similar to the case $\Omega = \mathbb{R}_+^3$, as for y close to ∂B_R the inversion of y behaves almost as the reflection by the tangent plane to the sphere at $Ry/|y|$. In fact, the derivative of the inversion at $\bar{y} \in \partial B_R$ is exactly the reflection by the plane tangent to the sphere at \bar{y} - this fact is useful to keep in mind when calculating the Poisson kernel by taking the normal derivative of G at the boundary.

We can now calculate the Poisson kernel. For $x \in B_R$ and $y \in \partial B_R$ we have

$$P(x, y) = \frac{\partial G(x, y)}{\partial \nu_y} = \frac{y_i}{R} \frac{\partial}{\partial y_i} G(x, y). \quad (88)$$

Carrying out the differentiation might first look as an unpleasant calculation, but it is simple if we use that $\frac{y_i}{R} \frac{\partial}{\partial y_i}$ is just $\frac{\partial}{\partial r}$ with $r = |y|$ and that at the boundary of B_R we have $\frac{\partial y^*}{\partial r} = -y$.

The end result is

$$P(x, y) = \frac{R^2 - |x|^2}{4\pi R|x - y|^3}. \quad (89)$$

We can now analyze this function in a similar way as we analyzed the Poisson kernel of the half-space. First we fix $y \in \partial B_R$ and note that $x \rightarrow P(x, y)$ is a harmonic function in B_R which vanishes at ∂B_R with the exception of the point y , where it has a singularity. The field generated by the singularity at y has a dipole component and a point-charge component. This is easily seen by moving the beginning of the coordinates to y and writing $x = y + \tilde{x}$. Then we have

$$P(x, y) = \frac{R^2 - |y + \tilde{x}|^2}{4\pi R|\tilde{x}|^3} = -\frac{2y\tilde{x}}{4\pi R|\tilde{x}|^3} - \frac{1}{4\pi R|\tilde{x}|}, \quad (90)$$

and we can see the dipole potential $-\frac{2y\tilde{x}}{4\pi R|\tilde{x}|^3}$ and the point-charge potential $-\frac{1}{4\pi R|\tilde{x}|}$. The dipole potential is independent of the radius of the ball, and the point-charge potential approaches 0 as $R \rightarrow \infty$, so that in the limit $R \rightarrow \infty$ we get, in the \tilde{x} coordinates, the Poisson kernel of the half-space, as we should.

We now fix $x \in B_R$ and look at the function $y \rightarrow P(x, y)$. This is a function on the sphere. When $x \neq 0$, it attains its maximum at the point $x' = x/R$. We can rotate the coordinates so that $x = (0, 0, x_3)$ and $x' = (0, 0, R)$. Similarly to the case of the half-space, we can describe the behavior of $y \rightarrow P(x, y)$ in terms of the parameter $\varepsilon = R - x_3$ and the function

$$\phi_\varepsilon(y) = P((0, 0, R - \varepsilon), y). \quad (91)$$

We have $\int_{\partial\Omega} \phi_\varepsilon = 1$. This time we do not have to check it (unless we are worried about a mistake in the calculations), because we are in a compact domain

and the definition of the Poisson kernel implies this property by integration by parts, as we have seen in the previous lecture. As $\varepsilon \rightarrow 0$, the functions ϕ_ε concentrate close to x' , and converge uniformly to zero on any compact subset of the complement of $\{x'\}$.

As in the case of $\Omega = \mathbb{R}_+^3$, we can see that for $\Omega = B_R$ and any continuous $g: \partial\Omega \rightarrow \mathbb{R}$, the potential

$$u(x) = \int_{\partial\Omega} P(x, y) g(y) dy \tag{92}$$

defines a harmonic function in Ω which is continuous up to the boundary of Ω (can be continuously extended to the closure of Ω) and agrees with g on $\partial\Omega$.

Lecture 11, 10/1/2010

So far we have only looked at our objects in dimension $n = 3$. We now check what the situation is in other dimensions.

Newton's law in n dimensions for the attraction of two point-masses m and M separated by distance r is

$$\text{force} \sim \frac{mM}{r^{n-1}}. \quad (93)$$

Repeating our considerations from lecture 1, we can try to define the potential u of mass M at $a \in \mathbb{R}^n$ by the requirement that $-u(x)$ is the work needed to "free" a particle of unit mass from the gravitational field produced by M , and move it from position x to ∞ . In dimensions $n \geq 3$ it is easy to calculate that

$$u(x) = \text{const.} \cdot \frac{-M}{|x - a|^{n-2}}, \quad (94)$$

and one can say without any exaggeration that one can repeat everything we did for $n = 3$ without any difficulties.⁵⁴

One can also approach things from the other end: postulate that the gravitational potential satisfies $\Delta u = \text{const.}$ and that the potential of a point-mass is radial, and calculate the force from there.⁵⁵

There is one calculation one has to do - we need to evaluate the right constant c_n so that the convolution $G * f$ with $G(x) = c_n/|x|^{n-2}$ inverts the laplacian, i. e.

$$\Delta(G * f) = f. \quad (95)$$

For this we only need to calculate $\int \Delta K(x) dx$ where K is smooth and $K = G$ in $\mathbb{R}^n \setminus B_1$. We have

$$\int_{B_1} \Delta K(x) dx = \int_{\partial B_1} \frac{\partial K}{\partial \nu} = \int_{\partial B_1} -\frac{\partial}{\partial r} \frac{c_n}{r^{n-2}} = c_n(n-2)|S^{n-1}|, \quad (96)$$

where $|S^{n-1}|$ denotes the volume of the $(n-1)$ -dimensional sphere

⁵⁴While from the point of the Laplace equation the case $n \geq 4$ is very similar to $n = 3$, the motion of planets would be very different in these dimensions. One can calculate easily the circular orbits of a planet around a star, and these are similar as for $n = 3$. However, when we look at the stability of these circular orbits for $n \geq 4$, we get a surprise. A typical small perturbation of the motion (coming from an interaction with another planet, say) will result in a catastrophic change of the orbit. It is a good exercise (although not in PDEs) to verify this classical observation.

⁵⁵It may look odd at first that in 1d the force is independent of distance. However, the following example shows that it is natural: in the 3d space consider a uniform "surface distribution" of mass in the plane $\{x_3 = 0\}$. As an exercise, you can show by direct integration that the force with which a point-mass is attracted to the plane is independent of the distance from the plane. In fact, the integral which comes up in this calculation is exactly the same as the one we calculated when proving that $\int_{\partial\Omega} P(x, y) dy = 1$ for the Poisson kernel of the half-space in 3d. One can see the independence of the force on x_3 from "dimensional analysis", without having to evaluate the integral.

$$S^{n-1} = \{x \in \mathbb{R}^n, |x| = 1\}.$$

We see that

$$G(x) = -\frac{1}{(n-2)|S^{n-1}||x|^{n-2}}. \quad (97)$$

Formula (23) for general $n \geq 3$ is

$$K_\varepsilon(x) = \frac{1}{\varepsilon^{n-2}} K\left(\frac{x}{\varepsilon}\right). \quad (98)$$

The cases $n = 2$ and $n = 1$ are somewhat different. In those cases the integral expressing the energy needed to move a point-mass from a gravitational field of a star to spatial infinity diverges: when $n \leq 2$ we have

$$\int_{r_0}^{\infty} \frac{dr}{r^{n-1}} = +\infty. \quad (99)$$

Therefore the gravitational potential cannot be bounded at ∞ .⁵⁶

The fundamental solution G in 2d is usually taken as

$$G(x) = \frac{1}{2\pi} \log|x|. \quad (100)$$

The formula (98) can be modified to

$$K_\varepsilon(x) = K\left(\frac{x}{\varepsilon}\right) + \log \varepsilon \quad (101)$$

where K is a smooth extension of $\frac{1}{2\pi} \log|x|$ from $\mathbb{R}^2 \setminus B_1$ to \mathbb{R}^2 .

For $n = 1$ we can again use (98), with K a smooth extension of $|x|/2$ from $\mathbb{R} \setminus (-1, -1)$ to \mathbb{R} .

The important thing to remember in dimension $n = 2$ is that the equation

$$\begin{aligned} \Delta u &= f && \text{in } \mathbb{R}^2, \\ u(x) &\rightarrow 0 && \text{as } |x| \rightarrow \infty, \end{aligned} \quad (102)$$

is not always solvable even when f is smooth and compactly supported. The necessary and sufficient condition for the solvability in this case is that $\int_{\mathbb{R}^2} f = 0$.⁵⁷ One way to see this is to use the following argument:

- a) The solution of the problem is unique in the class of functions with sub-linear growth at infinity, by the Liouville theorem.
- b) The potential $u = G * f$ with the kernel (100) is such a solution.

⁵⁶This can be interpreted as follows: in the 3d world, if one shoots an object straight up from a planet at a sufficiently high speed, it will escape the gravity of the planet, and will not return back to the surface. In the 2d world (or 1d world), the object will always fall back - it can never escape the gravity of the planet.

⁵⁷Exercise: what is the necessary and sufficient condition for $n = 1$ (when f is compactly supported)?

c) $u \rightarrow 0$ as $|x| \rightarrow \infty$ if and only if $\int_{\mathbb{R}^2} f = 0$.

The following example gives a good illustration of the situation. The boundary-value problem

$$\begin{aligned} \Delta u &= f & \text{in } B_R, \\ u &= 0 & \text{at } \partial B_R, \end{aligned} \tag{103}$$

has a unique solution for each $R > 0$. It can be given for example by the Green's function (which one calculates in the same way as in \mathbb{R}^3).

$$u(x) = \int_{B_R} G_{B_R}(x, y) f(y) dy. \tag{104}$$

If f is radially symmetric, u will also be radially symmetric, and outside of the support of f it has to be of the form

$$u(x) = A \log |x| + B, \tag{105}$$

with $A = (\int_{B_R} f)/2\pi$. Hence, denoting $c = \int_{B_R} f$ and using the boundary condition at ∂B_R , we see that in the radial case we have

$$u(x) = \frac{c}{2\pi} \log\left(\frac{|x|}{R}\right) \quad \text{outside the support of } f. \tag{106}$$

We now see that for a fixed x and $R \rightarrow \infty$ the value $u(x)$ converges to $-\infty$ or $+\infty$, depending of the sign of c , as $R \rightarrow \infty$.

The calculations of Green's functions $G(x, y)$ and the Poisson kernels $P(x, y)$ we did for $n = 3$ can be repeated for general dimension n without any difficulty. For example, for the Poisson kernel we get:

$$P(x, y) = \frac{2}{|S^{n-1}|} \frac{x_n}{|x - y|^n} \quad \text{for the half-space } \{x_n > 0\} \tag{107}$$

and

$$P(x, y) = \frac{R^2 - |x|^2}{|S^{n-1}| R |x - y|^n} \quad \text{for the ball } \{|x| < R\}. \tag{108}$$

Lecture 12, 10/3/2010

Today we will have a look at another explicit solutions of the laplace equation - harmonic polynomials. A polynomial p in \mathbb{R}^n is harmonic if $\Delta p = 0$. Our first observation is that, locally, any harmonic function is “build up” from harmonic polynomials. Let $u: B_R \rightarrow \mathbb{R}$ be harmonic, and consider its Taylor expansion at the origin. We will write the expansion as follows:

$$u(x) = u_0 + u_1(x) + u_2(x) + \dots, \quad (109)$$

where u_0 is a constant (= a zero-homogeneous polynomial), u_1 is a 1-homogeneous polynomial, u_2 is a two-homogeneous polynomial, etc. The radius of convergence of the series is R . This can be seen for example by representing u by the Poisson kernel.⁵⁸

We claim that $\Delta u = 0$ implies that $\Delta u_j = 0$ for each $j = 0, 1, 2, \dots$. Let u_k be the first term of (189) with $\Delta u_k \neq 0$. Note that $k \geq 2$. As the series is convergent, we can differentiate it term-by-term. We have $\Delta u_{k+1} + \Delta u_{k+2} + \dots = O(|x|^{k-1})$. On the other hand, Δu_k is a non-zero $(k-2)$ -homogeneous polynomial and therefore cannot be $O(|x|^{k-1})$. We see that all Δu_j have to vanish.

Let us denote by \mathcal{P}_m the space of polynomials of degree $\leq m$ in \mathbb{R}^n . We could also write \mathcal{P}_m^n if we wanted to emphasize the dependence on the dimension, but we will think of n as a fixed parameter, so there will be no confusion if we will not write this extra index. We also denote by \mathcal{H}_m the space of all harmonic polynomials of degree $\leq m$.

We denote by $\dot{\mathcal{P}}_m$ the space of m -homogeneous polynomials, and by $\dot{\mathcal{H}}_m$ the space of all harmonic m -homogeneous polynomials. We can write

$$\mathcal{P}_m = \dot{\mathcal{P}}_m \oplus \dot{\mathcal{P}}_{m-1} \oplus \dots \oplus \dot{\mathcal{P}}_0, \quad (110)$$

and

$$\mathcal{H}_m = \dot{\mathcal{H}}_m \oplus \dot{\mathcal{H}}_{m-1} \oplus \dots \oplus \dot{\mathcal{H}}_0. \quad (111)$$

We recall that

$$\dim \dot{\mathcal{P}}_m = \frac{(m+n-1)!}{m!(n-1)!}. \quad (112)$$

Clearly Δ maps $\dot{\mathcal{P}}_m$ into $\dot{\mathcal{P}}_{m-2}$.

Lemma.

Δ maps $\dot{\mathcal{P}}_m$ onto $\dot{\mathcal{P}}_{m-2}$ for each m .⁵⁹

Proof: This is a purely algebraic result, so we should not need any Analysis for the proof. Let us consider the mapping $A: \dot{\mathcal{P}}_m \rightarrow \dot{\mathcal{P}}_m$ defined by $A(p) = \Delta(|x|^2 p)$. As for $p \in \dot{\mathcal{P}}_m$ we have $|x|^2 p \in \dot{\mathcal{P}}_{m+2}$, the lemma will be proved if

⁵⁸For any $R_1 < R$ write $u(x) = \int_{\partial B_{R_1}} P_{B_{R_1}}(x, y) u(y) dy$, and then write $x \rightarrow P_{B_{R_1}}(x, y)$ as a power series. As an exercise, you can verify the details.

⁵⁹We can define $\dot{\mathcal{P}}_m = \{0\}$ for negative integers m .

we show that A has trivial kernel. Let us assume $\Delta(|x|^2 p) = 0$ for some $p \neq 0$. Let us write $p = |x|^{2l} q$, where q is not divisible by $|x|^2$. We have $\Delta(|x|^{2k} q) =$, with $k = l + 1$. Using $\Delta(rq) = (\Delta r)q + 2\nabla r \nabla q + r\Delta q$ (by Leibnitz rule) with $r = |y|^{2k}$ and $x\nabla p = mp$ (Euler's formula), we see that $cq = |x|^2 \Delta q$ for some constant $c \neq 0$, a contradiction.⁶⁰

Corollary:

$$\dim \dot{\mathcal{H}}_m = \dim \dot{\mathcal{P}}_m - \dim \dot{\mathcal{P}}_{m-2}. \quad (113)$$

The following table gives the dimensions of $\dim m$ for low n .⁶¹

$$\begin{array}{ll} n = 2 & \dim \dot{\mathcal{H}}_m = 2 \\ n = 3 & \dim \dot{\mathcal{H}}_m = 2m + 1 \\ n = 4 & \dim \dot{\mathcal{H}}_m = (m + 1)^2 \end{array} \quad (114)$$

For general n , the dimension of $\dot{\mathcal{P}}_m$ is of order $\sim m^{n-1}$ and the dimension of $\dot{\mathcal{H}}_m$ is of order $\sim m^{n-2}$.

We will now describe $\dot{\mathcal{H}}_m$ in dimensions $n = 2$ and $n = 3$.

Dimension $n = 2$.

Let $z = x_1 + ix_2$. Then a basis of $\dot{\mathcal{H}}_m$ is formed by

$$\operatorname{Re} z^m \quad \text{and} \quad \operatorname{Im} z^m. \quad (115)$$

If we consider polynomials with complex coefficients, we can take z^m and \bar{z}^m as a basis. We will discuss some of the special features of harmonic functions in dimension $n = 2$ in the next lecture.

You can also derive these polynomials by direct calculation. Let us consider polar coordinates

$$\begin{aligned} x_1 &= r \cos \varphi, \\ x_2 &= r \sin \varphi. \end{aligned}$$

An m -homogeneous polynomial is of the form $r^m f(\varphi)$. As an exercise, you can write the Laplacian in the polar coordinates⁶² and check that the equation $\Delta(r^m f(\varphi) = 0)$ reduces to $f'' = -m^2 f$, which can be easily solved, giving the polynomials above.

Dimension $n = 3$. In principle, in dimension $n = 3$ we could again do a direct calculation: convert the Laplacian to the polar coordinates (r, ϕ, φ) , write the unknown polynomial as $r^m f(\phi, \varphi)$ and get an equation for f . This can be done, but if one tries to do it purely "by calculation" without any guidance from

⁶⁰There are many other ways to prove the lemma.

⁶¹The numbers reflect some properties of the orthogonal groups in the corresponding dimensions. We will not go into the details.

⁶²You should get $\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{r^2\partial\varphi^2}$

geometry, it is not easy. Also, it is not how the polynomials were discovered historically.

We recall that the polar coordinates are given by

$$\begin{aligned}x_1 &= r \sin \phi \cos \varphi, \\x_2 &= r \sin \phi \sin \varphi, \\x_3 &= r \cos \phi.\end{aligned}\tag{116}$$

Let us set $e = (0, 0, 1)$ and consider the harmonic function

$$u(x) = \frac{1}{|x - e|} = \frac{1}{\sqrt{1 - 2r \cos \phi + r^2}}.\tag{117}$$

We now expand this function as in (189). We note that each term u_m in that expansion is invariant under rotations about the x_3 -axis, and therefore $u_m(x) = r^m P_m(\cos \phi)$, where P_m is a polynomial of order m .⁶³ Hence

$$\frac{1}{\sqrt{1 - 2r \cos \phi + r^2}} = 1 + rP_1(\cos \phi) + r^2P_2(\cos \phi) + \dots\tag{118}$$

The polynomials $P_0 = 1, P_1, P_2, \dots$ are called Legendre polynomials (and they were discovered by Legendre exactly from expansion (118)). They have many remarkable features, and a lot is known about them. We will not go in the direction of studying them in more detail at this time, but we mention at least the following:

- The polynomials are orthogonal to each other in $L^2(-1, 1)$. Therefore, they can be obtained (up to a multiple) by the usual orthogonalization of the sequence $1, t, t^2, \dots$.
- $(m + 1)P_{m+1}(t) - (2m + 1)P_m(t) + mP_{m-1}(t) = 0$,
- $[(1 - t^2)P_m']' = -m(m + 1)P_m$, where $'$ denotes the derivative $\frac{d}{dt}$.

The basis of \mathcal{H}_m in dimension $n = 3$ can now be given as follows:

$$\begin{aligned}&r^m P_m(\cos \phi), \\&r^m \sin \phi P_m'(\cos \phi) e^{\pm i\varphi}, \\&r^m \sin^2 \phi P_m''(\cos \phi) e^{\pm 2i\varphi}, \\&\dots, \\&r^m \sin^l \phi P_m^{(l)}(\cos \phi) e^{\pm li\varphi}, \\&\dots, \\&r^m \sin^m \phi P_m^{(m)} e^{\pm mi\varphi}.\end{aligned}\tag{119}$$

We will explain next time why this works.

⁶³Strictly speaking, one needs to justify that the order is exactly m , and not $< m$. To see this, note that if the order of P_m was $< m$, then $u_m(x)$ would be divisible by r^2 , and we have seen above that this cannot happen for a harmonic function.

Homework assignment 1, due October 20.

Give a proof of the Liouville Theorem for half-space:

Let v be harmonic in \mathbb{R}_+^3 , continuous up to the boundary, with $v = 0$ at the boundary. Assume that $v(x)/|x| \rightarrow 0$ as $x \rightarrow \infty$, $x \in \mathbb{R}_+^3$. Then $v = 0$ in \mathbb{R}_+^3 .

Hints:

- a) Prove the theorem first in the case when the second derivatives of v are continuous up to the boundary. One way to do it is to extend v suitably to all space so that it is harmonic. (When you check that the extended function is harmonic, be careful what happens near the plane $x_3 = 0$ - that is the key.)
- b) Let $\phi: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a mollifier, $\phi_\varepsilon = \varepsilon^{-2}\phi(x/\varepsilon)$, and $v_\varepsilon(x_1, x_2, x_3) = \int_{\mathbb{R}^2} v(x_1 - y_1, x_2 - y_2, x_3)\phi_\varepsilon(y_1, y_2) dy_1 dy_2$. Check that it is enough to prove that $v_\varepsilon = 0$ for each $\varepsilon > 0$.
- c) Check that on any compact subset of the closure of the half-space, v_ε has bounded derivatives up of any order in the x_1 and x_2 directions.
- d) Use the equation $\Delta v_\varepsilon = 0$ to check that all second derivatives of v_ε are continuous up to the boundary.
- e) Use a) to show that $v_\varepsilon = 0$.

Remarks:

1. There are other ways to do the proof, not using the extension to the whole space. The theorem is essentially equivalent to local regularity of harmonic functions near the boundary. The disadvantage of the proof above is that it is not easy to modify it once we replace the half-space by a domain with a boundary which is not flat.
2. As an optional part of the homework, you can prove the theorem in the context of the very weak solutions we discussed in lecture 10: instead of v being continuous up to the boundary, assume that v is bounded in \mathbb{R}_+^3 and satisfies the boundary condition $v|_{\partial\mathbb{R}_+^3} = 0$ in the very weak sense introduced in lecture 10.

We will first return briefly to formulae (119). One way to derive that is to express the laplacian in polar coordinates, and work with suitable the ODEs. This is not hard in principle, but the calculations are somewhat lengthy and perhaps not very illuminating if one does not know what is going on “behind the scenes”. There is some general principles underlying those calculations, and we are not yet familiar with them. It might be interesting to go in the direction of this topic, but maybe it would not be the best option at the moment. However, there is a nice way to derive (119) based on the symmetries of the equation. That is a calculation which we can do with a reasonable level of understanding, and it shows some typical features of using symmetries of equations to generate new solutions from those which we already have. Such topics by themselves are a large area of research, and we will see only a very tiny and elementary sample. Nevertheless, I think that the calculation is illuminating, and gives a good opportunity of seeing symmetries in action.

The group of rotations of \mathbb{R}^n acts on functions in \mathbb{R}^n is the following way: given a rotation R (which we identify of an orthogonal matrix) and a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$, the rotated function Rf is usually defined by

$$Rf(x) = f(R^{-1}x). \quad (120)$$

The definition is chosen in such a way that the action $f \rightarrow Rf$ rotates the graph of the function in the natural way: $(x, f(x))$ goes to $(Rx, f(x))$.

An important property of the laplacian is that it commutes with the rotations:

$$\Delta(Rf) = R(\Delta f). \quad (121)$$

This is an elementary calculation.⁶⁴ The space of $\dot{\mathcal{P}}_m$ of m -homogeneous polynomials is also obviously invariant under rotations, and hence $\dot{\mathcal{H}}_m$ is also invariant under rotations.

We now take the polynomial

$$h(x) = r^m P_m(\cos \theta). \quad (122)$$

This polynomial in $\dot{\mathcal{H}}_m$ is obviously invariant under rotations invariant about the x_3 -axis, and it is – up to a multiple – the only such polynomial in $\dot{\mathcal{H}}_m$.⁶⁵

One can now generate other polynomials in $\dot{\mathcal{H}}_m$ from h by rotations. However, it may not be immediately clear how to find enough rotations for which Rh can be easily written down. It is much easier to work with “infinitesimal rotations”.

⁶⁴It is equivalent to the fact that trace is an invariant of a symmetric matrix.

⁶⁵The uniqueness can be shown by using the ODE satisfied by the polynomial, or just by using the fact that the monomials $x_3^m, x_3^{m-2}(x_1^2 + x_2^2), x_3^{m-4}(x_1^2 + x_2^2)^2, \dots$ form a basis of the m -homogeneous polynomials invariant under the rotations about the x_3 axis. A linear combination of such terms can only be harmonic if it contains x_3^m with a non-zero coefficient, as one can easily check directly from the equation.

These are defined as follows: Let $R_1(\alpha)$ be rotation by α about the x_1 axis, given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix}. \quad (123)$$

The function

$$\frac{d}{d\alpha} \Big|_{\alpha=0} R_1(\alpha)h \quad (124)$$

is clearly again in $\dot{\mathcal{H}}_m$.

Carrying out the differentiation, we get

$$\frac{d}{d\alpha} \Big|_{\alpha=0} R_1(\alpha)h = L_1h, \quad (125)$$

where

$$L_1 = x_3\partial_2 - x_2\partial_3. \quad (126)$$

Note that L_1 can be identified with a vector field in \mathbb{R}^3 , and the vector field is tangent to the spheres centered at the origin. We know we must have

$$\Delta L_1 = L_1\Delta \quad (127)$$

by the construction of L_1 , and it is also easy to check it directly.

In a similar way, using the rotations about the x_2 -axis we define

$$L_2 = -x_3\partial_2 + x_2\partial_3. \quad (128)$$

We can now use L_1 and L_2 to generate a basis of $\dot{\mathcal{H}}_m$ from the single polynomial h . The formulae come out particularly simple if we work with $L = L_1 + iL_2$, and $z = x_1 + ix_2$. Note that $L|x| = 0$, and therefore it is enough to work with the restriction of the polynomials to the sphere $\{x, |x| = 1\}$. On the sphere we have

$$LP_m = (iz)P'_m, L^2P_m = (iz)^2P''_m, \dots, L^mP_m = (iz)^mP_m^{(m)}. \quad (129)$$

We obtain additional polynomials by taking conjugation:

$$\overline{(iz)^k} P_m^{(k)}, \quad k = 1, 2, \dots, m. \quad (130)$$

We have obtained $2m + 1$ polynomials in $\dot{\mathcal{H}}_m$. It is easy to see that they are linearly independent, as they behave differently under rotations about the x_3 -axis. We have seen in the last lecture that for $n = 3$ we have $\dim \dot{\mathcal{H}}_m = 2m + 1$, and hence we have found a basis of $\dot{\mathcal{H}}_m$.

We know that the restriction of any polynomial to the sphere ∂B_R can be extended to a harmonic function in B_R . The following argument shows that the harmonic function is actually a polynomial of the same or lower degree.

Lemma.⁶⁶

The map $p \rightarrow \Delta[(|x|^2 - R^2)p]$ maps \mathcal{P}_m onto \mathcal{P}_m .

Proof: A polynomial in the kernel is harmonic and vanishes on ∂B_R . Hence it has to vanish identically by the maximum principle.⁶⁷

The lemma shows that we have the (algebraical) decomposition

$$\mathcal{P}_m = \mathcal{H}_m \oplus (|x|^2 - R^2)\mathcal{P}_{m-2}. \quad (131)$$

One way to interpret the statement is that every polynomial on the boundary of the ball can be extended to a harmonic polynomial in the ball.

Harmonic functions and holomorphic functions in dimension $n = 2$.

There is close connection in dimension $n = 2$ between harmonic functions and holomorphic functions. While talking about specific harmonic functions, we should recall some of the details of this connection.

Let us for now use the classical complex analysis notation. The coordinates in the plane are x, y , we denote $z = x + iy$ and $\bar{z} = x - iy$. We define

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \quad (132)$$

Sometimes we will write

$$\partial = \frac{\partial}{\partial z}, \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}}. \quad (133)$$

One way to think about these operators is the following: we can identify the complex numbers \mathbb{C} with \mathbb{R}^2 and vice versa. Then any linear map from \mathbb{R}^2 to \mathbb{R}^2 can be considered as a map from \mathbb{C} to \mathbb{C} which is linear over \mathbb{R} . Such a linear map can be written as

$$l(z) = az + b\bar{z}, \quad (134)$$

where a, b are complex numbers. Then we have

$$\frac{\partial}{\partial z} l = a, \quad \frac{\partial}{\partial \bar{z}} l = b. \quad (135)$$

⁶⁶See also the book "Lectures on Elliptic and Parabolic Equations in Hölder Spaces" by N. V. Krylov, p. 24.

⁶⁷Although the result is purely algebraic, the proof is not algebraic - it uses the maximum principle. As an exercise, you can try to find a purely algebraic proof. It should work also when we replace $-R^2$ by R^2 , in which case the above proof does not work without further arguments. We remark that the lemma can also be used to determine the dimensions of \mathcal{H}_m , as an alternative to the lemma we used in the last lecture.

Let $a = a_1 + ia_2$, $b = b_1 + ib_2$. In terms of linear maps $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ and their matrices, the map l can be identified with the matrix

$$\begin{pmatrix} a_1 + b_1 & -a_2 + b_2 \\ a_2 + b_2 & a_1 - b_1 \end{pmatrix}. \quad (136)$$

Then $\frac{\partial}{\partial z}l$ can be identified with the “ a -part” of this matrix (which is the “holomorphic part”) and $\frac{\partial}{\partial \bar{z}}l$ can be identified with the “ b -part” of this matrix (which is the “anti-holomorphic part”). We can think of $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ as projections of the full gradient onto certain two-dimensional subspaces of the space of 2×2 matrices. The subspaces are orthogonal to each other (if we use the standard scalar product in the space of matrices).

We will consider the equation

$$\frac{\partial}{\partial \bar{z}}u = f \quad (137)$$

in \mathbb{R}^2 . We claim that the equation can be solved (under some natural assumptions) as follows:

$$u = G * f \quad (138)$$

where

$$G(z) = \frac{1}{\pi z}. \quad (139)$$

We also have

$$u = G * \left(\frac{\partial}{\partial \bar{z}}u \right) \quad (140)$$

for all compactly supported and sufficiently regular functions.

For the proofs we can follow more or less line-by-line the proofs we did for the laplacian. Let K be any smooth extension of G from $\mathbb{R}^2 \setminus B_1$ to \mathbb{R}^2 . Set

$$K_\varepsilon(z) = \frac{1}{\varepsilon} K\left(\frac{z}{\varepsilon}\right). \quad (141)$$

The key point is that

$$\int_{\mathbb{R}^2} \frac{\partial}{\partial \bar{z}} K = 1, \quad (142)$$

and

$$\frac{\partial}{\partial \bar{z}} K_\varepsilon(z) = \frac{1}{\varepsilon^2} K\left(\frac{z}{\varepsilon}\right) \quad (143)$$

can be considered as a mollifier. We see that the situation is the same as in our proofs of inverting the laplacian.⁶⁸

⁶⁸Later we will see that all this generalizes to much more general situations.

Lecture 14, 10/8/2010

We recall that

$$\partial = \frac{\partial}{\partial z}, \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}}. \quad (144)$$

As is customary, a function satisfying $\bar{\partial}u = 0$ will be called holomorphic. Any holomorphic function is harmonic, as $\Delta = 4\partial\bar{\partial}$.

Let us consider a bounded domain Ω with smooth boundary $\Gamma = \partial\Omega$, and a holomorphic function u defined in a neighborhood of the closure of Ω . Let φ be a cut-off function which is 1 in Ω and vanishes outside of a small neighborhood of a closure of Ω . For our purposes here we can take

$$\varphi_\varepsilon(x) = \max\left(1 - \frac{\text{dist}(x, \Omega)}{\varepsilon}, 0\right). \quad (145)$$

We have

$$u\varphi_\varepsilon = G * (\bar{\partial}(u\varphi_\varepsilon)) = G * (u\bar{\partial}\varphi_\varepsilon). \quad (146)$$

Strictly speaking we should first approximate φ_ε by smooth functions and pass to the limit to obtain (146) - this step is left as an exercise. When $\varepsilon \rightarrow 0$ we have

$$\int G(x-y)u(y)\varphi_\varepsilon(y) dy \rightarrow \int_\Gamma G(x-y)u(y)(n_1(y) + in_2(y))/2 dy. \quad (147)$$

Here $n = (n_1, n_2) \sim n_1 + in_2$ denotes the vector field given by the normal to Γ . As Γ is smooth, the vector field n is well-defined in a small neighborhood of Γ . We use the notation $y = (y_1, y_2) \sim y_1 + iy_2, x \sim x_1 + ix_2$, etc. The tangent to Γ can be taken as $\tau(y) = in(y)$. We obtain from (146) and (147) as $\varepsilon \rightarrow 0$

$$u(x) = -\frac{1}{2} \int_\Gamma G(x-y)u(y)n(y) dy = \frac{1}{2\pi i} \int_\Gamma \frac{u(y)}{y-x} \tau(y) dy, \quad (148)$$

which is the Cauchy formula.⁶⁹

When $\Omega = R_+^2$, one gets for $\text{Re } x > 0$ - assuming the formula also works in this case⁷⁰ -

$$u(x) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{u(y)}{y-x} dy. \quad (149)$$

⁶⁹Our notation is that dy denotes the 1-dimensional measure on Γ . It might be more accurate to write $|dy|$ for this measure, so that one could write $\tau(y)|dy| = dy$, and $\int_\Gamma \dots dy$ would denote the usual curve integral.

⁷⁰As we derived the formula only for bounded domains, one should check that the "contributions from ∞ " vanish, as implicitly assumed by applying the formula. A sufficient condition is that $u(x) = O(|x|^{-\alpha})$ for some $\alpha > 0$ as $x \rightarrow \infty$, for example.

It is interesting to compare this formula with (65), the Poisson formula for the laplacian. Both formulae were derived as representation formulae of functions satisfying certain equation. However, we found that (65) is in some sense even stronger: we can start with any continuous function g on the boundary, without knowing it comes from a harmonic function, plug it into the formula, and we get a continuous extension of g to a harmonic function.⁷¹ It is interesting to see what happens in the case of the Cauchy formula (149) if we start with a sufficiently regular $g: \mathbb{R} \rightarrow \mathbb{C}$ with some decay at ∞ ⁷² and set

$$u(x) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{g(y)}{y-x} dy. \quad (150)$$

Clearly u will be holomorphic in $\mathbb{R}^2 \setminus \Gamma$, where $\Gamma = \{x, x_2 = 0\}$. We would like to investigate the behavior of u near Γ . Let us assume that g is real-valued. (The general case can be reduced to this case by writing $g = g_1 + ig_2$ with real-valued g_1 and g_2 .)

Let us first look at the kernel

$$C(x, y) = \frac{1}{2\pi i} \frac{1}{x-y} \quad y \in \Gamma, x \neq y. \quad (151)$$

As a function of x this is a holomorphic function with a pole at y . When $x \in \Gamma, x \neq y$, then $\operatorname{Re} C(x, y) = 0$, but the imaginary part does not vanish – we have $\operatorname{Im} C(x, y) = (2\pi(y-x))^{-1}$.

To check the behavior of C as a function of y for a fixed x , it is enough to consider the case $x = i\varepsilon$. The condition $y \in \Gamma$ simply means that y is real.

$$C(i\varepsilon, y) = \frac{1}{2\pi} \frac{\varepsilon}{y^2 + \varepsilon^2} + \frac{1}{2\pi i} \frac{y}{y^2 + \varepsilon^2}. \quad (152)$$

We know what the first term of the left-hand side is: it is a multiple (by $1/2$) of the Poisson kernel (107) in dimension 2. Hence (keeping in mind that g is real-valued) we can conclude that

$$\text{in } \mathbb{R}_+^2 \text{ the function } \operatorname{Re} u \text{ is the harmonic extension of } g/2. \quad (153)$$

Similarly,

$$\text{in } \mathbb{R}_-^2 = \{x, x_2 < 0\} \text{ the function } \operatorname{Re} u \text{ is the harmonic extension of } -g/2. \quad (154)$$

Let us now look at

$$\operatorname{Im} u(i\varepsilon) = \int_{\mathbb{R}} \frac{1}{2\pi} \frac{-y}{y^2 + \varepsilon^2} g(y) dy. \quad (155)$$

⁷¹Strictly speaking, so far we only verified it when the domain is a half-space or a ball.

⁷²You can assume g is compactly supported at this stage.

As $\varepsilon \rightarrow 0$, formally we should get

$$\operatorname{Im} u(0) = \int_{\mathbb{R}} \frac{1}{2\pi} \frac{1}{-y} g(y) dy. \quad (156)$$

The last integral needs to be interpreted with some care, as the integrand is not in $L^1(\mathbb{R})$, at least when $g(0) \neq 0$. However, note that there is no problem when $g(0) = 0$ (as we assume that g is regular). Also, if $g(0) \neq 0$, but g is constant on some interval $(-\delta, \delta)$, we can ignore the contribution to the integral from $(-\delta, \delta)$, due to the cancelation of the positive part and the negative part, and the integral is again well-defined (although not as the Lebesgue integral). The general case is a combination of these two cases: g can always be written as a sum of a function vanishing at 0 and a function which is constant near 0. So we see that the integral can be well-defined for any regular g (with some decay at ∞). More formally, one can define

$$\int_{\mathbb{R}} \frac{1}{2\pi} \frac{1}{-y} g(y) dy = \lim_{\delta \rightarrow 0} \int_{\mathbb{R} \setminus (-\delta, \delta)} \frac{1}{2\pi} \frac{1}{-y} g(y) dy. \quad (157)$$

To indicate that this particular interpretation has been invoked, often the notation p.v. (standing for “principal value”) is used. In this notation we write

$$\text{p.v.} \int_{\mathbb{R}} \frac{1}{2\pi} \frac{1}{-y} g(y) dy = \lim_{\delta \rightarrow 0} \int_{\mathbb{R} \setminus (-\delta, \delta)} \frac{1}{2\pi} \frac{1}{-y} g(y) dy \quad (158)$$

and, returning to the situation above,

$$\operatorname{Im} u(0) = \text{p.v.} \int_{\mathbb{R}} \frac{1}{2\pi} \frac{1}{-y} g(y) dy. \quad (159)$$

In fact, taking the limit $\varepsilon \rightarrow 0$ in (155) is another way to “regularize” the divergent integral defining $\operatorname{Im} u(0)$. It is not hard to check that these two different regularizations lead to the same result (for sufficiently regular g):

$$\lim_{\varepsilon \rightarrow 0} \operatorname{Im} u(i\varepsilon) = \operatorname{Im} u(0), \quad (160)$$

or

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} \frac{1}{2\pi} \frac{-y}{y^2 + \varepsilon^2} g(y) dy = \text{p.v.} \int_{\mathbb{R}} \frac{1}{2\pi} \frac{1}{-y} g(y) dy. \quad (161)$$

As the principal value is defined by (158), the last identity has to be proved. We leave the proof to the reader as an exercise.

Above we worked at $x = 0$. For other points $x \in \mathbb{R}$ we get

$$\operatorname{Im} u(x) = \lim_{\varepsilon \rightarrow 0} \operatorname{Im} u(x + i\varepsilon) = \text{p.v.} \int_{\mathbb{R}} \frac{1}{2\pi} \frac{g(y)}{x - y} dy. \quad (162)$$

Definition.

Given a sufficiently regular $g: \mathbb{R} \rightarrow \mathbb{R}$, the *Hilbert transform* Hg of g is defined by

$$Hg(x) = \text{p.v.} \int_{\mathbb{R}} \frac{1}{\pi} \frac{g(y)}{x-y} dy. \quad (163)$$

Summarizing the above, we see that for a sufficiently regular $g: \mathbb{R} \rightarrow \mathbb{R}$ the function u defined by the Cauchy integral (149) satisfies

$$\lim_{\varepsilon \rightarrow 0_+} u(x + i\varepsilon) = \frac{1}{2}g(x) + \frac{i}{2}Hg(x), \quad x \in \mathbb{R}. \quad (164)$$

In a similar way one shows that

$$\lim_{\varepsilon \rightarrow 0_+} u(x - i\varepsilon) = -\frac{1}{2}g(x) + \frac{i}{2}Hg(x), \quad x \in \mathbb{R}. \quad (165)$$

We see that the “jump” of u across Γ , given by $\lim_{\varepsilon \rightarrow 0}(u(x + i\varepsilon) - u(x - i\varepsilon))$ is exactly $g(x)$. So we can write

$$g(x) = u_+(x) - u_-(x), \quad (166)$$

where u_+ has a holomorphic extension to the upper half-plane and u_- has a holomorphic extension to the lower half-plane.

One can now use these identities to derive a number of properties of the Hilbert transform H . We will not need them in the immediate future. However, they are useful for the study of some deeper estimates for PDEs. The Hilbert transform is studied in depth in Harmonic Analysis. In what follows we only sketch the arguments, without giving the detailed proofs. However, we encourage the reader to fill in the details as an exercise.

Although we have assumed so far that g is real-valued, it is easy to see that (164) and (165) are also true for a complex-valued g .⁷³ In particular, if u is a restriction of a holomorphic function in the upper half-plane to the real axis and u has some decay at ∞ , we have

$$u = \frac{1}{2}u + \frac{i}{2}Hu, \quad (167)$$

or

$$Hu = -iu. \quad (168)$$

Similarly, if u is a restriction of a holomorphic function in the lower half-plane to the real axis (and u has some decay at ∞), we have

$$Hu = iu. \quad (169)$$

⁷³For example, write $g = g_1 + ig_2$, and use that the formulae are true for u_1 given by g_1 and u_2 given by g_2 .

In terms of the real and imaginary part, we have for the restriction to \mathbb{R} of a holomorphic function $u = u_1 + iu_2$ (with real u_1, u_2) in the upper half-plane (with some decay at ∞ and regular up to the boundary),

$$\begin{aligned}Hu_1 &= u_2, \\Hu_2 &= -u_1.\end{aligned}\tag{170}$$

and hence

$$\begin{aligned}H^2u_1 &= -u_1, \\H^2u_2 &= -u_2.\end{aligned}\tag{171}$$

It is not hard to check that the function obtained as u_1 and u_2 above form a dense subspace of $L^2(\mathbb{R})$.⁷⁴ We also note that for sufficiently regular $f, g \in L^2(\mathbb{R})$, for which the above definition of Hf and Hg work, we have

$$(Hf, g)_{L^2} = -(f, Hg)_{L^2},\tag{172}$$

where $(f, g)_{L^2} = \int_{\mathbb{R}} f(x)\bar{g}(x) dx$ is the canonical scalar product on $L^2(\mathbb{R})$. Hence

$$(Hf, Hg) = (-H^2f, g) = (f, g)\tag{173}$$

and we see that H can be considered as an isometry (or a unitary operator) on $L^2(\mathbb{R})$, with $H^{-1} = H^* = -H$. (Once we know that the above identities are true for a dense subspace of $L^2(\mathbb{R})$, we can extend H to all $L^2(\mathbb{R})$ by continuity.)

Note also that H commutes with translations and derivatives. Defining $T_a f(x) = f(x - a)$, we have $HT_a = T_a H$ and, taking derivatives, we have $H \frac{d}{dx} = \frac{d}{dx} H$.

We know that every holomorphic function is harmonic. If we have a real harmonic function u_1 in the upper half-plane given by the Poisson integral, we can use the Cauchy integral to obtain a harmonic u_2 in the upper half-plane, so that $u = u_1 + iu_2$ is holomorphic. This is possible in general simply connected domains: if u_1 is a real-valued harmonic function in a simply connected domain Ω , we can find a harmonic u_2 such that $u = u_1 + iu_2$ is holomorphic. The equation $\bar{\partial}u = 0$ gives the Cauchy-Riemann conditions:

$$\begin{aligned}u_{2,x_1} &= -u_{1,x_2}, \\u_{2,x_2} &= u_{1,x_1}.\end{aligned}\tag{174}$$

Note that any function u_2 satisfying these conditions will be harmonic, by the commutativity of the mixed derivatives. In a simple connected domain, the existence of u_2 is equivalent the vanishing of the curl of the vector field on the right-hand side. Let us denote the field by $b = (b_1, b_2)$. We have to check

$$b_{2,x_1} - b_{1,x_2} = 0.\tag{175}$$

This is exactly the equation $\Delta u_1 = 0$, and our statement is proved.

⁷⁴It is good exercise to prove this.

In a simply connected domain, every harmonic function is a real part of a holomorphic function.

In the Complex Analysis course you have probably learned the Riemann mapping theorem that every simply connected domain in the plane (the complement of which is larger than just one point) is an image of the unit disc by a conformal mapping. There is a relatively rich list of domains for which such mappings can be written down fairly explicitly, and this can be used when solving the Laplace equation.

Let $\phi: \Omega_1 \rightarrow \Omega_2$ be a conformal mapping and let $h_2: \Omega_2 \rightarrow \mathbb{C}$ be a function. Let $h_1 = h_2 \circ \phi$. As an exercise, show that

$$\Delta h_1 = (\Delta h_2 \circ \phi) |\phi'|^2. \tag{176}$$

In particular, when h_2 is harmonic then h_1 is also harmonic. (This is related to the fact that a composition of holomorphic functions is holomorphic and that – locally – any harmonic function is a real part of a holomorphic function, as we have seen.)

Today we start discussing methods which are used for solving the boundary value problem

$$\Delta u = 0 \quad \text{in } \Omega, \text{ and} \tag{177}$$

$$u = g \quad \text{at the boundary } \partial\Omega. \tag{178}$$

At this stage we will assume that Ω is a bounded smooth domain in \mathbb{R}^n and g is a continuous function defined on the boundary $\partial\Omega$. The solution u is required to be continuous up to the boundary. We can think of it as a continuous function on the closure $\bar{\Omega}$ which extends harmonically the function g into Ω . By the maximum principle we know that such an extension is unique, if it exists. Once we know how to solve the problem above, we also know how to solve the inhomogeneous problem in which (177) is replaced by $\Delta u = f$ (with some assumptions on f). This is because the potential $\tilde{u} = K_0 * f$ gives a solution of the equation, and we only need to find a harmonic function to adjust the boundary condition.

There are several methods for approaching the problem. I will mention four of them.

1. The method of boundary potentials and integral equations at the boundary.

This traditional method, going back to the second half of the 19th century, is based on the fact that all harmonic function inside Ω can be generated by means of fundamental solutions with poles outside of Ω , see lecture 4. We can try to distribute the poles⁷⁵ at the boundary in such a way that they generate our solution. The densities giving the right distribution of the poles will satisfy some integral equations at the boundary. These equations can be solved by applying classical Fredholm theorems.

The method has some attractive features: it is quite straightforward, and does not require a lot of theory, beyond the Fredholm theorems. On the other hand, we have to do some calculations with the potentials, which can seem somewhat lengthy, if we wish to cover all the details. The equations we get are on a manifold of dimension $n - 1$. (This lowering of the dimension is relevant when solving the equation numerically, although we do have to pay a price for it in that the equations are more complicated.) The method is easily generalized to systems of equations, the maximum principle is not important. Some of today's fast numerical algorithms are based on the method. One disadvantage of the method is that it is not easy to generalize to equations with variable coefficients.

2. The "Direct Method", sometimes also called the Variational Method.

This method is based on the minimization property of the harmonic function which we mentioned in lecture 6: the solution u minimizes the integral

⁷⁵These can include dipoles.

$$\int_{\Omega} |\nabla v|^2 \tag{179}$$

among all (“sufficiently regular”) functions with the boundary condition g . This method is very flexible, can be generalized to a large class of equations, including systems (as it does not rely on the maximum principle) and equations with variable coefficients. It requires some elementary theory of Sobolev spaces.⁷⁶ However, the parts of this theory which are needed can nowadays be covered quite easily. Also, the method is ideally suited for formulating numerical approximations.

3. The method of a-priori estimates and building up the global solution from local solutions.

We know how to construct solutions in all space and half space. We can try to localize to that situation, build local solutions, and somehow coordinate these local solutions into a global one. This can be done and the method works very well for large classes of equations and systems of equations. It does not need the maximum principle.

4. The method of approaching the solution via super-solutions or sub-solutions.

This is another classical method going back to Poincaré (around 1890). It is also known as “Perron’s method”. It provides a relatively quick way of proving the existence result for our problem above, and works well for quite general classes of scalar second-order equations. It does not generalize to systems of equations nor to higher order equations, as it relies on the maximum principle in an essential way.

Our exposition will start with the integral equation method. For this method we do not really need any significantly new ideas to what we have already covered.⁷⁷ So far we have relied quite a bit on the use of potentials, and some exposition of the boundary integral equations approach to boundary value problems seems to be a natural part of such an approach. In the study of PDE one eventually does not want to rely on the potentials too much, for many reasons. However, there are also good reasons to learn how to work with the potentials before abandoning them in favor of other approaches.

The key to the success of the boundary potentials method is to take a good “first guess” at how the solution should look like, and then adjust the guess by solving some equation. We have seen in lecture 9 that we can solve the Dirichlet problem in the half space by means of the explicit representation formula with the Poisson kernel, see (75). One interpretation of the Poisson formula is that we

⁷⁶If one had to start from scratch, this would be no easy task - it took several decades of research (most of which took place between 1900 and 1960), before the probably more or less optimal ways we use today have been found.

⁷⁷At least if we take the Fredholm Theorems for granted.

put on the boundary a field of dipoles oriented perpendicularly to the boundary, with density g (= the desired boundary value) We can try to do the same in a general domain. The precise analogy to the Poisson kernel for a half-space in the case of the general bounded domain Ω is formula (64), which uses Green's function G_Ω . However, this function is not accessible to us, as its existence is more or less equivalent to what we are trying to prove. We will therefore follow the recipe of the Poisson formula in the half-space, and put dipoles oriented along the normal along the boundary, with some given density, which we will denote by ρ . (For the half-space we took $\rho = g$, but it is easy to see that this choice will typically not give us the precise solution in case of a bounded domain.)

The potential of a unit dipole located at $y \in \partial\Omega$ and oriented along the normal $\nu(y)$ is

$$u_{\text{dipole at } y}(x) = -2 \frac{\partial K_0}{\partial \nu_y}(x-y) = \frac{2}{|S^{n-1}|} \frac{(y-x, \nu(y))}{|y-x|^n} = K(x, y). \quad (180)$$

The kernel $K(x, y)$, viewed as a function of x , vanishes at the tangent plane to $\partial\Omega$ through y . However, it does not vanish on $\partial\Omega$. If we watch it as a function of $y \in \partial\Omega$ for $x \in \Omega$ approaching some $\bar{x} \in \partial\Omega$, we see that the local behavior of $y \rightarrow K(x, y)$ near \bar{x} is similar to what we have seen for the Poisson kernel in half-space. There is a mollifier-like component in the function $y \rightarrow K(x, y)$ concentrating near \bar{x} as $x \in \Omega$ approaches $\bar{x} \in \partial\Omega$. By contrast with the half space, there is also another component: the function $y \rightarrow K(\bar{x}, y)$. For the half-space this function vanishes identically, but for bounded domains it cannot vanish.

The scalar product $(y-\bar{x}, \nu(y))$ is of order $O(|y-x|^2)$ and hence $y \rightarrow K(\bar{x}, y)$ is bounded in dimension $n = 2$ and has a singularity of order at most $O(|y-x|^{-(n-2)})$ when $n \geq 3$. In fact, in dimension $n = 2$ the function $(\bar{x}, y) \rightarrow K(\bar{x}, y)$ is continuous on $\partial\Omega \times \partial\Omega$, if $\partial\Omega$ is of class C^2 .

Our ‘‘dipole potential’’ with density ρ (sometimes also called the ‘‘double layer potential’’) is

$$u(x) = \int_{\partial\Omega} K(x, y) \rho(y) dy, \quad x \in \Omega. \quad (181)$$

We will show that for $\bar{x} \in \partial\Omega$ we have

$$\lim_{\substack{x \rightarrow \bar{x} \\ x \in \Omega}} u(x) = \rho(\bar{x}) + \int_{\partial\Omega} K(\bar{x}, y) dy. \quad (182)$$

Denoting by $K\rho$ the function given by the integral on the right-hand side, we see that for $u|_{\partial\Omega} = g$ we need

$$g = \rho + K\rho. \quad (183)$$

This is an integral equation of the Fredholm type. We would like to show that for each continuous g there exist a continuous solution ρ of (263). For this we

invoke the Fredholm Theorem: equation (263) is uniquely solvable in continuous functions for each continuous g if and only if the homogeneous equation

$$\rho + K\rho = 0 \tag{184}$$

has only the trivial solution $\rho \equiv 0$.

We will look in more detail at the kernel $K(x, y)$ introduced in (180). To investigate the properties of the kernel when x is near $\partial\Omega$, we can without loss of generality restrict our attention to the following situation: Assume that the boundary passes through the origin of the coordinates, and at the origin is tangent to the plane $\{x_n = 0\}$. Near the origin the boundary is described by

$$y_n = g(y_1, \dots, y_{n-1}) \quad (185)$$

for some C^2 -function g . Assume the interior of the domain near the origin is given by $\{g > 0\}$. The unit outer normal to the domain at a point $y \in \partial\Omega$ near the origin is

$$\nu(y) = \frac{(g_{,1}, \dots, g_{,n-1}, -1)}{\sqrt{1 + |\nabla g|^2}}, \quad (186)$$

where we use the shorthand notation $g_{,i}$ for g_{x_i} .

We will investigate $K(x, y)$ for $x = (0, \dots, 0, \varepsilon)$ for small $\varepsilon > 0$. The situation at any point sufficiently close to the boundary can be brought to this form by a suitable choice of coordinates. If we wish to get estimates which depend only on the distance of x to the boundary, then the function g should not be considered as fixed, and the estimates we obtain should depend only on the parameters of g which are controlled uniformly for the various local coordinate charts we use along the boundary. Since we assume that the boundary is regular, this is not a problem. As always in similar situations, the main point is that each point $x \in \Omega$ in a sufficiently small neighborhood of $\partial\Omega$ has a uniquely defined projection $\bar{x} \in \partial\Omega$, characterized as the unique point of $\partial\Omega$ which minimizes the distance to x . The line $\bar{x}x$ is perpendicular to $\partial\Omega$ at \bar{x} , and its direction is given by the normal $\nu(\bar{x})$. For any given $x \in \Omega$ we can choose \bar{x} as the origin of the coordinate system in which we are observing the boundary, with $n - 1$ axes tangent to $\partial\Omega$ and the remaining axis perpendicular to $\partial\Omega$. That is exactly the situation described above with the help of the function g . As we move x , the function g can change, of course. However, if we assume that $\partial\Omega$ is of class C^2 , say, then we have uniform control of the parameters of g on which our estimates depend.

Let us first look at the values of the kernel $K(x, y)$ when $x = \bar{x}$. We denote $y' = (y_1, \dots, y_{n-1})$. We have

$$K(\bar{x}, y) \sim \frac{(y - \bar{x}, \nu(y))}{|y - \bar{x}|^n} = \frac{(y, \nu(y))}{|y|^n} = \frac{y_1 g_{,1} + \dots + y_{n-1} g_{,n-1} - g(y')}{|y|^n}, \quad (187)$$

where \sim is used to indicate that we have not included the normalization constant in the formulae. Note that the expression

$$g(y') - y_1 g_{,1}(y') - \dots - y_{n-1} g_{,n-1}(y') \quad (188)$$

is exactly the first two terms of the Taylor series at y' for evaluating $g(0)$. We know that $g(0) = 0$. We can use the following form of Taylor's formula

$$g(y' + z) = g(y') + g'(y')z + \int_0^1 (1-t)g''(y' + tz)z^2 dt. \quad (189)$$

In higher dimensions the notation has to be interpreted appropriately, of course. Applying (189) with $z = -y'$ and using $g(0) = 0$, we obtain

$$y_1 g_{,1}(y') + \cdots + y_{n-1} g_{,n-1}(y') - g(y') = \int_0^1 (1-t) g''((1-t)y')(y')^2 dt. \quad (190)$$

It is now easy to see that in dimension $n = 2$ the expression (187) converges to $g''(0)/2$ as $y \rightarrow 0$, and the convergence is uniform for the functions g having the same modulus of continuity of the second derivative. We conclude that when $n = 2$, the kernel K is continuous on $\partial\Omega \times \partial\Omega$.

In dimensions $n \geq 3$ we get from (187) and (190) that

$$K(\bar{x}, y) \leq \frac{C}{|\bar{x} - y|^{n-2}}, \quad \bar{x}, y \in \partial\Omega. \quad (191)$$

The kernel is singular, at the diagonal, but from the point of view of integral equations the singularity is weak. There is no problem with the Fredholm theory for kernels with such singularities⁷⁸.

Let us now consider $K(x, y)$ for $x \in \Omega$. We continue to use the notation above, with $x = (0, \dots, 0, \varepsilon)$ and $y \in \partial\Omega$, close to \bar{x} . We have

$$|y - x|^2 = |y'|^2 + \varepsilon^2 + 2g(y')\varepsilon + g(y')^2. \quad (192)$$

We note that

$$g(y') = O(|y'|^2), \quad \text{and } g(y')\varepsilon = O(|y'|)(|y'|^2 + \varepsilon^2). \quad (193)$$

Using (192) and (193) is easy to check that

$$K(x, y) = \frac{2}{|S^{n-1}|} \frac{\varepsilon + O(|y'|^2)}{(|y'|^2 + \varepsilon^2)^{\frac{n}{2}} (1 + O(|y'|))}. \quad (194)$$

We see that as $y' \rightarrow 0$ the kernel $K(x, y)$ looks (in the y' -coordinates) increasingly as the Poisson kernel of the tangent half-space.

We also note that $dy = \sqrt{1 + |\nabla g|^2} dy'$. When evaluating the integral for

$$\int_{\partial\Omega} K(x, y) \rho(y) dy \quad (195)$$

for small ε , we write it as

$$\int_{\partial\Omega} \cdots = \int_{\partial\Omega \cap B_{\bar{x}, r}} \cdots + \int_{\partial\Omega \setminus B_{\bar{x}, r}} \cdots = I + II \quad (196)$$

There is no problem in passing to the limit $\varepsilon \rightarrow 0$ in II , as there is no singularity in the domain of integration. For the integral I we get, using its similarity with the Poisson kernel

$$\rho(\bar{x}) - o(r) \leq \liminf_{\varepsilon \rightarrow 0_+} I \leq \limsup_{\varepsilon \rightarrow 0_+} I \leq \rho(\bar{x}) + o(r). \quad (197)$$

⁷⁸Sometimes the terminology “weakly singular kernels” is used in this context

The key point here is that for $x = (0, \dots, 0, \varepsilon)$ the expression $K(x, y)$ and $P(x, y')$ differ only by a function with uniformly bounded integral over y' as $\varepsilon \rightarrow 0$, and that ρ is bounded⁷⁹. As $o(r) \rightarrow 0$ when $r \rightarrow 0$, we see that

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\partial\Omega} K((0, \dots, 0, \varepsilon), y) \rho(y) dy = \rho(\bar{x}) + \int_{\partial\Omega} K(\bar{x}, y) \rho(y) dy. \quad (198)$$

Moreover, it is easy to see that the above estimates are uniform in the class of the functions g with continuous second derivatives and a common modulus of continuity of the second derivatives. Therefore we conclude that the function

$$u(x) = \int_{\partial\Omega} K(x, y) \rho(y) dy \quad (199)$$

defined in Ω can be continuously extended to the boundary $\partial\Omega$, and that

$$\lim_{\substack{x \rightarrow \bar{x} \\ x \in \Omega}} u(x) = \rho(\bar{x}) + \int_{\partial\Omega} K(\bar{x}, y) \rho(y) dy. \quad (200)$$

We see that to solve the Dirichlet problem, it is enough to solve the integral equation

$$g = \rho + K\rho. \quad (201)$$

The formula (200) can be also explained from the geometrical meaning of $K(x, y)$. Let us first consider the case $n = 2$. For a fixed $x \in \Omega$, let us consider the quantity $K(x, y)dy$ as y moves along the boundary. If y and $y + dy$ are two “infinitesimally close” points at $\partial\Omega$, the quantity $\pi K(x, y)dy$ is exactly the (infinitesimal) angle $d\alpha = d\alpha(x, y)$ under which we see the segment $[y, y + dy]$ when we watch it from the point x . Heuristically it is clear that we should have

$$\int_{\partial\Omega} d\alpha(x, y) dy = 2\pi, \quad (202)$$

independently of $x \in \Omega$.

When x is very close to some point $\bar{x} \in \partial\Omega$, we note that about half of the contribution to the integral (202) comes from an immediate neighborhood of \bar{x} . When x reaches \bar{x} , this contribution will disappear, and we will have

$$\int_{\partial\Omega} d\alpha(\bar{x}, y) dy = \pi. \quad (203)$$

So we have a jump in the integral as we reach the boundary, because when we are at the boundary, we lose half of the total angle under which the boundary

⁷⁹This rough estimate would have to be refined if we were dealing with unbounded densities ρ . This issue comes up for example when we try to solve the Dirichlet problem with $g \in L^p(\partial\Omega)$. Then the natural space for ρ is also $L^p(\partial\Omega)$, and one must be more careful with the estimates.

⁸⁰The g in this formula is the boundary condition, and not the function used to parametrize the boundary, of course. Our notation is not optimal here, but hopefully there is no danger of confusing the two meanings of g used above.

is seen. When we integrate $\int_{\partial\Omega} \rho(y) d\alpha(x, y)$, the loss will be $\pi\rho(\bar{x})$, as all the points from nearby \bar{x} contribute about $\rho(\bar{x})d\alpha$ to that part of the integral, due to the continuity of ρ .

In higher dimension the situation is similar if replace the angle by the “solid angle”.

Using this heuristics, we can see that we should have

$$\int_{\partial\Omega} K(x, y) dy = \begin{array}{ll} 2 & \text{when } x \in \Omega, \\ 1 & \text{when } x \in \partial\Omega, \\ 0 & \text{when } x \notin \bar{\Omega}. \end{array} \quad (204)$$

The first and the last identity can be easily proved rigorously by integration by parts. As an exercise, you can try to find a simple proof of the middle identity. These identities provide another heuristic explanation for (200) (which can be turned into a proof, with some work).

Lecture 17, 10/18/2010

Last time we reduced the Dirichlet boundary-value problem

$$\begin{aligned}\Delta u &= 0 && \text{in } \Omega \\ u &= g && \text{at } \partial\Omega\end{aligned}\tag{205}$$

to an integral equation of the form

$$g = \rho + K\rho,\tag{206}$$

at the boundary, where K is an integral operator with a weakly singular kernel. When $n = 2$ the kernel is in fact continuous (when the domain is of class C^2). We wish to find a continuous solution ρ of this equation. The function g is assumed to be continuous. Let us denote by X the space of continuous functions at $\partial\Omega$, with the norm $\|f\| = \sup_{\partial\Omega} |f|$. (We could also work with other spaces, but the choice $X = C(\partial\Omega)$ is probably the most straightforward choice.)

The Fredholm theory says that the range of the operator $I + K$ has finite codimension, which is equal to the dimension of the kernel of $I + K$. We can see from this that our “guess” to seek the solution as a dipole potential with density ρ was quite good: even without doing anything else we now know that we can solve our problem if we impose finitely many linear conditions on g . Such result is not obvious from the original formulation (205). Another way to look at the situation is the following: we know from the abstract spectral theory of compact operators that we can write $X = X_1 \oplus X_2$ ⁸¹ with X_1, X_2 invariant under K and X_1 finite-dimensional, so that that some power K^m is small on X_2 .⁸² For $g \in X_2$ we can invert $I + K$ by the usual power series:

$$\rho = (I + K)^{-1}g = (I - K + K^2 - K^3 + \dots)g\tag{207}$$

This is another illustration that our “guess” to seek the solution as dipole potential was good. In some sense, it solves the problem modulo finite-dimensional adjustments. The difficulties with solving equations in infinite-dimensional spaces disappeared.⁸³

By Fredholm theory we will know that the equation (206) is solvable in continuous functions for each continuous g if and only if the homogeneous equations

$$\rho + K\rho = 0\tag{208}$$

has only the trivial solution $\rho = 0$ in continuous functions.

⁸¹Here \oplus means “direct sum, algebraically and topologically”, not orthogonality, in general

⁸²For self-adjoint operators on a Hilbert space we can take $m = 1$. In our special case it is very likely that we still can take $m = 1$, even though K is not self-adjoint. This should be in the literature, with high probability.

⁸³This is of course one of the main points of the Fredholm theory. However, note that before we got to the stage when the Fredholm theory can be applied, we had to find the right set-up and make some good choices which were not obvious. This is quite typical.

At first this might look as an easy consequence of the uniqueness we have for the Dirichlet problem: $\rho + K\rho = 0$ means that the potential u defined by ρ approaches 0 as we approach $\partial\Omega$ from Ω . As u is harmonic in Ω , we now that $u \equiv 0$ in Ω . However, we need to show that $\rho = 0$, not only u inside Ω . In fact the formula

$$u(x) = \int_{\partial\Omega} K(x,y)\rho(y) dy \quad (209)$$

defines u also outside $\partial\Omega$. Let us denote Ω_{ext} the open set $\mathbb{R}^n \setminus \bar{\Omega}$. If $\rho = 0$ we not only have $u = 0$ in Ω , but we also have $u = 0$ in Ω_{ext} . The calculation of the limit

$$\lim_{\substack{x \rightarrow \bar{x} \in \partial\Omega \\ x \in \Omega}} u(x) = \rho(\bar{x}) + K\rho(\bar{x}) \quad (210)$$

we have done last time can be also applied to calculate the limit from Ω_{ext} . One gets, exactly by the same method,

$$\lim_{\substack{x \rightarrow \bar{x} \in \partial\Omega \\ x \in \Omega_{\text{ext}}}} u(x) = -\rho(\bar{x}) + K\rho(\bar{x}). \quad (211)$$

By subtracting (261) from (260) we see that $\rho = 0$ is equivalent to the condition that the jump of u across the boundary is zero.

In our situation when $\rho + K\rho = 0$, to verify that $\rho = 0$, we need to show that the exterior limit (261) vanishes.

However, we have a small complication. The jump is not always zero, and the kernel is not always trivial. For $0 < R_1 < R_2$ consider the domain

$$\Omega = \{x, |x| < R_2\} \setminus \{x, |x| \leq R_1\}. \quad (212)$$

We have

$$\partial\Omega = \Gamma_1 \cup \Gamma_2, \quad (213)$$

with $\Gamma_1 = \{x, |x| = R_1\}$ and $\Gamma_2 = \{x, |x| = R_2\}$.

Take $\rho_0 = 0$ on Γ_2 and $\rho = 1$ on Γ_1 .

We can use formula (204) to see that the dipole potential u of ρ_0 will vanish in Ω , and

$$\rho_0 + K\rho_0 = 0. \quad (214)$$

One can show⁸⁴ that in this case we have

$$\text{Ker}(I + K) = \mathbb{R}\rho_0 \quad (215)$$

i. e. the kernel of $I + K$ is one dimensional, consisting of multiples of ρ_0 .

Since the kernel is non-trivial, the range of $(I + K)$ cannot be all space of continuous functions, and the integral equation $\rho + K\rho = g$ is not always solvable. This does not mean that the original Dirichlet problem (205) is not solvable.

⁸⁴This will follow from our further considerations

It only means that it is not solvable by dipole potentials. In this respect our initial guess was not perfect - there is a small defect here. It can be easily corrected⁸⁵, but for now we will focus on domains where the complication does not arise. The complication is related to the fact that in the example Ω_{ext} was not connected.

Note that at this point we are essentially stuck. We know that in some cases the kernel can be non-trivial. We are guessing that it might be related to the topology of the domain, but it is not clear how to proceed. We need a new idea. One way to proceed⁸⁶ is based on the following trick: look at the normal derivative

$$\frac{\partial u}{\partial \nu}(x) \quad \text{as } x \rightarrow \partial\Omega, \quad x \in \Omega_{\text{ext}}. \quad (216)$$

Here we use the fact that the normal ν can be thought of as a vector field defined in a neighborhood of the boundary. If $d(x)$ is the signed distance of x to $\partial\Omega$ (with the convention that $d(x) < 0$ inside Ω), then in a small neighborhood of $\partial\Omega$ we have $\nu(x) = \nabla d(x)$.

At first the idea of looking at the normal derivative at the boundary might not look very promising: for general continuous ρ the potential u may not be C^1 up to the boundary, and it is not even clear that the normal derivative of u at $\partial\Omega$ is well-defined.

The trick is that rather than looking at $\frac{\partial u}{\partial \nu}(x)$ we will compare the values of the normal derivatives on the two sides of the boundary. To be precise, let us define for each point $x \in \Omega_{\text{ext}}$ which is close to the boundary its ‘‘reflection by the boundary’’, denoted by x^* . To define it, we denote by \bar{x} the projection of x to $\partial\Omega$, i. e. the point of $\partial\Omega$ which is the closest to x ,⁸⁷ and set $x^* = x - 2(x - \bar{x})$. We will now investigate the function

$$b(x) = \frac{\partial u}{\partial \nu}(x) - \frac{\partial u}{\partial \nu}(x^*). \quad (217)$$

Our goal is to show that

$$\lim_{\substack{x \rightarrow \bar{x} \in \partial\Omega \\ x \in \Omega_{\text{ext}}}} b(x) = 0. \quad (218)$$

Note that in case when ρ is in the kernel of $I + K$, the function u vanishes in Ω , hence $b(x) = \frac{\partial u}{\partial \nu}(x)$, and (218) will imply that the normal derivative of u in Ω_{ext} is continuous up to the boundary and vanishes at $\partial\Omega$.

Once we know this, we will be able to conclude that u has to be constant on each connected component of Ω_{ext} , and has to vanish on the connected components

⁸⁵In fact, dealing with this defect is a very good exercise in Fredholm theory, as the theory also gives us necessary and sufficient conditions for solvability in terms of the kernel of the adjoint operator.

⁸⁶We will later mention an alternative way, using the adjoint operator $I + K^*$.

⁸⁷Recall that \bar{x} is well-defined and unique when x is close to $\partial\Omega$.

of Ω_{ext} containing ∞ . If Ω_{ext} has only one connected component, this will show that u vanishes in Ω_{ext} , and hence by (260) and (261) $\rho = 0$ on $\partial\Omega$.

Checking (218) requires some calculation. We have

$$b(x) = \int_{\partial\Omega} \left(\frac{\partial K}{\partial \nu_x}(x, y) - \frac{\partial K}{\partial \nu_x}(x^*, y) \right) \rho(y) dy. \quad (219)$$

Taken individually, each term of the integral looks quite dangerous, for example

$$\frac{\partial K}{\partial \nu_x} \sim \frac{1}{|x - y|^n}, \quad (220)$$

which not bounded in L^1 (as a function of y) when $x \rightarrow \partial\Omega$. However, there is a lot of cancelation between the two terms.

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We continue with the calculation of function b given by (219). We use the same set-up as in lecture 16, see (185) and (186). In the coordinate system set up that way, the points x and x^* will have coordinates

$$x = (0, \dots, 0, -\varepsilon), \quad x^* = (0, \dots, 0, \varepsilon). \quad (221)$$

On the line xx^* the normal direction is parallel to the x_n -axis. So the normal derivatives coming into (219) are

$$-\frac{\partial}{\partial x_n} \frac{(y-x, \nu(y))}{|y-x|^n}, \quad -\frac{\partial}{\partial x_n} \frac{(y-x^*, \nu(y))}{|y-x^*|^n}. \quad (222)$$

Carrying out the differentiation, the first expression is

$$-\frac{1}{\sqrt{1+|\nabla g|^2}} \frac{|y-x|^2 + n(y-x, \bar{\nu}(y))(y_n - x_n)}{|y-x|^{n+2}}. \quad (223)$$

with

$$\bar{\nu} = (g_1, \dots, g_{n-1}, -1). \quad (224)$$

Writing g_i rather than $g_{\cdot i}$ or g_{x_i} , we have

$$(y-x, \bar{\nu}(y)) = (y_1 g_1 + \dots + y_{n-1} g_{n-1} - g(y') + x_n). \quad (225)$$

Using that

$$y_1 g_1 + \dots + y_{n-1} g_{n-1} - g(y') = O(|y'|^2), \quad (226)$$

and $x_n = -\varepsilon$ we obtain

$$|y-x|^2 + n(y-x, \bar{\nu}(y))(y_n - x_n) = \varepsilon^2 + O(|y'|^2) + 2\varepsilon O(|y'|^2). \quad (227)$$

Using the same calculation with x replaced by x^* and letting

$$d = |x-y| \quad d^* = |x^*-y|, \quad (228)$$

we see that

$$\int_{\partial\Omega} \left| -\frac{\partial}{\partial x_n} \frac{(y-x, \nu(y))}{|y-x|^n} + \frac{\partial}{\partial x_n} \frac{(y-x^*, \nu(y))}{|y-x^*|^n} \right| |\rho(y)| dy. \quad (229)$$

can be estimated by

$$\int_{\partial\Omega} \left((\varepsilon^2 + O(|y'|^2)) \left(\frac{1}{d^{n+2}} - \frac{1}{(d^*)^{n+2}} \right) + 2\varepsilon O(|y'|^2) \left(\frac{1}{d^{n+2}} + \frac{1}{(d^*)^{n+2}} \right) \right) |\rho(y)| dy. \quad (230)$$

We have

$$\begin{aligned} d^2 &= |y'|^2 + (y_n - \varepsilon)^2, \\ (d^*)^2 &= |y'|^2 + (y_n + \varepsilon)^2, \\ d^2 - (d^*)^2 &= -4\varepsilon y_n = 4\varepsilon O(|y'|^2). \end{aligned} \quad (231)$$

We see from (233) that for small $|y'|$ we have

$$\frac{d^2}{(d^*)^2} = 1 + \frac{4\varepsilon O(|y'|^2)}{(d^*)^2} = 1 + O(|y'|) \frac{\varepsilon|y'|}{(d^*)^2} = 1 + O(|y'|). \quad (232)$$

It is important that the implied constants in the O expressions above are independent of ε .

We have

$$\frac{1}{d^{n+2}} - \frac{1}{(d^*)^{n+2}} = \frac{(d^*)^{n+2} - d^{n+2}}{d^{n+2}(d^*)^{n+2}} \quad (233)$$

and using

$$(d^*)^{n+2} - d^{n+2} = (d - d^*)((d^*)^{n+1} + (d^*)^n d + \dots + d^{n+1}) \quad (234)$$

together with (233) and (234) we see that

$$\left| \frac{1}{d^{n+2}} - \frac{1}{(d^*)^{n+2}} \right| \lesssim \frac{|d^* - d|}{d^{n+3}} \lesssim \frac{\varepsilon|y'|^2}{d^{n+4}}, \quad (235)$$

where we use the usual notation \lesssim to imply that the inequality is true up to some unimportant multiplicative factor.

From (233)–(235) we see by some elementary estimates that to estimate (230), we need to bound, for some $R > 0$, the integral

$$I = \int_{B_R} \frac{\varepsilon|y'|^2}{(|y'|^2 + \varepsilon^2)^{\frac{n+2}{2}}} dy_1 dy_2 \dots dy_{n-1}, \quad (236)$$

where B_R denotes a ball of radius R in $(n-1)$ -dimensional space.

Let us now make substitution

$$y' = \varepsilon z. \quad (237)$$

In the new variables the integral becomes

$$I = \int_{B_{\frac{R}{\varepsilon}}} \frac{|z|^2}{(|z|^2 + 1)^{\frac{n+2}{2}}} dz_1 \dots dz_{n-1} \leq \int_{\mathbb{R}^{n-1}} \frac{|z|^2}{(|z|^2 + 1)^{\frac{n+2}{2}}} dz = M < +\infty. \quad (238)$$

Given that ρ is bounded, we see that we estimated the expression (229) by a constant independent of ε . However, this is not enough, we need to show that it approaches 0 as $\varepsilon \rightarrow 0$. To obtain such an estimate, we can use the fact that for $\rho = \rho_0 = \text{const}$.

$$\int_{\partial\Omega} \frac{\partial}{\partial\nu_x} K(x, y) \rho_0 dy = \frac{\partial}{\partial\nu_x} \int_{\partial\Omega} K(x, y) \rho_0 dy = 0, \quad (239)$$

for $x \notin \partial\Omega$ by (204). This saves the day, as we can freely change ρ by any constant. When we calculate the limit (218) at $\bar{x} \in \partial\Omega$, we can change $\rho(y)$ to $\rho(y) - \rho(\bar{x})$, and since ρ is continuous, in (230) we can assume that for each $\delta > 0$ there exists $r > 0$ such that $|\rho(y)| \leq \delta$ in $B_{\bar{x}, 2r}$.

We now write the integral (229) as

$$\int_{\partial\Omega} \cdots = \int_{\partial\Omega \cap B_{\bar{x},r}} \cdots + \int_{\partial\Omega \setminus B_{\bar{x},r}} \cdots = I_1 + I_2. \quad (240)$$

The integral I_2 converges to 0 as $x \rightarrow \bar{x}$, as we stay away from the singularity. The integral I_1 will be bounded by $\sim M\delta$. We see that by choosing r sufficiently small, the values of I_1 will stay close to 0 uniformly as $x \rightarrow \bar{x}$. Repeating the argument in (197), we see that we have proved (218).⁸⁸

Let us summarize the situation. We wish to show that the kernel of $I + K$ is trivial. We assume that for some continuous ρ we have $\rho + K\rho = 0$. This means that the potential (209) vanishes in Ω and we have just shown that when we approach $\partial\Omega$ from Ω_{ext} , the normal derivative $\frac{\partial u}{\partial \nu}$ of u approaches 0.

Claim: In the situation above, u vanishes in the connected component of Ω_{ext} containing (a neighborhood of) ∞ .

Proof: When we know more about the Neumann problem, we will see that there are many reasons for this statement to be true, but for now we will do what might first seem like an *ad hoc* proof by a trick.

We do the following formal calculation

$$\int_{\Omega_{\text{ext}}} |\nabla u|^2 = \int_{\partial\Omega_{\text{ext}}} u \frac{\partial u}{\partial \nu_{\text{ext}}} - \int_{\Omega_{\text{ext}}} u \Delta u, \quad (241)$$

where $\nu_{\text{ext}} = -\nu$. As both terms on the right-hand side clearly vanish, we see that u must be constant.

Since Ω_{ext} is unbounded, the formal calculation needs to be justified. However, this is quite easy: We replace Ω_{ext} by $\mathcal{O}_R = \Omega_{\text{ext}} \cap B_R$ for some large radius R , and integrate by parts on \mathcal{O}_R . In comparison with (241) we get an extra term

$$\int_{\partial B_R} u \frac{\partial u}{\partial \nu}. \quad (242)$$

We know that u is a dipole potential, and therefore

$$u = O\left(\frac{1}{|x|^{n-1}}\right), \quad \frac{\partial u}{\partial \nu} = O\left(\frac{1}{|x|^n}\right), \quad |x| \rightarrow \infty, \quad (243)$$

which is more than enough to show that (242) approaches 0 as $R \rightarrow \infty$. In fact, one should also be somewhat careful with the calculation at $\partial\Omega$, and we did not really show that u is C^1 in Ω_{ext} up to the boundary. However, we showed that the normal derivative is continuous up to the boundary, with the boundary value being 0. As u is smooth in Ω_{ext} , we can do the fully rigorous calculation

⁸⁸As an exercise you can check that the limit is uniform for all $\bar{x} \in \partial\Omega$. Note that we need $\partial\Omega$ to be of class C^2 for the above proof.

by approaching $\partial\Omega$ from Ω_{ext} by some smooth surface. The details are left to the reader as an exercise.

So we see that u is constant in each connected component of Ω_{ext} . Since in the unbounded component u approaches 0 as $x \rightarrow \infty$, the claim is proved.

Now if Ω_{ext} has only one component, that u must vanish in Ω_{ext} , and we have seen that this means, together with the fact that $u = 0$ in Ω , that the density ρ generating the potential vanishes. Therefore we have shown that the kernel of the operator $I + K$ vanishes if Ω_{ext} is connected.⁸⁹

As you have seen, the direct proof we have just finished that for a connected Ω_{ext} the kernel of $I + K$ is trivial is quite laborious. There is one non-obvious idea in it, namely that we should look at the normal derivatives, but the rest is just calculation. In this sense the proof is simple, even though we cannot really say that it is short.

By the Fredholm theorem, the dimension of the kernel of $I + K$ is the same as the dimension of the kernel of the adjoint operator $I + K^*$. As we will see, the calculation of the kernel of $I + K^*$ is easier, once we know how the operator $I + K^*$ should be interpreted, so working with $I + K^*$ is probably a more efficient way to calculate the dimension of the kernels. Nevertheless, the direct way of dealing with $I + K$ also has some value.

⁸⁹We recall that we have indicated before that if Ω_{ext} is not connected when the kernel does not vanish. You can show as an exercise that the dimension of the kernel is equal to the number of bounded connected components of Ω_{ext} .

Let us look in more detail at the boundary value problem

$$\Delta u = 0 \quad \text{in } \Omega, \text{ and} \quad (244)$$

$$\frac{\partial u}{\partial \nu} = g \quad \text{at the boundary } \partial\Omega, \quad (245)$$

where ν is the outward unit normal to $\partial\Omega$. We assume that Ω is a bounded smooth domain. This is usually called the *interior Neumann problem*.⁹⁰ Note that the solution is determined only up to a constant: the homogeneous problem (with $g = 0$) has a non-zero solution $u = \text{const}$. This suggests that the non-homogeneous problem will not be always solvable⁹¹, and one can indeed easily identify a non-trivial necessary condition for the solvability of the problem. We note that for a solution we must have

$$\int_{\partial\Omega} g = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} = \int_{\Omega} \Delta u = 0. \quad (246)$$

One can also consider the *exterior Neumann problem*

$$\Delta u = 0 \quad \text{in } \Omega_{\text{ext}}, \text{ and} \quad (247)$$

$$\frac{\partial u}{\partial \nu} = g \quad \text{at the boundary } \partial\Omega, \quad (248)$$

$$u(x) = O\left(\frac{1}{|x|^{n-2}}\right) \quad \text{when } |x| \rightarrow \infty \text{ and } n \geq 3. \quad (249)$$

$$u(x) = c_1 \log |x| + O\left(\frac{1}{|x|}\right) \quad \text{when } |x| \rightarrow \infty \text{ and } n = 2. \quad (250)$$

where $\Omega_{\text{ext}} = \mathbb{R}^n \setminus \bar{\Omega}$. The normal ν in (248) is taken to be the same as in (245) – it is the outward unit normal of Ω . We do not change the orientation of the normal when we go from (245) to (248). There are other ways to formulate the boundary conditions (249) and (250). For example, (249) can be replaced by $u \rightarrow 0$ as $|x| \rightarrow \infty$ and (250) can be replaced by $u = c_1 \log |x| + o(1)$.⁹²

The main issue in choosing the right conditions at ∞ is that we get as close to both solvability and uniqueness for our problem as possible. You can check by the integration by parts that we used in the last lecture for proving that u vanishes in Ω_{ext} that our condition (249) in dimensions $n \geq 3$ gives uniqueness when Ω_{ext} is connected. Condition (250) also gives uniqueness. When $c_1 \neq 0$,

⁹⁰Various interpretations of the boundary condition $\frac{\partial u}{\partial \nu} = g$ will be discussed later.

⁹¹By analogy with linear algebra, which is usually safe to use with elliptic equations – we will formulate this more precisely later.

⁹²All this is easy to understand if we know what the asymptotic behavior of harmonic functions in $\mathbb{R}^n \setminus B_R$ with some growth conditions at ∞ is. We have not done this in detail, but we have all the necessary tools to investigate it. This might be a good topic for the next homework assignment.

the total “flux to infinity” given by $\int_{\partial B_R} \frac{\partial u}{\partial \nu}$ as $R \rightarrow \infty$ is $2\pi c_1$, which is not compatible with $g = 0$ (by integration by parts, using $\Delta u = 0$). So we see that for $g = 0$ we must have $c_1 = 0$. For $c_1 = 0$ one can again use the integration by parts, and obtain uniqueness when Ω_{ext} is connected.

An important point in our calculations will be that the normal is well-defined not just at the boundary, but also in a small neighborhood of it. In particular, when u is defined inside Ω , the normal derivative $\frac{\partial u}{\partial \nu}(x)$ is well defined for x sufficiently close to $\partial\Omega$, and similarly for Ω_{ext} .

We will try to solve both the interior and the exterior Neumann problems by suitable potentials. It turns out that in this case the right potential to consider is the Newton potential of a density ρ at $\partial\Omega$,

$$u(x) = \int_{\partial\Omega} 2K_0(x-y)\rho(y) dy, \quad x \in \mathbb{R}^n \setminus \partial\Omega. \quad (251)$$

We then have

$$\frac{\partial u}{\partial \nu}(x) = \int_{\partial\Omega} 2 \frac{\partial}{\partial \nu_x} K_0(x-y)\rho(y) dy, \quad x \text{ close to } \partial\Omega, x \neq \partial\Omega. \quad (252)$$

The key is again to calculate the limits of $\frac{\partial u}{\partial \nu}(x)$ as x approaches the boundary.

By similar calculation as those we have done for the Dirichlet problem, it is possible to show that for continuous ρ the normal derivative (252) has limits as x approaches the boundary from Ω or from Ω_{ext} . The limits can be expressed in terms of ρ and the kernel $K(x, y)$ defined earlier in connection with the Dirichlet problem, see (180). Recall that

$$K(x, y) = -2 \frac{\partial K_0}{\partial \nu_y}(x-y) = \frac{2}{|S^{n-1}|} \frac{(y-x, \nu(y))}{|y-x|^n}. \quad (253)$$

This should be compared with the expression

$$2 \frac{\partial}{\partial \nu_x} K_0(x-y) \quad (254)$$

in (252). We have

$$2 \frac{\partial}{\partial \nu_x} K_0(x-y) = -2 \frac{\partial K_0}{\partial \nu_y}(x-y), \quad (255)$$

and hence

$$2 \frac{\partial}{\partial \nu_x} K_0(\bar{x}-y) = K(y, \bar{x}), \quad \bar{x}, y \in \partial\Omega, \bar{x} \neq y. \quad (256)$$

We introduce

$$K^*(x, y) = K(y, x). \quad (257)$$

We will denote by K^* the operator defined on functions $\rho: \partial\Omega$ by

$$K^* \rho(\bar{x}) = \int_{\partial\Omega} K^*(\bar{x}, y) \rho(y) dy. \quad (258)$$

We note that K^* is the L^2 -adjoint of the operator K , which means that

$$\int_{\partial\Omega} (K\rho_1)(y) \rho_2(y) dy = \int_{\partial\Omega} \rho_1(y) (K^* \rho_2)(y) dy. \quad (259)$$

With this notation, the limits of $\frac{\partial u}{\partial \nu}(x)$ (given by (252)) as x approaches the boundary can be calculated as follows.

$$\lim_{\substack{x \rightarrow \bar{x} \in \partial\Omega \\ x \in \Omega}} \frac{\partial u}{\partial \nu}(x) = -\rho(\bar{x}) + K^* \rho(\bar{x}), \quad (260)$$

and

$$\lim_{\substack{x \rightarrow \bar{x} \in \partial\Omega \\ x \in \Omega_{\text{ext}}}} \frac{\partial u}{\partial \nu}(x) = \rho(\bar{x}) + K^* \rho(\bar{x}). \quad (261)$$

The proofs are similar to the calculations we have done in connection with the Dirichlet problem, see (200), and we will not go into details.

Let us now summarize the integral equation we get in connection with the interior and exterior Neumann problems, as well as the interior and exterior Dirichlet problems.

Strictly speaking, we have not talked much about the exterior Dirichlet problem, but it is exactly what one would expect: find a harmonic u in Ω_{ext} with $u = g$ at $\partial\Omega$ and suitable decay at ∞ . If we seek the solution of the exterior Dirichlet problem in terms of the same dipole potential (181), we get integral equation

$$g = -\rho + K\rho. \quad (262)$$

So we have four problems, the exterior/interior Dirichlet/Neumann problems, and four integral equations at $\partial\Omega$ associated with them. All four equations can be formulated in terms of just one integral operator, the operator K introduced in connection with the interior Dirichlet problem.

The situation can be summarized in the following table.

	interior	exterior
Dirichlet	$(I + K)\rho = g$	$(-I + K)\rho = g$
Neumann	$(-I + K^*)\rho = g$	$(I + K^*)\rho = g$

Let X be the space of all continuous functions and $\partial\Omega$. From Fredholm theorems we know

$$\begin{aligned} \dim \text{Ker}(\pm I + K) &= \dim \text{Ker}(\pm I + K^*), \\ (\pm I + K)(X) &= [\text{Ker}(\pm I + K^*)]^\perp, \\ (\pm I + K^*)(X) &= [\text{Ker}(\pm I + K)]^\perp, \end{aligned} \quad (263)$$

where in each line we always take the same signs, and

$$Y^\perp = \left\{ \rho \in X, \int_{\partial\Omega} \rho \eta = 0 \text{ for each } \eta \in Y \right\}. \quad (264)$$

We have seen in the last lecture that

$$\dim \text{Ker}(I + K) = \# \text{ of bounded connected components of } \Omega_{\text{ext}} \quad (265)$$

and that, denoting by Ω_{ext}^0 the unbounded component of Ω_{ext} ,

$$\text{Ker}(I + K) = \left\{ \rho \in X, \rho \text{ is locally constant at } \partial\Omega \text{ and } \rho|_{\partial\Omega_{\text{ext}}^0} = 0 \right\}. \quad (266)$$

Repeating the same arguments for $-I + K$, one gets

$$\dim \text{Ker}(-I + K) = \# \text{ of connected components of } \Omega_{\text{ext}} \quad (267)$$

and

$$\text{Ker}(-I + K) = \left\{ \rho \in X, \rho \text{ is locally constant at } \partial\Omega \right\}. \quad (268)$$

This shows for example (together with (263) and (246) that if Ω_{ext} is connected, then the Neumann problem is solvable if and only if $\int_{\partial\Omega} g = 0$.⁹³

We can also use (263) to an alternative way of finding the dimension of the kernels. Let us look for example at $\text{Ker}(I + K^*)$. If

$$\rho + K^* \rho = 0 \quad (269)$$

then the normal derivative $\frac{\partial u}{\partial \nu}$ of the potential (251) vanishes at $\partial\Omega$ as we approach $\partial\Omega$ from Ω_{ext} . This means that u is constant in each connected component of Ω_{ext} . We note that u is continuous across $\partial\Omega$,⁹⁴ and that $u(x) \rightarrow 0$ as $x \rightarrow \infty$, i. e. u vanishes in the unbounded component of Ω_{ext} . Hence if Ω_{ext} is connected, then u has to vanish and $\text{Ker}(I + K^*) = \{0\}$. We see that the proof that the kernels of $I + K$ and $I + K^*$ are trivial in the case when Ω_{ext} is connected is quite simpler if we work with $I + K^*$.

If Ω_{ext} has bounded connected components $\Omega_{\text{ext}}^1, \dots, \Omega_{\text{ext}}^m$, we expect that we can choose the value of the constant c_j for which $u = c_j$ in Ω_{ext}^j , so the dimension of the kernel will be m . However, the corresponding densities ρ which will give such potentials are non-trivial. So while the argument gives easily that $\dim \text{Ker}(I + K^*) \leq m$, establishing that the dimension is exactly m would probably not be simpler than the proof using $I + K$, as for $I + K$ the densities in the kernel are locally constant, and hence explicit.

In the case when Ω_{ext} is not connected, we can see that the method of boundary integral equations has a certain defect: the integral equations do not always

⁹³This is also the case when Ω_{ext} is not simply connected, but one needs to work some more to prove it.

⁹⁴This statement is much easier to see than the continuity of the normal derivative across $\partial\Omega$ in the case of the dipole potential.

have a solution, even when we expect that the original problem will be solvable. This is due to the fact that the solutions of the original interior Dirichlet problem (say), cannot be given by the dipole potential. One can work around this by using (263) in the following way. Let us consider the example of the interior Dirichlet problem, and assume Ω_{ext} has bounded connected components $\Omega_{\text{ext}}^1, \dots, \Omega_{\text{ext}}^m$. Let us choose $a_j \in \Omega_{\text{ext}}^j$, and set $h_j = K_0(x - a_j)$. These functions are harmonic in Ω . It is clear that if we can solve the exterior Dirichlet problem with g replaced by $g - \sum_{j=1}^m c_j h_j$ for some constants c_j , then we can also solve the original problem with g . We can now try to find suitable constants c_1, \dots, c_m so that the function $\tilde{g} = g - \sum_{j=1}^m c_j h_j$ will be perpendicular to the kernel of $I + K^*$, and hence the interior Dirichlet problem will be solvable by a dipole potential when g is replaced by \tilde{g} . This is possible, but you can see that the method is becoming a little unwieldy. We will not go further into the details, but it is a very good exercise to work them out.

Today we start with the Direct Method for solving elliptic problems. We first illustrate the main idea of the method in a finite-dimensional situation. Let us consider an electrical circuit consisting of resistors. We could also call it a network of resistors. The circuit has n nodes. We fix some numbering of the nodes and will call each node by its number. We assume that

- the voltage at node i is U_i
- the resistor between nodes i and j has resistance R_{ij} . We allow $R_{ij} = +\infty$, which is equivalent to saying that there is no resistor between i and j .
- The current flowing from node i to j through the resistor R_{ij} is I_{ij} . We have $I_{ij} = -I_{ji}$.
- Ohm's law: $I_{ij}R_{ij} = U_i - U_j$. (No summation implied by the repeated indices.)
- The nodes are divided into two groups: "interior nodes", which are not connected to any outside device, and "boundary nodes" which are connected to an outside device.
- Kirchhoff's law: For any "interior node" i we have $\sum_{j=1}^n I_{ij} = 0$.
- "Boundary conditions" are given at the boundary nodes. For a boundary node i the boundary condition can be either $U_i =$ given voltage V_i , (we can imagine that the node is connected to some device which keeps the voltage constant), or $\sum_{j=1}^n I_{ij} = I_j$, which means that the node is connected to a device which keeps the current coming to that node from the device constant.⁹⁵ The set of the boundary nodes where the voltage is given will be denoted by \mathcal{D} , the set of the boundary nodes where the current is given will be called \mathcal{N} , and the set of the interior nodes will be called \mathcal{F} . Note that we do not really have to distinguish between \mathcal{N} and \mathcal{F} if we set $I_i = 0$, $i \in \mathcal{F}$.

Given any "boundary conditions", we would like to find the voltages U_i for all i . There is more than one way to solve this problem. For example, one can write down the system of equations which we get from the Kirchhoff's law and the Ohm's law and show that it can be solved, under some natural assumptions. This is doable, but it is not the most appealing way to deal with the problem. A more elegant method is the following.

Let us introduce a quantity

$$P = \frac{1}{2} \sum_{i,j=1}^n \frac{(U_i - U_j)^2}{R_{ij}} - \sum_{i \in \mathcal{N}} U_i I_i. \quad (270)$$

⁹⁵One can also consider more general boundary conditions, such as "mixed boundary conditions", but for simplicity we will restrict our considerations to the two type of nodes described above.

The first term in P is related to the power consumed by the resistors, the second term is related to the power drawn from the outside devices connected to the \mathcal{N} nodes. (Some power might also be drawn from the devices connected to the \mathcal{D} nodes, but this part does not explicitly appear in P .)

Let us now consider the problem

Minimize P over all possible voltages at the nodes in $\mathcal{N} \cup \mathcal{F}$, with the voltages at the nodes in \mathcal{D} being fixed to the given values V_i , $i \in \mathcal{D}$:

$$\min \{P(U_1, \dots, U_n); U_i = V_i \text{ for } i \in \mathcal{D}, U_i \text{ arbitrary for } i \in \mathcal{N} \cup \mathcal{F}\}. \quad (271)$$

If P attains its minimum at (U_1, \dots, U_n) , we will have

$$\frac{\partial P}{\partial U_i} = 0, \quad i \in \mathcal{F} \cup \mathcal{N}, \quad (272)$$

and it is easy to check that, taking into account Ohm's law, (272) expresses the Kirchhoff's law when $i \in \mathcal{F}$ and the boundary condition $\sum_j I_{ij} = I_i$ for $i \in \mathcal{N}$. We see that our problem is equivalent to finding the voltages U_i , $i \in \mathcal{N} \cup \mathcal{F}$ for at which P has a critical point. As P is clearly convex, any critical point has to be a minimum. To prove that our problem is solvable, it is enough to show that P attains its infimum over the set of the admissible voltages. This is easily seen when $\mathcal{D} \neq \emptyset$. In that case we get the existence and uniqueness of the solution from the variational immediately, without any detailed analysis of the equations. For uniqueness, note that when $\mathcal{D} \neq \emptyset$, then P is strictly convex in $\{U_i\}_{i \in \mathcal{F} \cup \mathcal{N}}$ and use the fact that a strictly convex function cannot attain its minimum at two different points.

When $\mathcal{D} = \emptyset$, we note that when $U_i = A$ for all i , we have

$$P = -A \sum_{i \in \mathcal{N}} I_i \quad (273)$$

and we see that $\sum_i I_i = 0$ is a necessary condition for P to attain its infimum. One can also see directly that $\sum_{i \in \mathcal{N}} I_i = 0$ is a necessary condition for the solvability of the linear system of equations associated with our problem.

If $\mathcal{D} = \emptyset$ and the condition $\sum_{i \in \mathcal{N}} I_i = 0$ is satisfied, then one can check easily that P attains its infimum. In that case the function P does not change if we shift $U_i \rightarrow U_i + A$. On any subspace complementary to the $1d$ subspace (A, A, \dots, A) , $A \in \mathbb{R}$.⁹⁶ The function P will be strictly convex on such a subspace, and hence the solution will be unique modulo the shifts $U_i \rightarrow U_i + A$, $i \in \mathcal{N} \cup \mathcal{F}$.

⁹⁶One can also think of P as a function on a factor space $Y = \mathbb{R}^n / (\mathbb{R}(1, \dots, 1))$. One can check that P is strictly convex on Y .

The Laplace equation with Dirichlet/Neumann boundary conditions can be thought of as an infinite-dimensional version of the problem above. In fact, one can approximate the problem

$$\begin{aligned}\Delta u &= 0 && \text{in } \Omega, \\ u &= g_1 && \text{on } \Gamma_1 \subset \partial\Omega, \\ \frac{\partial u}{\partial \nu} &= g_2 && \text{on } \Gamma_2 = \partial\Omega \setminus \Gamma_1\end{aligned}\tag{274}$$

by the finite-dimensional problems above. In some sense, the PDE problem (274) is a continuous version of the circuit problem above, with a particular arrangement of the resistors. Consider a fine mesh (made of out of a conducting wire with some non-zero resistance) covering Ω . The nodes will be identified with the points where the wires of the mesh cross. The wires connecting the nodes will be identified with the resistors. The points of the mesh inside Ω will correspond to nodes \mathcal{F} , the meshpoints at Γ_1 will correspond to nodes \mathcal{D} , and the nodes at Γ_2 will correspond to nodes \mathcal{N} . The voltages V_i , $i \in \mathcal{D}$ correspond to g_1 and the currents I_i , $i \in \mathcal{N}$ correspond to g_2 . The analogue of the function P will be the functional

$$J(u) = \int_{\Omega} \frac{1}{2} |\nabla u|^2 - \int_{\partial\Omega} g_2 u,\tag{275}$$

which should be minimized over all (sufficiently regular) functions with $u = g_1$ on Γ_1 .

One can see easily that the finite-dimensional problem with the mesh will have a solution – the minimizer of P . In the case $\Gamma_2 = \partial\Omega$, we will have to have a compatibility condition $\sum_{i \in \mathcal{N}} I_i = 0$, corresponding to $\int_{\partial\Omega} g_2 = 0$, which is necessary when $\Gamma_2 = \partial\Omega$.

One way to prove the existence of (274) would be to show that the solutions of the finite-dimensional problems (with suitable normalization of the boundary currents) converge to a suitable limit as the mesh size approaches zero, and the limit is a solution of (274). This is doable, and provides one way of proving existence results for PDEs. Another method is to work directly in the infinite-dimensional setting, and this is what we will do. The results about the solutions obtained independently of particular approximations are in fact useful for the study of the convergence of the approximations to the PDE solution.

Consider a positive definite symmetric $n \times n$ matrix A and the corresponding quadratic form

$$P(x) = \frac{1}{2}(Ax, x).\tag{276}$$

Let us set for each $y \in \mathbb{R}^n$

$$Q(y) = \inf_x P(x) - yx.\tag{277}$$

For each $y \in \mathbb{R}^n$ the minimum of $P(x) - yx$ is attained for x which solves

$$Ax = y.\tag{278}$$

As an exercise, you can prove that Q is a quadratic form given by $-A^{-1}$, the negative inverse matrix of A . For each y the solution of $P'(x) = y$ is given by $x = -Q'(y)$. For this it is not so important that P be quadratic, the main assumption needed is that P be convex, although one needs a little more if we demand that $x \rightarrow P'(x)$ covers the whole \mathbb{R}^n . The function Q is then defined in the same way as above, and will be concave. The equation $P'(x) = y$ will be inverted by $y = -Q'(x)$. The function Q is called the Legendre transformation of P . (Sometimes the Legendre transformation is defined by $Q(y) = \sup_x (xy - P(x))$. With this definition it is convex and the inversion of $P'(x) = y$ is $x = Q'(y)$.) Legendre transformation is interesting even in dimension 1, and it is a good exercise to work out the details of the statements above in the 1d case.

Lecture 21, 10/27/2010

Let us consider the problem

$$\begin{aligned}\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{at } \partial\Omega,\end{aligned}\tag{279}$$

where f is a “sufficiently regular” function, and Ω is a smooth bounded domain. The condition of being “sufficiently regular” is somewhat ambiguous – its meaning can depend on the context. For example, we will see that for some purposes $f \in L^2(\Omega)$ is already “sufficiently regular”. For other aspects it is good to assume that f is Hölder continuous, i. e. $f \in C^{0,\alpha}$. If we look at higher derivatives of u , it is natural to have assumptions about derivatives of f . Each aspect of the problem has its natural assumptions on f . We say that f is sufficiently regular if we expect that some assumptions on f will be needed for our investigation, but we are leaving open for the moment what exactly those assumptions are. Of course, eventually we have to make some precise statement, and say exactly what the assumptions on f are.

Instead of (279) we could choose some other boundary conditions, such as those in (274) (and we will see later that it is easy to incorporate them), but for now we wish to illustrate the main idea of the Direct Method in a simple setting, and problem (279) seems to be well-suited for that.

Based on the last lecture, we know that the following functional will play an important rôle.

$$J(u) = \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 - fu \right) dx .\tag{280}$$

Let X denote some space of functions $u: \Omega \rightarrow \mathcal{R}$ which vanish at $\partial\Omega$ and are sufficiently regular in the sense above. These assumptions are not very precise yet, we will make them precise after it becomes clearer what is really important for our approach. For now we can say that we wish that if $u \in X$ then

- The functional (280) is well-defined, and
- $u \rightarrow 0$ in some sense as we approach $\partial\Omega$. The strongest sense would be to assume that u has a continuous extension to $\bar{\Omega}$ which vanishes at $\partial\Omega$. However, we will see that one can work with weaker notions, which are not defined by point-wise limits.

We would like to minimize J over X . There are several issues which we have to clarify.

A. Is J bounded from below on X ? (Boundedness from below is clearly a necessary condition for minimization.)

B. Is the infimum of J over X attained on some element $\bar{u} \in X$?

Note that once we know that there exists $\bar{u} \in X$ such that

$$J(\bar{u}) \leq J(u) \quad \text{for all } u \in X,\tag{281}$$

then we know that for each smooth, compactly supported $\varphi: \Omega \rightarrow \mathbb{R}$ the function

$$t \rightarrow J(\bar{u} + t\varphi), \quad t \in \mathbb{R} \tag{282}$$

attains its minimal value at $t = 0$. We note that $t \rightarrow J(\bar{u} + t\varphi)$ is a quadratic function of t , and since it attains minimum at $t = 0$, we must have

$$0 = \frac{d}{dt} \Big|_{t=0} J(\bar{u} + t\varphi) = \int_{\Omega} (\nabla \bar{u} \nabla \varphi - f\varphi) dx. \tag{283}$$

If we can integrate by part and put the derivatives on \bar{u} , we obtain

$$\int_{\Omega} (-\Delta \bar{u} - f) \varphi dx = 0 \quad \text{for each } \varphi: \Omega \rightarrow \mathbb{R} \text{ smooth and comp. supported in } \Omega.$$

This means that $-\Delta \bar{u} = f$, at least if \bar{u} is sufficiently regular, which we however do not know at this point.

As the function φ is smooth, we can integrate by parts and put the derivatives on φ without any problems to obtain

$$\int_{\Omega} (-\bar{u} \Delta \varphi - f\varphi) dx = 0. \tag{284}$$

We recall this is exactly the definition from lecture 5 that $-\Delta \bar{u} = f$ is satisfied weakly. Note that Weyl's Lemma which we proved in lecture 4 implies that in the interior of Ω the function \bar{u} has the same degree of smoothness as the potential $K_0 * f$ (where we extend f by zero outside Ω). This is because $\Delta(\bar{u} - K_0 * f) = 0$ weakly and therefore it is smooth in Ω by Weyl's Lemma.

Although it is worth mentioning these connections to some of the previous material we covered, the passage to the weak solution in the sense of lecture 4, and the use of Weyl's Lemma is somewhat unnatural in this situation, as the methods we used in lectures 4,5 are quite different from the Direct Method. It is much more natural to prove regularity of \bar{u} within the Direct Method, without resorting to the use of the potentials. This is what we will soon do. You will see that the proof via the Direct Method will have the advantage of covering the case of variable coefficients practically without any extra work.

In any case, we see that it is not so hard to justify that the minimizer \bar{u} should satisfy the equation $-\Delta \bar{u} = f$ in Ω , at least in some weak sense. How is it with the boundary condition? The condition $u|_{\partial\Omega} = 0$ is a part of the "definition" of X above, and therefore \bar{u} should vanish at the boundary by definition. However, this is where we have to do some work and make sure that the condition $u|_{\partial\Omega} = 0$ will "survive" the minimization procedure. We will see that this is indeed the case, but it should be emphasized that it is not automatic. It is not difficult to give examples of boundary conditions which will not "survive" the minimization procedure.⁹⁷ In fact, we will see that if we replace Ω by $\Omega \setminus \{a\}$ for some point

⁹⁷Such examples were already known to Weierstrass, who constructed them in order to show that – in comparison with the finite-dimensional situation – the minimization procedure in function spaces can have many more subtleties.

$a \in \Omega$ ⁹⁸ and try to impose the condition $u(a) = 0$ at this new part of the boundary, then typically this extra boundary condition will not survive the minimization, except in dimension $n = 1$.

We will now start addressing the points A, B above.

Lemma (weak Poincare inequality)

Assume Ω is contained in a cube of size A . Then for each C^1 function $u: \Omega \rightarrow \mathbb{R}$ vanishing at $\partial\Omega$ we have

$$\int_{\Omega} |u|^2 dx \leq A^2 \int_{\Omega} |\nabla u|^2 dx. \quad (285)$$

Proof: Choose coordinates so that the cube is $[0, A]^n$. Extend u by 0 outside Ω . For each $x \in \Omega \subset [0, A]^n$ we have, by Cauchy-Schwartz inequality,

$$|u(x)|^2 = \left(\int_0^{x_1} \frac{\partial u}{\partial x_1}(s, x_2, \dots, x_n) ds \right)^2 \leq A \int_0^A \left| \frac{\partial u}{\partial x_1}(s, x_2, \dots, x_n) \right|^2 ds. \quad (286)$$

Integrating this inequality over x , we get (285).

It is now easy to see that J is bounded from below:

$$J(u) = \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 - fu \right) dx \geq \frac{1}{2} \|\nabla u\|_{L^2}^2 - \|f\|_{L^2} \|u\|_{L^2} \geq \quad (287)$$

$$\frac{1}{2} \|\nabla u\|_{L^2}^2 - A \|f\|_{L^2} \|\nabla u\|_{L^2} \geq -\frac{1}{2} A^2 \|f\|_{L^2}^2. \quad (288)$$

Let

$$m = \inf_{u \in X} J(u), \quad (289)$$

and let u^j , $j = 1, 2, \dots$ be a minimizing sequence for J , in the sense that

$$J(u^j) \searrow m \quad j \rightarrow \infty. \quad (290)$$

One can see from (287) that the sequence ∇u^j is bounded in $L^2(\Omega)$.⁹⁹ In finite dimension we could now use compactness and choose a converging subsequence to complete the proof. This argument cannot be used in our infinite-dimensional function space. However, the structure of the functional J gives us what is needed:

Lemma

With the notation introduced above, let u^j be a minimizing sequence for J . Then ∇u^j is a Cauchy sequence in L^2 .

⁹⁸The new domain $\Omega \setminus \{a\}$ will not have smooth boundary, of course.

⁹⁹In this case the function space is the space of vector-valued functions, but we still denote it in the same way as the corresponding space of the scalar functions. As you will see, there is no danger of confusion from this.

Proof: Heuristically, the reason is the following: if f is a uniformly convex function on \mathbb{R} , $m = \inf_{t \in \mathbb{R}} f(t)$ and $f(t_1), f(t_2)$ are both close to m , then t_1, t_2 must be close to each other. The heuristics is justified by the following calculation. Assume

$$m \leq J(u_1) \leq m', \quad m \leq J(u_2) \leq m'. \quad (291)$$

One checks easily that

$$J\left(\frac{u_1 + u_2}{2}\right) + \frac{1}{2} \left\| \frac{\nabla u_2 - \nabla u_1}{2} \right\|^2 = \frac{1}{2} [J(u_1) + J(u_2)]. \quad (292)$$

Hence

$$\frac{1}{2} \left\| \frac{\nabla u_2 - \nabla u_1}{2} \right\|^2 \leq \frac{1}{2} [J(u_1) + J(u_2)] - J\left(\frac{u_1 + u_2}{2}\right) \leq m' - m, \quad (293)$$

and the statement easily follows.

We have established that the sequence ∇u^j is Cauchy in the space $L^2(\Omega)$. The Lebesgue theory now implies that ∇u^j must converge in L^2 to some limit, let us call it $w = (w_1, \dots, w_n)$. We expect that $w = \nabla \bar{u}$ for some \bar{u} and that the function \bar{u} will be the minimizer of our functional. Note that the existence of the limit w would be hard to establish without Lebesgue integration. Next time we introduce function spaces in which the limiting function \bar{u} will live. This will settle the still unanswered question what exactly the space X is.

Lecture 22, 10/29/2010

Recall that Ω is a bounded smooth domain. Let us set

$$\mathcal{D}(\Omega) = \{\varphi: \Omega \rightarrow \mathbb{R}; \quad \varphi \text{ is smooth and compactly supported in } \Omega.\} \quad (294)$$

We consider the norm $\|\cdot\|_{H_0^1}$ on $\mathcal{D}(\Omega)$ defined by

$$\|\varphi\|_{H_0^1}^2 = \int_{\Omega} |\nabla\varphi|^2 dx. \quad (295)$$

By the weak Poincaré inequality (285) we know that

$$\|\varphi\|_{L^2} \leq A \|\nabla\varphi\|_{H_0^1}. \quad (296)$$

This means that the completion of $\mathcal{D}(\Omega)$ in norm $\|\cdot\|_{H_0^1}$ can be considered as a certain subset of the space $L^2(\Omega)$. (We know that $\mathcal{D}(\Omega)$ is dense in the space $L^2(\Omega)$.) In particular, every set in the completion can be identified with an L^2 -function. We can therefore say that the completion is a subset of $L^2(\Omega)$. We set

$$H_0^1(\Omega) = \text{the completion of } \mathcal{D}(\Omega) \text{ in the norm } \|\cdot\|_{H_0^1}. \quad (297)$$

If $v \in H_0^1(\Omega)$ we have a sequence $v^j \in \mathcal{D}(\Omega)$ converging to v in $L^2(\Omega)$ such that the gradient ∇v^j converge in $L^2(\Omega)$ to some vector-valued function $w = (w_1, \dots, w_n)$. It should be the case that

$$w = \nabla v, \quad (298)$$

but in which sense? Easy examples show that $H_0^1(\Omega)$ is not contained in $\mathcal{C}^1(\Omega)$. The following definition addresses this point.

Definition

Let u, u_1, \dots, u_n be locally integrable functions in an open set Ω . We say that

$$\frac{\partial u}{\partial x_k} = u_k \quad (299)$$

in the sense of distributions if

$$\int_{\Omega} -u \frac{\partial \varphi}{\partial x_k} dx = \int_{\Omega} u_k \varphi dx \quad (300)$$

for each $\varphi \in \mathcal{D}(\Omega)$. In the above situation with $v^j \rightarrow v$ and $\nabla v^j = w^j \rightarrow w$ in $L^2(\Omega)$, it is easy to check that $\nabla v = w$ in the sense of distributions.

An important point now is that for the derivative defined in the sense of distributions, many of the basic rules we use for the point-wise derivative remain true for the distributional derivative. For example, it commutes with mollification. More precisely, if $\phi_\varepsilon(x) = \frac{1}{\varepsilon^n} \phi(\frac{x}{\varepsilon})$ and $\Omega_\varepsilon = \{x \in \Omega; \text{dist}(x, \partial\Omega) > \varepsilon\}$, then

$$\frac{\partial}{\partial x_j} (\phi_\varepsilon * v) = \phi_\varepsilon * \left(\frac{\partial}{\partial x_j} v \right) \quad (301)$$

in Ω_ε .

We leave the proof of (301) to the reader as an easy exercise. The main point is that the new definition of the derivative – just as the classical one – commutes with the shifts $v \rightarrow \tau_y v: x \rightarrow v(x - y)$ and, is linear in v .

One can use (301) to show that the derivative in the sense of distributions has many of the properties we usually associate with the derivative. For example, one has

If v is locally integrable in Ω and $\nabla v = 0$ in the sense of distributions, then v is constant in Ω .

Proof: Let $v_\varepsilon = \phi_\varepsilon * v$ and let Ω_ε be defined as above. Then

$$\nabla v_\varepsilon = \nabla(\phi_\varepsilon * v) = \phi_\varepsilon * (\nabla v) = 0 \quad \text{in } \Omega_\varepsilon,$$

and hence v_ε is constant in Ω_ε for each $\varepsilon > 0$. As $v_\varepsilon \rightarrow v$ in $L^1_{\text{loc}}(\Omega)$, we see that v will be a limit of v_ε , and hence it will be constant on Ω_ε .

Remark: The above proof can be interpreted in the following way. Assume first that we are in dimension $n = 1$ for simplicity and $\Omega = (a, b)$. Let $v \in L^1_{\text{loc}}(a, b)$, and let $x_1, x_2 \in (a, b)$, $x_1 \leq x_2$. Assume $\varepsilon < \min(x_1 - a, b - x_2)$. The integrals $\int_a^b v(x)\phi_\varepsilon(x - x_j) dx$, $j = 1, 2$, converge respectively to $v(x_1)$ and $v(x_2)$ at the Lebesgue points of v . We note that the function $x \rightarrow \phi_\varepsilon(x - x_1) - \phi_\varepsilon(x - x_2)$ is a derivative of a smooth compactly supported function $\psi: (a, b) \rightarrow \mathbb{R}$ defined simply by $\psi(x) = \int_a^x \phi_\varepsilon(x' - x_1) - \phi_\varepsilon(x' - x_2) dx'$. Hence we must have $\int_a^b v(x)\phi_\varepsilon(x - x_1) dx = \int_a^b v(x)\phi_\varepsilon(x - x_2) dx$ and in the limit $\varepsilon \rightarrow 0$ we see that $v(x_1) = v(x_2)$.

The main point here is that the condition $\int_a^b -v\psi_x = 0$ for each $\psi \in \mathcal{D}(a, b)$ is equivalent to saying that $\int_a^b v\eta = 0$ for each $\eta \in \mathcal{D}(a, b)$ with $\int \eta = 0$.

As an exercise, you can check that similar interpretation also works in higher dimensions. Note first that it is enough to do the proof when the segment $[x_1, x_2]$ belongs to Ω . One can then integrate on each line parallel to $[x_1, x_2]$ in the same way as above to obtain $\psi \in \mathcal{D}(\Omega)$ such that $b\nabla\psi(x) = \phi_\varepsilon(x - x_1) - \phi_\varepsilon(x - x_2)$, where $b = (x_2 - x_1)/|x_2 - x_1|$.¹⁰⁰

Returning back to our space H^1_0 and the question in which sense we have (298), it is easy to see that we have $w = \nabla v$ in the sense of distributions.

If $v \in H^1_0(\Omega)$, then $\nabla v = w$ in the sense of distributions for some $w = (w_1, \dots, w_n) \in L^2(\Omega)$. If $v = \lim \nabla\varphi_j$ in the norm $\|\cdot\|_{H^1_0}$ with $\varphi_j \in \mathcal{D}(\Omega)$, then $w = \lim \nabla\varphi_j$ in $L^2(\Omega)$.

¹⁰⁰The most natural approach here would be to show that each $\eta \in \mathcal{D}(\Omega)$ with $\int_\Omega \eta = 0$ can be written as $\eta = \text{div } \psi$ for some $\psi = (\psi_1, \dots, \psi_n) \in \mathcal{D}(\Omega)$. This is true and the above method more or less shows that, but some details would still have to be filled in.

The proof is immediate by using the standard convergence results from Lebesgue theory in our definitions.¹⁰¹

Let us now consider $v \in L^2(\Omega)$ such that $\nabla v = w \in L^2(\Omega)$ is the sense of distributions. When do we have $v \in H_0^1(\Omega)$? If we avoid questions about the behavior of v near the boundary $\partial\Omega$, the answer is simple:

If $v \in L^2(\Omega)$ is compactly supported in Ω and $\nabla v = w \in L^2(\Omega)$ in the sense of distributions, then $v \in H_0^1(\Omega)$.

Proof: Let us take $v_\varepsilon = \phi_\varepsilon * v$. These functions are well-defined and compactly supported in Ω for sufficiently small ε . We have $\nabla v_\varepsilon = \phi_\varepsilon * w$. As $\phi_\varepsilon * w \rightarrow w$ is L^2 as $\varepsilon \rightarrow 0$, we see that $v \in H_0^1(\Omega)$.

When we do not avoid the boundary, the answer the conditions characterizing $v \in H_0^1(\Omega)$ are still simple and natural, but one has to work a bit more. We will deal with that situation next time.

¹⁰¹As is the case with many other results about distributions, the proofs are simple. The non-trivial part of the theory is in choosing the right definitions. There are many ways to generalize the point-wise definitions of derivatives, but most of them are not suitable for PDEs. The definition above (which is a special case of a more general definition due to L. Schwartz) gives us all the flexibility we need, while preserving the basic properties we expect from the derivatives, e. g. that a function with vanishing derivatives must be constant.

Lecture 23, 11/1/2010

In addition to the space $H_0^1(\Omega)$ we have introduced last time, it is natural to introduce the following spaces.

$$W^{1,2}(\Omega) = \{v \in L^2(\Omega), \nabla v = w \text{ in distributions for some } w = (w_1, \dots, w_n) \in L^2(\Omega)\},$$

with the norm defined by

$$\|v\|_{W^{1,2}}^2 = \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2. \quad (302)$$

Let us denote by $\mathcal{E}(\Omega)$ the space of restrictions to Ω of smooth functions defined in some neighborhood of Ω . We define

$$H^1(\Omega) = \text{closure of } \mathcal{E}(\Omega) \text{ in } W^{1,2}(\Omega). \quad (303)$$

We clearly have

$$H_0^1(\Omega) \subset H^1(\Omega) \subset W^{1,2}(\Omega). \quad (304)$$

Our first goal is to show that

$$H^1(\Omega) = W^{1,2}(\Omega) \quad (305)$$

for bounded regular domains Ω .

We first describe an elementary localization technique which is useful for this purpose and also in other situations.

Let $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_m$ be open sets covering $\bar{\Omega}$, and assume $\mathcal{U}_j \subset \bar{\mathcal{U}}_j \subset \mathcal{O}_j, j = 1, \dots, m$ be open sets still covering Ω . Consider non-negative smooth functions $\tilde{\phi}_j$ which are compactly supported in \mathcal{O}_j with $\tilde{\phi}_j > 0$ on $\bar{\mathcal{U}}_j$, and set

$$\phi_j = \frac{\tilde{\phi}_j}{\tilde{\phi}_1 + \dots + \tilde{\phi}_m}. \quad (306)$$

The functions ϕ_j are obviously well-defined, smooth, and non-negative in some neighborhood \mathcal{O} of $\bar{\Omega}$ and

$$\phi_1 + \dots + \phi_m = 1 \quad \text{in } \mathcal{O}. \quad (307)$$

We mention one more step one can take, although we will not really need it for now. We can take a smooth function η compactly supported in \mathcal{O} with $\eta = 1$ in some neighborhood of $\bar{\Omega}$ and replace the functions ϕ_j by $\eta\phi_j$. With this new definition, the functions ϕ_j are smooth and globally defined in \mathbb{R}^n , their support is contained in \mathcal{O}_j , and one has (307) with \mathcal{O} replaced by a smaller neighborhood of $\bar{\Omega}$.

The system of functions ϕ_1, \dots, ϕ_m we have constructed is often called the partition the unity for $\bar{\Omega}$.

If $u \in W^{1,2}(\Omega)$ and a smooth function φ defined in a neighborhood of $\bar{\Omega}$, the function φu also belong to $W^{1,2}(\Omega)$, and the distributional derivative of φu is given by the usual Leibnitz rule:

$$\nabla(\varphi u) = (\nabla\varphi)u + \varphi\nabla u. \quad (308)$$

This is an immediate consequence of the definitions and the Leibnitz rule for smooth functions.¹⁰²

For any function $u \in W^{1,2}(\Omega)$ we can write

$$u = u\phi_1 + \cdots + u\phi_m = u_1 + \cdots + u_m, \quad (309)$$

and – as we have just seen – we have $u_j \in W^{1,2}(\Omega)$, $j = 1, 2, \dots, m$.

The support of the functions u_j is contained in the set \mathcal{O}_j of the original cover of $\bar{\Omega}$. By taking \mathcal{O}_j to be small balls, we can achieve that each u_j is either compactly supported in Ω , or is supported in a neighborhood of a boundary point, such that the boundary $\partial\Omega$ in that neighborhood is very close to being flat. This is not the case if the boundary is merely Lipschitz.¹⁰³ However, even for Lipschitz functions this localization can achieve that each u_j which does not vanish close to $\partial\Omega$ has all its support near $\partial\Omega$ and in an area where $\partial\Omega$ is described in a single coordinate frame by a graph of a Lipschitz function.

Let us now go back to the proof of $H^1(\Omega) = W^{1,2}(\Omega)$. The only non-trivial part is to show that each $u \in W^{1,2}(\Omega)$ can be approximated in the $W^{1,2}$ -norm by functions smooth in $\bar{\Omega}$. It is clear that for this it is enough to approximate each function u_j in the decomposition (309). If the support of u_j is contained in Ω , we can approximate just by mollification, as we did in the last lecture. It remains to deal with the case when the support of u_j intersects the boundary. Assume that u_j is supported in a small area near $\partial\Omega$, and that $\partial\Omega$ is described by $x_n = g(x')$, with $x' = (x_1, \dots, x_{n-1})$ and g Lipschitz, and Ω is locally described by $x_n > g(x')$. In this picture the approximation of u_j is simple. The only obstacle to straightforward mollification is the possible interference coming from the boundary, when we would need the values of u_j outside of Ω for the mollification. This is however easily fixed by slightly shifting u_j in the negative direction of the x_n axis: we replace u_j by u_j^δ with $u_j^\delta(x', x_n) = u_j(x', x_n + \delta)$ for some small $\delta > 0$. Now we can mollify the shifted function u_j^δ . We let $u_j^{\delta,\varepsilon} = \phi_\varepsilon * u_j^\delta$, where ϕ_ε is the usual mollifier. It is easily seen that after we have chosen δ , we can choose a small $\varepsilon > 0$ so that all the values of $u_j^{\delta,\varepsilon}$ in Ω are well-defined only in terms of u_j^δ , without a need for extending it. Moreover, the restriction of $u_j^{\delta,\varepsilon}$ to Ω is close to the restriction of u_j^δ to Ω in the $W^{1,2}$ -norm. As u_j^δ is close to u_j in $W^{1,2}$ -norm for small δ , the proof is easily finished.

¹⁰²We notice again how well the definition of the distributional derivative works. The proof of the Leibnitz rule is more or less automatic.

¹⁰³By definition, this means that $\partial\Omega$ is locally described by a graph of a Lipschitz function. We recall that a function g is Lipschitz if $|g(x_1) - g(x_2)| \leq L|x_1 - x_2|$ for some constant L for all x_1, x_2 in the domain of the function.

Lecture 24, 11/3/2010

Recall that our goal is to minimize functional (280) over a suitable space X of functions on Ω for which J is well-defined and which vanish at $\partial\Omega$. From what we have seen so far it is clear what X should be. The minimizing sequence of J will always be Cauchy in $H_0^1(\Omega)$ and hence it is natural to take

$$X = H_0^1(\Omega). \quad (310)$$

With this choice any minimizing sequence u^j for J will converge to an element $\bar{u} \in H_0^1(\Omega)$, and we have seen in lecture 21 that \bar{u} will satisfy (283) and (284). So \bar{u} satisfies the right equation, at least weakly. We now address the question if \bar{u} also satisfies the correct boundary condition, at least in some weak sense at first. (Later we will see that for smooth data our minimizer will be smooth in $\bar{\Omega}$ and vanish at $\partial\Omega$.)

It may be useful to look first at some simple examples of (irregular) domains for which the boundary condition is not preserved everywhere by the minimization procedure, i. e. the minimizer \bar{u} does not vanish everywhere where the members of the minimizing sequence u^j do.

Let take a point $a \in \Omega$ and consider the domain $\Omega_a = \Omega \setminus \{a\}$ for $n \geq 2$. We claim that $H_0^1(\Omega \setminus \{a\}) = H_0^1(\Omega)$, unless $n = 1$. This implies that if we minimize our functional over smooth functions in Ω_a with the boundary condition that $u = 0$ in $\partial\Omega_a$ in dimension $n \geq 2$, the limit \bar{u}_a of any minimizing sequence will be the same as if we do not take the boundary condition $u(a) = 0$ into account. Therefore, unless the minimizer \bar{u} of the problem in Ω happens to satisfy $\bar{u}(a) = 0$, the problem in Ω_a does not have a minimizer, in the sense that no sufficiently regular function with $u(a) = 0$ can be a minimizer. If we take a smooth minimizing sequence u^j of J in Ω_a which will satisfy $u^j(a) = 0$, its limit \bar{u} may not satisfy $\bar{u}(a) = 0$. Again, all this is true in dimensions $n \geq 2$. For $n = 1$ the condition $u(a) = 1$ will be preserved under convergence in $H_0^1(\Omega)$. (It is a good exercise to prove this.)

The examples are especially easy in dimensions $n \geq 3$. We can clearly assume $a = 0$ without loss of generality. Let $\psi: \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function supported in the unit ball B_1 such that $\psi = 1$ in some neighborhood of 0, and for $\varepsilon > 0$ set

$$\psi_\varepsilon(x) = \psi\left(\frac{x}{\varepsilon}\right). \quad (311)$$

The support of ψ_ε is contained in B_ε . We still have $\psi_\varepsilon = 1$ in some (even smaller) neighborhood of 0. We can calculate

$$\int_{\mathbb{R}^n} |\nabla \psi_\varepsilon|^2 dx = \varepsilon^{n-2} \int_{\mathbb{R}^n} |\nabla \psi|^2 dx \rightarrow 0, \quad n \geq 3. \quad (312)$$

For any $u \in \mathcal{D}(\Omega)$ we can consider the function $\tilde{u} = u(1 - \psi_\varepsilon)$. We have

$$\|\tilde{u} - u\|_{H_0^1} = \|\nabla u \psi_\varepsilon + u \nabla \psi_\varepsilon\|_{L^2} \leq \|\nabla u \psi_\varepsilon\|_{L^2} + \|u \nabla \psi_\varepsilon\|_{L^2} \rightarrow 0, \quad \varepsilon \rightarrow 0. \quad (313)$$

We any $u \in \mathcal{D}(\Omega)$ can be approximate as closely the H_0^1 -norm as we wish by $\tilde{u} \in \mathcal{D}(\Omega \setminus \{a\})$. We have shown the case $n \geq 3$ of the following

Proposition

$\mathcal{D}(\Omega \setminus \{a\})$ is dense in $H_0^1(\Omega)$ when $n \geq 2$.

Proof: Above we proved the case $n \geq 3$. It remains to do the proof for $n = 2$. The idea is the same, except we have to be more careful with the choice of ψ_ε . Let us consider a smooth non-negative radial function $\phi: R^2 \rightarrow R$ supported in $B_1 \setminus B_{\frac{1}{2}}$ with $\int \phi = 1$. Set $\phi_\delta = \frac{1}{\delta^2} \phi(\frac{x}{\delta})$ and $K_\delta(x) = \phi_\delta * K_0$, where $K_0(x) = \frac{1}{2\pi} \log|x|$. Note that K_δ is constant near 0 and coincides with K_0 outside a ball of radius δ . To prove the proposition, assume without loss of generality that $a = 0$ and that $B_r \subset \Omega$. Let η be a cut-off function such that is $\eta \equiv 1$ in $B_{\frac{r}{4}}$ and support $\eta \subset B_{\frac{r}{2}}$. Finally, set

$$\psi_\delta(x) = \eta(x) \frac{K_\delta(x)}{K_\delta(0)}. \quad (314)$$

The function ψ_δ is supported in $B_{\frac{r}{2}}$, is $\equiv 1$ in $B_{\frac{\delta}{2}}$ for $\delta < \frac{r}{4}$, and a simple direct calculation shows that $\|\nabla \psi_\delta\|_{L^2} \rightarrow 0$ as $\delta \rightarrow 0$. We can now proceed with the proof in the same way as in the case $n = 3$ above, replacing ψ_ε by ψ_δ .

As an exercise you can show that for $n \geq 2$ we also have

$$W^{1,2}(\Omega) = W^{1,2}(\Omega \setminus \{a\}) \text{ and } H^1(\Omega) = H^1(\Omega \setminus \{a\}). \quad (315)$$

The functions (314) and (311) can be used to prove theorems about removable singularities for harmonic functions, which are closely related to the above proposition. For example, one has

Lemma¹⁰⁴

Assume u is a bounded harmonic function in $\Omega \setminus \{a\}$. Then u can be continuously extended to Ω and the extended function is harmonic in Ω .

Proof (a sketch): We assume that $a = 0$. By Weyl's lemma (lecture 4), it is enough to show that

$$\int_{\Omega} u \Delta \varphi = 0 \quad (316)$$

for each $\varphi \in \mathcal{D}(\Omega)$. Our assumption implies that (316) holds for $\varphi \in \mathcal{D}(\Omega \setminus \{a\})$. Let $\varphi \in \mathcal{D}(\Omega)$ and let ψ_δ be as in (314) (when $n=2$). We know that

$$\int_{\Omega} u \Delta(\varphi(1 - \psi_\delta)) = 0 \quad (317)$$

¹⁰⁴The assumptions of the lemma can be weakened in various ways. For example, instead of assuming that u is bounded, we can make a weaker assumption that $|u|$ grows slower than $K_0(x - a)$ (the fundamental solution) as $x \rightarrow a$, or that $u \in L^p$ for $p \geq \frac{n}{n-2}$ (when $n \geq 3$), etc. Our point here is not to get an optimal result, but to illustrate the effect in a simple situation.

We see that it is enough to show that

$$\lim_{\delta \rightarrow 0} \int_{\Omega} u \Delta(\varphi\psi_{\delta}) = 0. \quad (318)$$

We write

$$\Delta(\varphi\psi_{\delta}) = (\Delta\varphi)\psi_{\delta} + 2\nabla\varphi\nabla\psi_{\delta} + \varphi\Delta\psi_{\delta} \quad (319)$$

and in (318) we estimate the contribution from these three terms one-by-one, using that u is bounded. (In fact, u growing slightly slower than the fundamental solution at a is sufficient.) The most dangerous term is the third one, but we note that $\Delta\psi_{\delta}$ is large only in B_{δ} , with $\int_{B_{\delta}} |\Delta\psi_{\delta}| \leq \frac{1}{K_{\delta}(0)}$.

All which we did above for the one-point set $\{a\}$ can be investigated for more general compact sets. It is immediately clear that the one-point sets can be replaced by finite sets. Also, in dimension n we can replace the one-point sets by $(n-2)$ -dimensional surfaces. A necessary and sufficient condition for a compact set $K \subset \Omega$ to be negligible in the same sense as the one-point sets above is that the *capacity* of the set K be zero. This is equivalent to

$$\inf\left\{ \int_{\Omega} |\nabla\varphi|^2, \varphi \in \mathcal{D}(\Omega), \varphi|_K \geq 1 \right\} = 0. \quad (320)$$

We have seen that in some cases, when the boundary is not smooth, we can lose the boundary condition at some part of the boundary in the process of minimization. We will now show that “in the mean” this cannot happen when the boundary is regular. Later we will still improve this when we prove boundary regularity, but for now our goal is to show that the boundary condition $u|_{\partial\Omega}$ is preserved in a L^2 -sense.

Let us recall that the space $\mathcal{E}(\Omega)$ of restrictions to Ω of smooth functions defined in a neighborhood of $\bar{\Omega}$ is dense in $W^{1,2}(\Omega)$ when the boundary of Ω is regular. (For example, Lipschitz boundary is sufficient.)

For $u \in \mathcal{E}(\Omega)$ the boundary values $u|_{\partial\Omega}$ are of course well-defined. We will show that for each $u \in \mathcal{E}(\Omega)$ we have

$$\|u|_{\partial\Omega}\|_{L^2(\partial\Omega)} \leq C\|u\|_{W^{1,2}(\Omega)}, \quad (321)$$

with C independent of u . This means that the map $u \rightarrow u|_{\partial\Omega}$, first defined on $\mathcal{E}(\Omega)$, extends continuously to $W^{1,2}(\Omega)$. Therefore for each $u \in W^{1,2}(\Omega)$ the restriction $u|_{\partial\Omega}$ is well-defined as an L^2 -function on $\partial\Omega$. The function $u|_{\partial\Omega}$ is sometimes called the trace of the function $u \in W^{1,2}(\Omega)$ and the linear map $u \rightarrow u|_{\partial\Omega}$ defined on $W^{1,2}(\Omega)$ is sometimes called the trace operator. For $u \in \mathcal{D}(\Omega)$ one of course has $u|_{\partial\Omega} = 0$ and therefore $u|_{\partial\Omega} = 0$ for any $u \in H_0^1(\Omega)$.¹⁰⁵ We

¹⁰⁵In fact, we will see that when $u \in H_0^1(\Omega)$, then “on average” $u \rightarrow 0$ as we approach the boundary. This will be made more precise.

will see later that this condition characterizes $H_0^1(\Omega)$ as a subset of $W^{1,2}(\Omega)$, at least when Ω is regular (e. g. Lipschitz).

For the proofs of the properties of $u \in W^{1,2}(\Omega)$ near the boundary $\partial\Omega$ we can again use localization and partition of unity, similarly to what we did in the last lecture. We can assume without loss of generality that u is a neighborhood of some point on the boundary $\partial\Omega$ in which the boundary is described by $x_n = g(x_1, \dots, x_{n-1}) = g(x')$ where g is a Lipschitz function defined in some open ball $B_r^{n-1} \subset \mathbb{R}^{n-1}$. We assume that, locally near the graph of g , our domain Ω is described by $x_n > g(x')$. Moreover, we assume that for some $b > \sup_{B_r} g$ the set $\{x = (x', x_n), x' \in B_r^{n-1}, g(x') < x_n < b\}$ is contained in Ω and that the support of u is contained in $\{x = (x', x_n), x' \in B_r^{n-1}, g(x') \leq x_n < b\}$.

For a smooth u we can write and hence

$$|u(x', g(x'))|^2 \leq (b - g(x')) \int_{g(x')}^b \left(\frac{\partial u(x', s)}{\partial x_n} \right)^2 ds. \quad (322)$$

Integrating over B_r^{n-1} , we obtain (321).¹⁰⁶

Remarks

1. A simple modification of the proof above shows that in fact

$$\|u|_{\partial\Omega}\|_{L^2(\partial\Omega)} \leq C \|u\|_{W^{1,2}(\Omega)}^{1/2} \|u\|_{L^2(\Omega)}^{1/2}. \quad (323)$$

This form of the trace inequality will be useful later.

2. When dealing with the boundary, sometimes it is efficient to change coordinates so that in the new coordinates the boundary becomes flat. In the picture above that would correspond to $g(x') = \text{const}$. It is not hard to check that in the context of $W^{1,2}(\Omega)$ the changes of variable $x = h(\tilde{x})$ where h is a Lipschitz homeomorphism with a Lipschitz inverse can be used, and those are sufficient to “straighten up” (locally) any Lipschitz boundary. If our boundary is smoother, the map h can be more regular, of course.

3. Let Σ_δ be the surface given by $x_n = g(x') + \delta$. The function $u|_{\Sigma_\delta}$ can be identified with a function on $\Sigma_0 \subset \partial\Omega$ in the obvious way. We can now consider the map

$$u \rightarrow u|_{\Sigma_\delta} \quad (324)$$

as a map from $W^{1,2}(\Omega)$ to $L^2(\Sigma_\delta) \sim L^2(\Sigma_0)$. The above proof shows that the mapping is uniformly continuous for $\delta \in (0, \delta_0)$ for some $\delta_0 > 0$. In particular it has a limit as $\delta \rightarrow 0$. We see that in this L^2 sense any function from $W^{1,2}(\Omega)$ approaches its boundary value as we approach the boundary.

¹⁰⁶Note that if we wish to express the surface integral $\int_{\partial\Omega} \dots dx$ in the coordinates x' , we should replace dx by $\sqrt{1 + |\nabla g(x')|^2} dx'$. When g is Lipschitz, the factor $\sqrt{1 + |\nabla g(x')|^2}$ is bounded from above and below, so that the norms which we get if we leave it out will be equivalent.

Lecture 25, 11/5/2010

We will characterize the space

$$H_0^1(\Omega) \subset H^1(\Omega) = W^{1,2}(\Omega) \quad (325)$$

in terms of the behavior of its function near or at the boundary. The domain Ω is assumed to be boundary and “sufficiently regular”. For example, Lipschitz boundary is sufficient.

Theorem

Let $u \in W^{1,2}(\Omega)$. The following conditions are equivalent:

- (i) $u \in H_0^1(\Omega)$,
- (ii) $u|_{\partial\Omega} = 0$ (in the sense of the trace operator),
- (iii) $\int_{\Omega} \left[\frac{u(x)}{\text{dist}(x, \partial\Omega)} \right]^2 dx < +\infty$.

Proof: We first prove that (i) is equivalent to (ii). The non-trivial part is (ii) \implies (i). To prove this implication, we note that we can use the partition of unity to localize the problem, similarly to what we did in lecture 23 when proving the density of $H^1(\Omega)$ in $W^{1,2}(\Omega)$. We can therefore assume without loss of generality that u is either supported away from the boundary, in which case we can directly use mollification to obtain the approximating smooth functions (see also the last part of lecture 22), or u is supported in a small area near $\partial\Omega$, and that $\partial\Omega$ is described by $x_n = g(x')$, with $x' = (x_1, \dots, x_{n-1})$, g Lipschitz, and Ω locally described by $x_n > g(x')$. Let us extend u by 0 outside Ω , and let \tilde{u} denote this extended function. Let \mathcal{O} be some open neighborhood of $\bar{\Omega}$. We claim that the function \tilde{u} belongs to $W^{1,2}(\mathcal{O})$ and that the distributional derivative $\tilde{w} = \nabla \tilde{u}$ is given as

$$\tilde{w} = \begin{cases} 0 & \text{in } \mathcal{O} \setminus \Omega \\ \nabla u & \text{in } \Omega. \end{cases} \quad (326)$$

To prove this, we have to verify the identity

$$\int_{\mathcal{O}} -\tilde{u} \nabla \varphi = \int_{\Omega} \nabla u \varphi \quad (327)$$

for $\varphi \in \mathcal{D}(\mathcal{O})$. From lecture 23 we know that we can approximate u by smooth functions u_j defined in a neighborhood of $\bar{\Omega}$. We have

$$\int_{\Omega} \nabla u_j \varphi = \int_{\partial\Omega} u_j \varphi n + \int_{\Omega} -u_j \nabla \varphi, \quad (328)$$

where in the first term on the right-hand side we use n to denote the normal to the boundary. As the map $v \rightarrow v|_{\partial\Omega}$ is continuous from $W^{1,2}(\Omega)$ to $L^2(\Omega)$,

$u_j \rightarrow u$ in $W^{1,2}(\Omega)$ and $u|_{\partial\Omega} = 0$, we see that the first term on the right-hand side converges to 0 as $j \rightarrow \infty$. The left-hand side of (328) converges to the left-hand side of (327), the passage to the limit in $\int_{\Omega} -u_j \varphi$ is obvious, and (327) is

established.

The approximation of u is now easy. We shift the extended function \tilde{u} up along the x_n -axis, so that the shifted function is compactly supported in Ω , and then we mollify the shifted function with a mollifier with sufficiently small support. The details are left to the reader as an exercise. This establishes the equivalence of (i) and (ii).

It remains to establish the equivalence of (i) and (iii). We first note that (iii) \implies (i) is easy. Let $\Omega_\varepsilon = \{x \in \Omega, \text{dist}(x, \partial\Omega) > \varepsilon\}$, and let $\mathcal{U}_\varepsilon = \Omega \setminus \Omega_\varepsilon$. Let $\eta_\varepsilon \in \mathcal{D}(\Omega)$ be such that $\eta_\varepsilon = 1$ in Ω_ε and $|\nabla \eta_\varepsilon| \leq 2/\varepsilon$ in Ω . Set $u_\varepsilon = u\eta_\varepsilon$. Clearly $u_\varepsilon \in H_0^1(\Omega)$, and we only have to show that $\|\nabla u - \nabla u_\varepsilon\|_{L^2} \rightarrow 0$ for $\varepsilon \rightarrow 0$. However, this is a simple consequence of condition (iii), as it implies that

$$\int_{\mathcal{U}_\varepsilon} \frac{u^2}{\varepsilon^2} dx \rightarrow 0 \quad \text{when } \varepsilon \rightarrow 0. \quad (329)$$

For (i) \implies (iii) we will use (a special case of) Hardy's inequality for smooth functions $v: [0, \infty) \rightarrow \mathbb{R}$ vanishing at 0:

$$\int_0^\infty \left(\frac{v}{x}\right)^2 dx \leq 4 \int_0^\infty |v'|^2 dx. \quad (330)$$

This shows that, in situation when u is localized near boundary and in an area where $\partial\Omega$ is described by $x_n = g(x')$, with $g: B_r^{n-1} \rightarrow \mathbb{R}$ Lipschitz, we have for a suitably chosen b (so that $\{x_n = b\}$ is "above" the support of u)

$$\int_{g(x')}^b \frac{|u(x', s)|^2}{(s - g(x'))^2} ds \leq 4 \int_{g(x')}^b \left|\frac{\partial u(x', s)}{\partial x_n}\right|^2 ds \quad (331)$$

As the function is Lipschitz, it is not hard to see that for $g(x') < s < b$ we have $\text{dist}((x', s), \partial\Omega) \geq c(s - g(x'))$ for some $c > 0$, and we see that (iii) follows by integrating (331) over x' . The proof of the theorem is finished.

Remark. An alternative approach would be to locally straighten the boundary to $\{x_n = 0\}$ by a suitable change of variables which will change the distance to the boundary only by a multiplicative factor bounded from above and below. Then we can take $g(x') \equiv 0$ in the above arguments.

The above theorem clarifies what happens with the boundary values, and we now have a good set-up for the minimization procedure of the functional

$$J(u) = \int_{\Omega} \left(\frac{1}{2}|\nabla u|^2 - fu\right) dx \quad (332)$$

over functions with $u|_{\partial\Omega} = 0$. The above results about the space $H_0^1(\Omega)$ show that it is natural to minimize J over this space. In other words the natural

choice for the space X which we did not quite identify in lecture 21 is

$$X = H_0^1(\Omega). \quad (333)$$

Our investigations in lecture 22 show that J has a unique minimizer in $\bar{u} \in H_0^1(\Omega)$. As J is quadratic, it is differentiable in any direction $v \in H_0^1(\Omega)$ and $J'(u)v = 0$ translates to

$$\int_{\Omega} \nabla \bar{u} \nabla v = \int_{\Omega} f v \quad \text{for each } v \in H_0^1(\Omega). \quad (334)$$

We have seen that the boundary condition $\bar{u}|_{\partial\Omega} = 0$ is guaranteed at least in the L^2 - sense by $\bar{u} \in H_0^1(\Omega)$. It is also easy to verify that u satisfies the equation $\Delta u = f$ in Ω in the weak sense introduced in lecture 4. We now adjust our terminology so that we can distinguish between the various notions of weak solutions.

1. A *weak solution* of the boundary-value problem $\Delta u = f$ in Ω and $u|_{\partial\Omega} = 0$ is a function $\bar{u} \in H_0^1(\Omega)$ satisfying (334) for each $v \in H_0^1(\Omega)$.
2. A local *very weak solution* of the equation $\Delta u = f$ in Ω is a locally integrable function u satisfying

$$\int_{\Omega} u \Delta \varphi = \int_{\Omega} f \varphi \quad \text{for each } \varphi \in \mathcal{D}(\Omega). \quad (335)$$

3. A *very weak solution* of the boundary value problem $\Delta u = f$ in Ω and $u|_{\partial\Omega} = 0$ is an integrable function u in Ω satisfying

$$\int_{\Omega} u \Delta \varphi = \int_{\Omega} f \varphi \quad (336)$$

for each smooth $\varphi: \bar{\Omega} \rightarrow \mathbb{R}$ with $\varphi|_{\partial\Omega} = 0$.

In the context of the minimization of the functional J the notion of the weak solution is very natural. Note that for our problem we get existence and uniqueness of the weak solution of a boundary-value problem quite easily. As we have seen, the existence can be obtained from the direct minimization of J over $H_0^1(\Omega)$. The uniqueness follows from the strict convexity of the functional J and can also be seen directly from the equations: if we had two different weak solutions u_1, u_2 to the problem $\Delta u = f$, $u|_{\partial\Omega} = 0$, then their difference $v = u_1 - u_2 \in H_0^1(\Omega)$ would satisfy equation (334) with $f = 0$, which means that $\int_{\Omega} \nabla v \nabla w = 0$ for each $w \in H_0^1(\Omega)$. Taking $w = v$ we see that $v = 0$.

The definition of the very weak solution is also quite natural in case we wish to work with minimal possible regularity of f . However, the proof of uniqueness is not as straightforward in this case.

In general, a good generalization of a notion of a solution should preserve uniqueness for problems where uniqueness is expected.

So far we have dealt with the equation $\Delta u = f$ and the boundary condition $u|_{\partial\Omega} = 0$. It is easy to adapt the “Direct Method” to much more general situation. As we will see, the method is very flexible, and can accommodate easily more general situations, such as variable coefficients, systems of equations, various types of boundary conditions, etc. Also, variational formulations are very suitable for setting up numerical schemes: roughly speaking one just needs to replace $X = H_0^1(\Omega)$ by a finite-dimensional subspace.¹⁰⁷

¹⁰⁷Of course, details concerning to choice of the subspace can lead to non-trivial questions. These are studied in Numerical Analysis.

Homework assignment 2

due November 22

Prove the following statement.

Let $u: \mathbb{R}^3 \setminus B_R \rightarrow \mathbb{R}$ be a bounded harmonic function. Then for each $m \in \mathbf{N}$ we can write

$$u(x) = A + a \frac{1}{|x|} + \sum_j a_j \frac{\partial}{\partial x_j} \frac{1}{|x|} + \sum_{jk} a_{jk} \frac{\partial^2}{\partial x_j \partial x_k} \frac{1}{|x|} + \dots \\ + \sum_{j_1, \dots, j_m} a_{j_1 \dots j_m} \frac{\partial^m}{\partial x_{j_1} \dots \partial x_{j_m}} \frac{1}{|x|} + O\left(\frac{1}{|x|^{m+2}}\right) \quad |x| \rightarrow \infty,$$

where $A, a, a_j, \dots, a_{j_1, \dots, j_m}$ are suitable constants.

Hint:

Extend u from $\mathbb{R}^3 \setminus B_{2R}$ to \mathbb{R}^3 to a smooth function \tilde{u} on \mathbb{R}^3 . Then consider the function

$$v(x) = \int_{\mathbb{R}^3} -\frac{1}{4\pi|x-y|} \Delta \tilde{u}(y) dy. \quad (337)$$

Show that $v - \tilde{u}$ is constant. Note that for large x one can expand the expression $\frac{1}{|x-y|}$ into a power series in y . Use the necessary number of the terms of this power series in (337).

Remark

Consider the following transformation taking a function u in $\mathbb{R}^n \setminus B_1$ to function v on B_1 given by

$$u(x) = K_0(x) v\left(\frac{x}{|x|^2}\right), \quad (338)$$

where K_0 is the fundamental solution, $K_0(x) = -\frac{1}{(n-2)|S^{n-1}||x|^{n-2}}$. We can write $v = Tu$. The transformation T is called the Kelvin transformation. It can be shown that T takes harmonic functions into harmonic functions. (You can prove this as an optional part of the homework.) The series above for u at $x \rightarrow \infty$ is related via the Kelvin transform to the Taylor series of v at $x = 0$. This can be used as an alternative way to establish the series for u at ∞ .

Lecture 26, 11/8/2010

Let X be a Hilbert space over \mathbb{R} with scalar product $x, y \rightarrow (x, y)$ and let $l: X \rightarrow \mathbb{R}$ be a linear functional. We recall that

$$\|l\| = \sup_{\|x\| \leq 1} |l(x)|. \quad (339)$$

A classical theorem of F. Riesz says that l can be represented by an element of X via the scalar product: there exists a unique $a \in X$ such that

$$l(x) = (x, a) \quad x \in X. \quad (340)$$

We have $\|l\| = \|a\|$.

We have de facto proved this theorem when we established the existence of a minimizer for the functional J in lecture 21: we can just go through the proof with

$$J(x) = \frac{1}{2} \|x\|^2 - l(x) \quad (341)$$

and we note that the minimizer a of J satisfies (340).

We can also take the Riesz theorem as a starting point, and apply it to the linear functional $v \rightarrow \int_{\Omega} f v$ in the Hilbert space $X = H_0^1(\Omega)$ with the scalar product $(u, v)_{H_0^1} = \int_{\Omega} \nabla u \nabla v$. This immediately gives the existence and uniqueness of the weak solution for problem (279).

Important classes of equations with variable coefficients are also immediately covered by the Riesz theorem, or the minimization procedure used for J . Let $a_{ij} = a_{ij}(x)$, $i, j = 1, \dots, n$ be bounded measurable functions satisfying

$$a_{ij}(x) \xi_i \xi_j \geq \nu |\xi|^2 \quad \xi \in \mathbb{R}^n, \quad (342)$$

where we sum over the repeated indices, as usual. Let also $c = c(x)$ be a non-negative bounded measurable function in Ω .

Assume

$$a_{ij} = a_{ji} \quad (343)$$

and consider the functional

$$J(u) = \int_{\Omega} \frac{1}{2} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} + \frac{1}{2} c(x) u^2 - f(x) u. \quad (344)$$

Thanks to the assumptions on a_{ij} and c , one can show again, in the same way, that J has a minimizer u on $H_0^1(\Omega)$ and that

$$\int_{\Omega} (a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + c u v - f v) dx = 0 \quad v \in H_0^1(\Omega). \quad (345)$$

This means that

$$-\frac{\partial}{\partial x_i} (a_{ij} \frac{\partial u}{\partial x_j}) + c u = f \quad (346)$$

in the sense of distributions, and we also note that $u|_{\partial\Omega} = 0$ (in the sense of the trace operator) since $u \in H_0^1(\Omega)$.

Alternatively, we could define a new scalar product

$$(u, v)_{a,c} = \int_{\Omega} (a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + cuv) dx \quad (347)$$

on $H_0^1(\Omega)$, which is easily seen to be equivalent to the standard one, and apply the Riesz theorem. We leave the details for the reader as an easy exercise.

There are important classes of equations which are not covered by the above. For example, let us consider

$$-\Delta u + b_j \frac{\partial u}{\partial x_j} = f \quad (348)$$

where $b = (b_1, \dots, b_n) = (b_1(x), \dots, b_n(x))$ is a bounded measurable vector field.¹⁰⁸ The term $b_j \frac{\partial u}{\partial x_j}$ is often called “drift”. Such equations arise in many situations and in general are not covered by the simple application of the Riesz theorem we saw above. The following generalization of the Riesz theorem addresses this problem.

Let us first introduce some terminology. Let $A = A(x, y)$ be a continuous bilinear form on a Hilbert space X . This means that A is linear in each variable and $|A(x, y)| \leq c\|x\|\|y\|$ for some $c \geq 0$. The form A is not assumed to be symmetric, i. e. $A(x, y)$ may not be equal to $A(y, x)$. We say that A is *coercive* if there exists $\alpha > 0$ such that

$$A(x, x) \geq \alpha\|x\|^2. \quad (349)$$

Lemma (Lax-Milgram)

Let $A = A(x, y)$ be a continuous bilinear form on a Hilbert space X . Assume that A is coercive and let α be as in (349). Then each continuous linear functional $l: X \rightarrow \mathbb{R}$ can be uniquely represented as

$$l(x) = A(x, a) \quad \text{for all } x \in X, \quad (350)$$

with $a \in X$ and $\|a\| \leq \frac{1}{\alpha}\|l\|$.

Proof: For each $y \in X$ define Ty by

$$A(x, y) = (x, Ty), \quad x \in X. \quad (351)$$

The existence of Ty follows from the continuity of A and the Riesz theorem. It is easy to see that the map $y \rightarrow Ty$ is a bounded linear operator on X . We

¹⁰⁸The assumptions on b (and also on c above) can still be relaxed, we are not striving for the greatest possible generality at this point.

need to show that T is invertible. Once the invertibility is established than we can write $l(x) = (x, b)$ for some $b \in X$ by the Riesz theorem, and using the invertibility of T we see that

$$l(x) = (x, b) = A(x, T^{-1}b), \quad x \in X, \quad (352)$$

and we take $a = T^{-1}b$.

We note that T satisfies

$$(Tx, x) \geq \alpha \|x\|^2. \quad (353)$$

We show that a bounded linear operator T satisfying (353) is invertible, with $\|T^{-1}\| \leq \frac{1}{\alpha}$.

We note that the condition (353) gives the required bound: $Tx = y$ implies $\alpha \|x\|^2 \leq (Tx, x) = (y, x) \leq \|x\| \|y\|$ and hence $\|x\| \leq \frac{1}{\alpha} \|y\|$. This shows that the range of T is closed. No vector non-trivial vector $y \in X$ can be perpendicular to $T(X)$, as this would mean $(Ty, y) = 0$, and hence $T(X) = X$.¹⁰⁹

As an example of an application of the Lax-Milgram lemma, let us consider a bounded vector field $b = (b_1, \dots, b_n) = (b_1(x), \dots, b_n(x))$ with $\operatorname{div} b = 0$. For $f \in L^2(\Omega)$ consider the problem

$$\begin{aligned} -\Delta u + b \nabla u &= f & \text{in } \Omega, \\ u &= 0 & \text{at } \partial\Omega. \end{aligned} \quad (355)$$

The weak formulation is: find $u \in H_0^1(\Omega)$ such that

$$A(u, v) = l(v) = \int_{\Omega} f v \quad \text{for each } v \in H_0^1(\Omega), \quad (356)$$

where

$$A(u, v) = \int_{\Omega} [\nabla u \nabla v + (b \nabla u) v]. \quad (357)$$

Since $\operatorname{div} b = 0$ it is easy to check that

$$A(u, u) = \int_{\Omega} |\nabla u|^2 \quad (358)$$

¹⁰⁹One can also see the invertibility of T directly in the following way. Let us write $T = S + B$ where S is symmetric and B is anti-symmetric, i. e. $B(x, y) = -B(y, x)$. We recall that if M is a continuous invertible operator, then $M + N$ is invertible for $\|N\| < \|M^{-1}\|^{-1}$. We note that $(Sx, x) \geq \alpha \|x\|^2$ and since S is symmetric, we know it is invertible by the Riesz theorem, with

$$\|S^{-1}\| \leq \frac{1}{\alpha}. \quad (354)$$

The operator $S + tB$ will be still invertible for $|t| < t_0 = \frac{\alpha}{\|B\|}$, and will satisfy the same bound

$$\|(S + tB)^{-1}\| \leq \frac{1}{\alpha}.$$

This means that we can repeat the argument with S replaced by $S + tB$ and we get that $S + tB$ is invertible for $t \leq 2t_0$. It is clear that this can be continued to reach any t , and in particular $t = 1$.

and hence A is coercive in $H_0^1(\Omega)$. We conclude that the problem (355) will have a unique weak solution for each $f \in L^2(\Omega)$.

It turns out that the result is true even without the assumption $\operatorname{div} b = 0$, but we will need to develop the theory a little further to see that.

Remark

Note that when $\operatorname{div} b = 0$, we can write $-\Delta u + b \nabla u = -\frac{\partial}{\partial x_i} (a_{ij} \frac{\partial u}{\partial x_j})$ where $a_{ij} = \delta_{ij} + c_{ij}$ with $c_{ij} = -c_{ji}$. This form of the equation is sometimes useful.

Up to now we have dealt only with the Dirichlet boundary condition. Next time we will consider other boundary conditions.

Lecture 27, 11/10/2010

Let us first indicate how the problem

$$\begin{aligned}\Delta u &= 0 && \text{in } \Omega, \\ u &= g && \text{at } \partial\Omega,\end{aligned}\tag{359}$$

fits into the “Direct Method” setting.

One to solve the problem is to minimize the Dirichlet integral

$$\int_{\Omega} |\nabla u|^2\tag{360}$$

over

$$W_g^{1,2}(\Omega) = \{u \in W^{1,2}(\Omega), u|_{\partial\Omega} = g\}.\tag{361}$$

Using the techniques we developed, there is no difficulty in showing that the problem has a unique minimizer as long as $W_g^{1,2}(\Omega)$ is non-empty. This is clearly so for “sufficiently regular” g , for example $g \in W^{1,2}(\partial\Omega)$. In general, the requirement $W_g^{1,2}(\Omega) \neq \emptyset$ does represent a restriction on g . The precise result is that $W_g^{1,2}(\Omega) \neq \emptyset$ if and only if $g \in W^{\frac{1}{2},2}(\partial\Omega)$, which is a space we have not really defined yet. For now we will not go into these questions in more detail. Instead we will illustrate where restrictions may come from with the following example. Consider $\Omega = B_1^2 = \{x \in \mathbb{R}^2, |x| < 1\}$. The boundary $\partial\Omega$ is the unit circle S^1 . Let r, θ be the polar coordinates in the plane. The angle θ parametrizes the circle S^1 in the obvious way. The function g describing the boundary condition can be thought of as a function of θ , and we can consider its Fourier series

$$g(\theta) = \sum_k g_k e^{ik\theta}.\tag{362}$$

The solution of (359) can be written in polar coordinates as

$$u = u(r, \theta) = \sum_{k \in \mathbb{Z}} g_k r^{|k|} e^{ik\theta}.\tag{363}$$

We can calculate the Dirichlet integral (360) in terms of the Fourier coefficients g_k :

$$\int_{\Omega} |\nabla u|^2 = \int_{\Omega} \left[\left(\frac{\partial u}{\partial r} \right)^2 + \left(\frac{\partial u}{r \partial \theta} \right)^2 \right] r \, d\theta \, dr \sim \sum_k k |g_k|^2.\tag{364}$$

We expect that g belongs to $W_g^{1,2}(\Omega)$ if and only if $\sum_k k |g_k|^2 < +\infty$, which is indeed the case. This condition can be used to define the space $W^{\frac{1}{2},2}(S^1)$. As an exercise, you can give an example of a continuous function g on S^1 which does not belong to $W^{\frac{1}{2},2}(S^1)$. For such a function we cannot solve the problem (359) directly by the minimization of the Dirichlet integral (360), because the integral cannot be finite. On the other hand the solution clearly exists, we can write it as the Poisson integral, for example. So there are some constraints on the

applicability of the direct method, but in practice they rarely present a serious concern. For example, in the case at hand we can approximate g by smooth functions g_ε for which we do get a solution u_ε by minimization, and then show that u_ε converge as $g_\varepsilon \rightarrow g$.

Assume $W_g^{1,2}(\Omega)$ is not empty and let $\tilde{g} \in W^{1,2}(\Omega)$ be an extension of g . We can use the theorem we proved last time to conclude that

$$W_g^{1,2}(\Omega) = \tilde{g} + H_0^1(\Omega), \quad (365)$$

and hence minimizing (360) over $W_g^{1,2}(\Omega)$ is the same as minimizing

$$J(v) = v \rightarrow \int_{\Omega} |\nabla v + \nabla \tilde{g}|^2 \quad (366)$$

over $H_0^1(\Omega)$. Note that

$$J(v) = \int_{\partial\Omega} (|\nabla v|^2 + 2\nabla v \nabla \tilde{g}) + \int_{\partial\Omega} |\tilde{g}|^2 \quad (367)$$

The last term is constant, and it does not play any role in the minimization. The term $v \rightarrow \int_{\Omega} 2\nabla \tilde{g} \nabla v$ represents a continuous linear functional on $H_0^1(\Omega)$, and the equation $J'(v)\varphi = 0$ is

$$\int_{\Omega} (2\nabla v \nabla \varphi + 2\nabla \tilde{g} \nabla \varphi) = 0, \quad \varphi \in H_0^1(\Omega), \quad (368)$$

which means that $\Delta v + \Delta \tilde{g} = 0$ in the sense of distributions, which is what we needed.

The above is an example that, in addition to the linear functionals $v \rightarrow \int_{\Omega} f v$ on $H_0^1(\Omega)$, one can also consider the linear functionals of the form

$$v \rightarrow \int_{\Omega} f_j \frac{\partial v}{\partial x_j} = \int_{\Omega} f \nabla v \quad (369)$$

where $f = (f_1, \dots, f_n)$ is an $L^2(\Omega)$ vector field. The equation we get from minimization of

$$u \rightarrow \int_{\Omega} \frac{1}{2} |\nabla u|^2 - f \nabla u \quad (370)$$

is

$$\Delta u = \operatorname{div} f = \frac{\partial f_j}{\partial x_j}. \quad (371)$$

For sufficiently regular f and $u \in H_0^1(\Omega)$ we have

$$\int_{\Omega} f \nabla u = \int_{\Omega} -(\operatorname{div} f) u \quad (372)$$

so for regular f and minimization over $H_0^1(\Omega)$ we do not get anything new. However, when we minimize over spaces where the boundary value is not fixed, the integration by parts in (372) results in an additional boundary term $\int_{\partial\Omega} f_j n_j u$ which will affect the boundary condition.

Let us now consider other boundary value problems. Let us consider the functional

$$J(u) = \int_{\Omega} \left(\frac{1}{2} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} + \frac{1}{2} c u^2 - f u \right) + \int_{\partial\Omega} \left(\frac{1}{2} \gamma u^2 - g u \right), \quad (373)$$

where the coefficients a_{ij}, c, γ and the functions f, g satisfy the following assumptions:

- $a_{ij} = a_{ij}(x)$ are bounded measurable and satisfy the ellipticity condition (342). Note that only the symmetric part of a_{ij} plays a role, so we can assume that $a_{ij} = a_{ji}$.
- $c = c(x)$ and $\gamma = \gamma(x)$ are bounded measurable,
- f, g are L^2 -functions on Ω and $\partial\Omega$ respectively,
- The quadratic part of J is coercive, in the sense that for some $\kappa > 0$ we have

$$J(u) = \int_{\Omega} \left(\frac{1}{2} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} + \frac{1}{2} c u^2 \right) + \int_{\partial\Omega} \frac{1}{2} \gamma u^2 \geq \kappa \|u\|_{W^{1,2}(\Omega)}^2. \quad (374)$$

We will discuss the coercivity assumption and its variants in some detail later on, for now let us just note that it is trivially satisfied what $c \geq \delta > 0$ and $\gamma \geq 0$.

It is clear from the continuity of the trace operator $u \rightarrow u|_{\partial\Omega}$ that J is continuous on $W^{1,2}(\Omega)$. We can use minimization or the Riesz representation theorem to show that J has a unique minimizer in $W^{1,2}(\Omega)$ and the equation $J'(u) = 0$ has a unique solution on $W^{1,2}(\Omega)$. The condition $J'(u) = 0$ is

$$\int_{\Omega} \left(a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + c u v - f v \right) + \int_{\partial\Omega} (\gamma u - g) v = 0, \quad v \in W^{1,2}(\Omega). \quad (375)$$

Note that no boundary values for u were fixed in advance. The minimization procedure will “choose” the boundary conditions for us. Let us now interpret what (375) says in terms of an equation for u and the boundary conditions. As we have done before in similar situations, we integrate by parts to obtain an expression in which v comes with no derivatives. This time we have to pay attention also to the boundary terms. In fact, the only integration by parts we have to do is in the first term:

$$\int_{\Omega} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} = \int_{\partial\Omega} a_{ij} \frac{\partial u}{\partial x_j} n_i v + \int_{\Omega} - \left(\frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j} \right) v, \quad (376)$$

where n_i denotes the components of the outward unit normal to Ω .

At this point this identity is “formal”, in the sense that it is true for “sufficiently regular” functions, but we are not sure if we can justify it for $u \in W^{1,2}(\Omega)$, even when v is smooth. More precisely, while the term on the left-hand side is clearly well-defined and if we did not have the boundary term on the right-hand side we could say that the identity actually defines the term $\int_{\Omega} -(\frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j})v$, when v does not vanish at the boundary, we have two terms on the right-hand side of (376) which are not well-defined (even when v is smooth), and for a general $u \in W^{1,2}(\Omega)$ we cannot really define them individually. However, our u is not a general function in $W^{1,2}(\Omega)$, we know that it satisfies (375) for each $v \in W^{1,2}(\Omega)$, and this can be used to define the two terms on the right-hand side of (376) individually.

We first use (375) with $v \in H_0^1(\Omega)$ to conclude that

$$-\frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j} + cu = f \quad \text{in } \Omega \quad (377)$$

in the sense of distributions. That is the equation associated with the functional J . Once we know that u satisfies this identity inside Ω , we can use it to define the second term on the right-hand side of (376) by simply substituting $f - cu$ for $-\frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j}$. Then the first term on the right-hand side is defined by the other two terms. In particular, for the solutions of (377) the normal derivative $a_{ij} \frac{\partial u}{\partial x_j} n_i$ is well defined, at least in the sense that $\int_{\partial\Omega} a_{ij} \frac{\partial u}{\partial x_j} n_i v$ is defined for every smooth function v (and, in fact, for every function which can be obtained as trace of a $W^{1,2}(\Omega)$ function).

The resulting boundary-value problem associated with the functional J will be

$$\begin{aligned} -\frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j} + cu &= f & \text{in } \Omega, \\ a_{ij} \frac{\partial u}{\partial x_j} n_i + \gamma u &= g & \text{at } \partial\Omega. \end{aligned} \quad (378)$$

When $\gamma = 0$ the boundary condition is called the Neumann boundary condition. The Dirichlet boundary condition $u|_{\partial\Omega} = 0$ can be obtained in the limit $\gamma \nearrow +\infty$.

Note that the derivative which comes into the Neumann boundary condition is closely associated with the operator. In principle one can consider boundary conditions of the type

$$b_j \frac{\partial u}{\partial x_j} + \gamma u = g \quad \text{at } \partial\Omega, \quad (379)$$

where (b_1, \dots, b_n) is a given vector field at $\partial\Omega$, whose non-tangential part does not vanish at $\partial\Omega$. This is possible, and the problems have been studied (the term “oblique derivative boundary condition” is often used in this connection), but the simple variational approach above does not work for such problems, except for the special case when $b_j = a_{ij} n_i$. This might seem restrictive, but in practice

the Neumann condition with $a_{ij} \frac{\partial u}{\partial x_j} n_i$ (or the Dirichlet condition) is often the correct boundary condition for the model one is considering. This is exactly because the correct model is often governed by some type of functional (such as energy) which should be minimized. The minimization then gives the right boundary condition. This can also be used in the numerical approximations. In the situation above with the functional J , we have no trouble designing the right boundary condition for numerical approximations: one just takes J on a suitable finite dimensional subspace of $Y \subset W^{1,2}(\Omega)$, and minimizes J over Y . The boundary condition “takes care of itself”. This is why the boundary condition in (378) is often called the “natural boundary condition”.

The Neumann boundary condition $a_{ij} \frac{\partial u}{\partial x_j} n_i$ often has a clear physical meaning. For example, if we talk about electric potential u in a body with electrical conductivity given by the tensor a_{ij} , then the quantity $I_i = a_{ij} \frac{\partial u}{\partial x_j}$ corresponds to the electric current. The Neumann condition then is that $I_i n_i = 0$, which says that there is no flux of electricity through the boundary. This will be the case when the boundary is “left alone” and no external devices or fields are employed to influence the electric potential there.

We note that we can affect the boundary condition by other terms in the integral $\int_{\Omega} \dots$. For example, if we add to the functional the term

$$\int_{\Omega} f_j \frac{\partial u}{\partial x_j}, \quad (380)$$

then the expression for $J'(u)v$ will change by

$$\int_{\Omega} -\frac{\partial f_j}{\partial x_j} v + \int_{\partial\Omega} f_j n_j v \quad (381)$$

and the last term will contribute to the boundary condition. Note that if $\frac{\partial f_j}{\partial x_j} = 0$, then the effect of the term will be only on the boundary condition, the equation will remain the same. Such terms in the functional are often called “null lagrangians”. We should mention more terminology: The expression defining J , namely

$$L(x, u, \nabla u) = \frac{1}{2} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} + \frac{1}{2} c u^2 - f u + f_j \frac{\partial u}{\partial x_j} + b_j \frac{\partial u}{\partial x_j} u \quad (382)$$

(where we added the extra two terms with f_j and b_j) is called the lagrangian. The equation $J'(u) = 0$ is called the Euler-Lagrange equation. Note that the boundary terms in (373) can be added by taking adding $f_j \frac{\partial u}{\partial x_j}$ and $b_j \frac{\partial u}{\partial x_j} u$ with suitable f_j, g_j satisfying $\frac{\partial f_j}{\partial x_j} = 0$ and $\frac{\partial b_j}{\partial x_j} = 0$ to the lagrangian.

Lecture 28, 11/12/2010

We plan to address the issue of coercivity of the functional (373) and also of more general forms $A(u, v)$ appearing in the variational formulation of boundary value problem. An important tool will be the Poincarè inequality below. We first introduce the following notation.

$$\begin{aligned} \int_{\mathcal{O}} f &= \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} f, \\ (f)_{\mathcal{O}} &= \int_{\mathcal{O}} f. \end{aligned}$$

Lemma (Poincarè inequality for cubes)

Let $Q = Q_R = [0, R]^n$. Then for each $u \in W^{1,2}(\Omega)$

$$\int_Q |u - (u)_Q|^2 \leq nR^2 \int_Q |\nabla u|^2. \quad (383)$$

Proof: It is enough to prove the inequality for smooth functions, due to the density results in lecture 25. We note that by a simple application of the Cauchy-Schwartz (or, alternatively, Jensen's) inequality we have

$$\int_Q |u - (u)_Q|^2 = \int_Q \left| \int_Q (u(x) - u(y)) dy \right|^2 dx \leq \int_Q \int_Q |u(x) - u(y)|^2 dx dy. \quad (384)$$

We can write

$$u(y) - u(x) = \int_{x_1}^{y_1} \frac{\partial u}{\partial x_1}(s, x_2, \dots, x_n) ds + \dots + \int_{x_n}^{y_n} \frac{\partial u}{\partial x_n}(y_1, \dots, y_{n-1}, s) ds. \quad (385)$$

Writing the right-hand side as $I_1 + \dots + I_n$, we see that

$$|u(y) - u(x)|^2 \leq (I_1 + \dots + I_n)^2 \leq n(I_1^2 + \dots + I_n^2). \quad (386)$$

A simple application of the Cauchy-Schwartz inequality shows that

$$I_j^2 \leq R \int_0^R \left| \frac{\partial u}{\partial x_j}(x_1, \dots, x_{j-1}, s, y_{j+1}, \dots, y_n) \right|^2 ds. \quad (387)$$

Using this, integrating (386) over x and y , and dividing by $|Q| = R^n$ we obtain

$$\int_Q \int_Q |u(x) - u(y)|^2 dx dy \leq nR^2 \int_Q |\nabla u(x)|^2 dx, \quad (388)$$

which – keeping in mind (384) – completes the proof.

Remark

It is clear that the previous estimate is not optimal, note that in (387) we are throwing away a lot of information. A more careful calculation shows that in fact

$$\int_Q \int_Q \frac{|u(x) - u(y)|^2}{|x - y|^2} dx dy \leq 2^n \int_Q |\nabla u(x)|^2 dx \quad (389)$$

for any convex set Q . This can be seen as follows. We have

$$u(y) - u(x) = \int_0^1 \nabla u((1-s)x + sy)(y-x) ds \quad (390)$$

and a simple application of Cauchy-Schwartz inequality (or Jensen's inequality) gives

$$\frac{|u(x) - u(y)|^2}{|x - y|^2} \leq \int_0^1 |\nabla u((1-s)x + sy)|^2 ds. \quad (391)$$

Letting $F(x) = |\nabla u(x)|^2$ and integrating over x, y , we see that it is enough to show

$$\int_Q \int_Q \int_0^1 F((1-s)x + sy) dx dy ds \leq 2^n \int_Q F(x) dx. \quad (392)$$

This is not hard: we can split the integration over s as

$$\int_0^1 ds = \int_0^{1/2} ds + \int_{1/2}^1 ds. \quad (393)$$

It is enough to estimate $\int_{1/2}^1 ds$, the estimate of the other part being similar. We have

$$\int_Q F((1-s)x + sy) dy = s^{-n} \int_{sQ} F((1-s)x + \eta) d\eta \leq s^{-n} \int_Q F \leq 2^n \int_Q F, \quad (394)$$

and integrating over $s \geq 1/2$ and $x \in Q$ we obtain the required estimate.¹¹⁰

The proof of the Poincarè inequality for cubes/convex sets can be generalized to other domains, but it is easier to handle the case of a general domain somewhat indirectly, via a compactness argument based on the following compactness result which is a relatively easy consequence of the Poincarè inequality for cubes. The result is crucial in many other situations.

Theorem (Rellich)

The set $\{u \in W^{1,2}(\Omega); \|u\|_{W^{1,2}(\Omega)} \leq 1\}$ is compact in $L^2(\Omega)$.

¹¹⁰An alternative proof can be done by evaluating the left-hand side of (392) with the co-area formula. For the co-area formula see for example the book of H. Federer "Geometric Measure Theory", Theorem 3.2.12, p 249. That approach is not simpler than the elementary estimate above, but it provides a good exercise for calculations with the co-area formula.

Corollary (The Poincarè inequality for bounded domains)¹¹¹
 Assume Ω is bounded and $u \in W^{1,2}(\Omega)$. Then

$$\int_{\Omega} |u - (u)_{\Omega}|^2 \leq C(\Omega) \int_{\Omega} |\nabla u|^2. \quad (395)$$

Proof: Let us first show that

$$\|u\|_{W^{1,2}(\Omega)}^2 \leq C \left[\int_{\Omega} |\nabla u|^2 + \left(\int_{\Omega} u \right)^2 \right] \quad (396)$$

for a suitable $C = C(\Omega)$. We argue by contradiction. If the statement failed, there would exist a sequence $u^k \in W^{1,2}(\Omega)$ with $\|u^k\|_{W^{1,2}(\Omega)} = 1$ and $\int_{\Omega} |\nabla u^k|^2 + (\int_{\Omega} u^k)^2 \rightarrow 0$. By the compactness theorem, we can assume that u^k converge strongly in L^2 to some function \bar{u} . As $\|\nabla u^k\|_{L^2} \rightarrow 0$, the distributional derivative of \bar{u} vanishes, and hence $\bar{u} \equiv \text{const}$. Also, $\int_{\Omega} (u^k)^2 \rightarrow 1$, and therefore, due to the $L^2(\Omega)$ convergence of the sequence u^k we have $\int_{\Omega} (\bar{u})^2 = 1$. However, as $\int_{\Omega} u^k \rightarrow 0$, we see that $\bar{u} \equiv 0$, a contradiction.

If we use (395) for $u - (u)_{\Omega}$, we obtain (395).

¹¹¹Recall that, by definition, all domains are connected.

Before going to the proof of Rellich's compactness theorem, let us give another example where it can be applied to prove coercivity. As usual, we assume that Ω is bounded domain with sufficiently regular boundary. (Lipschitz boundary is sufficient, for example.)

Let $\gamma: \partial\Omega \rightarrow \mathbb{R}$ be a bounded non-negative measurable function with

$$\int_{\partial\Omega} \gamma > 0. \quad (397)$$

Then there exists $C > 0$ such that

$$\|u\|_{W^{1,2}(\Omega)}^2 \leq C \left(\int_{\Omega} |\nabla u|^2 + \int_{\partial\Omega} \gamma u^2 \right). \quad (398)$$

The proof is similar to the proof of (396) from the last lecture. Assuming that the statement fails, we get a sequence u^k of functions in $W^{1,2}(\Omega)$ with $\|u^k\|_{W^{1,2}} = 1$ such that $\int_{\Omega} |\nabla u^k|^2 + \int_{\partial\Omega} \gamma (u^k)^2 \rightarrow 0$ as $k \rightarrow \infty$. By Rellich's theorem we can assume $u^k \rightarrow \bar{u}$ in $L^2(\Omega)$ as $k \rightarrow \infty$. As in the last lecture, we can conclude that \bar{u} is constant, $\int_{\Omega} |\bar{u}|^2 = 1$, and $u^k \rightarrow \bar{u}$ in $W^{1,2}(\Omega)$. This also shows that $u^k|_{\partial\Omega}$ converge in $L^2(\partial\Omega)$ to the constant $c = \bar{u}$, contradicting to $\int_{\partial\Omega} \gamma (u^k)^2 \rightarrow 0$.

Proof of the Rellich's Theorem.

Let $X = \{u \in W^{1,2}(\Omega), \|u\|_{W^{1,2}} \leq 1\}$. We wish to show that X is a compact subset of $L^2(\Omega)$. We note that X is closed in $L^2(\Omega)$. This is because if $u^k \in X$ converge in L^2 to u , then the (distributional) gradients ∇u^k are bounded in L^2 and satisfy for each $\varphi \in \mathcal{D}(\Omega)$

$$\int_{\Omega} \nabla u^k \varphi = \int_{\Omega} -u^k \nabla \varphi \rightarrow \int_{\Omega} -u \nabla \varphi, \quad k \rightarrow \infty. \quad (399)$$

This shows that the sequence ∇u^k is weakly convergent in $L^2(\Omega)$, the distributional gradient of u is the weak limit of ∇u^k and $u \in X$.¹¹²

The main point of the proof is to show that X is *totally bounded* in $L^2(\Omega)$, which means that for each $\varepsilon > 0$ we can cover X by finitely many L^2 -balls of radius ε . Let us consider the following statement:

(*) For each $\varepsilon > 0$ there exists a finite-dimensional subspace $Y_\varepsilon \subset L^2(\Omega)$ such that the orthogonal projection $P_\varepsilon: L^2(\Omega) \rightarrow Y_\varepsilon$ satisfies

$$\|u - P_\varepsilon u\|_{L^2} \leq \varepsilon \quad (400)$$

¹¹²You should go through this argument carefully, as we have not used weak convergence before.

for each $u \in X$.

It is not hard to see that (*) implies the total boundedness of X : the set $\{u \in Y_\varepsilon, \|u\|_{L^2} \leq 1\}$ is compact, hence we can cover it by finitely many balls of radius ε . Now (400) shows that if double the radius of these balls, they will also cover X .

We prove (*) when Ω is a cube Q . This will establish compactness of the unit ball of $W^{1,2}(Q)$ in $L^2(Q)$. The compactness for general Lipschitz Ω then follows from the usual localization and boundary straightening technique.¹¹³ Alternatively, one can also prove (*) directly for Lipschitz domains. From the proof for Q it will be clear how to proceed in that case.

Let $Q = Q_R$. Set $r = R/m$, and divide Q into $N = m^n$ cubes Q_1, \dots, Q_N of size r . For $u \in L^2(Q)$ we denote by Pu the piece-wise constant function which is equal to $(u)_{Q_j} = \int_{Q_j} u$ on Q_j . Clearly P is the orthogonal projection of $L^2(\Omega)$ onto the subspace consisting of functions constant on each cube Q_j . By the Poincarè inequality from lecture 28, we know that

$$\int_{Q_j} |u - (u)_{Q_j}|^2 \leq nr^2 \int_{Q_j} |\nabla u|^2. \quad (401)$$

Summing over the cubes, we obtain

$$\int_Q |u - Pu|^2 \leq nr^2 \int_Q |\nabla u|^2, \quad (402)$$

which gives (400) with $\varepsilon = \sqrt{nr}$.

Remarks

1. The proof above is quite similar to the proof of the classical Arzela-Ascoli theorem about characterization of relatively compact sets in the space of continuous functions, except that estimates of oscillation of the functions are replaced by the Poincarè inequality.
2. The proof can also be done by using Kolmogorov's criterion of relative compactness for subsets of L^p spaces in (subsets of) \mathbb{R}^n .

¹¹³It is a good exercise to do this in detail.

We consider a Lax-Milgram form

$$A(u, v) = \int_{\Omega} \left(a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + b_j \frac{\partial u}{\partial x_j} v + \tilde{b}_j u \frac{\partial v}{\partial x_j} + cuv \right) + \int_{\partial\Omega} \gamma uv, \quad (403)$$

where the coefficients are bounded measurable functions, and, in addition, the a_{ij} satisfy the ellipticity condition

$$a_{ij} \xi_i \xi_j \geq \nu |\xi|^2. \quad (404)$$

The form is considered on some closed space X , with $H_0^1(\Omega) \subset X \subset W^{1,2}(\Omega)$. The space X is determined by the boundary conditions. Here are some examples.

1. Let $\Gamma_0 \subset \partial\Omega$ and set $X = \{u \in W^{1,2}(\Omega), u|_{\Gamma_0} = 0\}$. This space corresponds to prescribing the Dirichlet boundary condition on Γ_0 and the Neumann/mixed boundary condition on the rest of the boundary.
2. Let $\partial\Omega$ consist of connected components $\Gamma_0, \Gamma_1, \dots, \Gamma_m$ and assume that Γ_0 is also the boundary of the unbounded connected component of the complement of Ω .

$$X = \{u \in W^{1,2}(\Omega), u|_{\Gamma_0} = 0, u|_{\Gamma_j} = c_j, \text{ with } c_1, c_2, \dots, c_m \in \mathbb{R} \text{ not fixed}\}. \quad (405)$$

Similar space could be used when dealing with problems of electrostatics mentioned in lecture 7. The condition $u|_{\Gamma_0} = 0$ corresponds to the system being enclosed with a grounded conducting surface, and the condition that u is constant on each Γ_j corresponds to Γ_j being made of conductors. If the total charge contained in Γ_j is given for each j , we add $\int_{\Gamma_j} u g_j$ (for suitable constants g_j) to the linear functional l which represents the “right-hand side” in the abstract formulation of problem, see below. (In the “PDE formulation” this part of the functional will not contribute to the right-hand side in the interior, but it will specify the integral of the normal derivative over each Γ_j , which corresponds to the charge.)

3. $X = \{u \in W^{1,2}(\Omega), \int_{\partial\Omega} u = 0\}$. This is a good space to take when considering the Neumann problem and the coefficients \tilde{b}_j, c vanish.

The differential operator associated with A is formally given by

$$(Lu, v)_{L^2} = A(u, v), \quad v \in H_0^1(\Omega) \quad (406)$$

and one has

$$Lu = -\frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right) + b_j \frac{\partial u}{\partial x_j} - \frac{\partial}{\partial x_j} (\tilde{b}_j u) + cu. \quad (407)$$

If we take $v \in X$ we obtain (formally)

$$A(u, v) = (Lu, v)_{L^2} + \text{boundary terms} \quad (408)$$

The boundary terms will be

$$\int_{\partial\Omega} a_{ij} \frac{\partial u}{\partial x_j} n_i v + \tilde{b}_j n_j u v + \gamma u v \quad (409)$$

and they will contribute to the boundary conditions, in addition with the boundary terms coming from the linear functional l below, and the possible Dirichlet boundary conditions incorporated in the definition of X . (The boundary condition covered by this set-up include only the “natural boundary condition”, we do not consider the oblique derivative problems mentioned in lecture 27, (379).

Let l be a continuous linear functional on X . We will denote by X^* the space of continuous linear functionals on X . A typical l is

$$l(v) = \int_{\Omega} f_0 v + f_j \frac{\partial v}{\partial x_j} + \int_{\partial\Omega} g v, \quad (410)$$

where f_0, f_1, \dots, f_n are L^2 -functions in Ω and g is an L^2 function on $\partial\Omega$.

The abstract problem of finding $u \in X$ with

$$A(u, v) = l(v), \quad v \in X \quad (411)$$

corresponds to

$$Lu = f_0 - \frac{\partial f_j}{\partial x_j} \quad \text{in } \Omega, \quad + \quad \text{boundary conditions.} \quad (412)$$

Note that the term $\int_{\Omega} f_j \frac{\partial v}{\partial x_j}$ can also contribute to the boundary conditions, as already discussed at the end of lecture 27.

Let us first consider the case of a coercive form A . We assume

$$A(u, u) \geq \alpha \|u\|_{W^{1,2}}^2, \quad u \in X, \quad (413)$$

for some $\alpha > 0$. With this assumption the problem (436) is uniquely solvable (by the Lax-Milgram Lemma) and the solution u satisfies

$$\|u\| \leq \frac{1}{\alpha} \|l\|_{X^*}, \quad (414)$$

where

$$\|l\|_{X^*} = \sup_{\|v\|_{W^{1,2}} \leq 1, v \in X} l(v). \quad (415)$$

Let $G_A: X^* \rightarrow X$ be the solution operator which takes l to the solution u . As each $f \in L^2(\Omega)$ defines an element of X^* by $v \rightarrow \int_{\Omega} f v$, the operator is defined

on $L^2(\Omega)$. By (438) G_A maps the unit ball of $L^2(\Omega)$ into a bounded subset of $W^{1,2}(\Omega)$. In view of the Rellich compactness theorem, this means that the operator G_A , considered as an operator from $L^2(\Omega)$ into $L^2(\Omega)$ is compact. (It is still useful to keep in mind that G_A is defined on the larger space X^* .)

We can repeat the above consideration with A replaced by A^* defined by

$$A^*(u, v) = A(v, u). \quad (416)$$

(The space X remains the same.) Obviously $A(u, u) = A^*(u, u)$ and hence A^* is coercive, with the same constant α .

Remark

If we work with spaces of complex-valued functions, then we have to make the usual adjustments to the definition of scalar products and the adjoint operator. For example the form A could be defined by

$$A(u, v) = \int_{\Omega} \left(a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \bar{v}}{\partial x_i} + b_j \frac{\partial u}{\partial x_j} \bar{v} + \tilde{b}_j u \frac{\partial \bar{v}}{\partial x_j} + cu\bar{v} \right) + \int_{\partial\Omega} \gamma u \bar{v}, \quad (417)$$

and the adjoint A^* by $A^*(u, v) = \overline{A(v, u)}$. If we use this definition and keep our other conventions, then $v \rightarrow A(u, v)$ gives *anti-linear* functionals, i. e. functionals l with $l(zv) = \bar{z}l(v)$ for $z \in \mathbb{C}$, and we should work with these. For example, for $f \in L^2(\Omega)$ we consider $l(v) = \int_{\Omega} f \bar{v}$. We also need to generalize the Lax-Milgram Lemma to the complex Hilbert space situation, which is an easy exercise. In what follows we will continue working with the real-valued functions, but we still will use the notation A^* , L^* etc., even though it could perhaps invoke spaces over the complex numbers. An alternative would be to use the notation A' , L' for the adjoints when we work over the real numbers. These notational issues are not a serious concern, there will be no danger of misunderstanding.

Let L, L^* be the differential operators associated with the forms $A(u, v), A^*(u, v)$. It is easy to see that, formally $(Lu, v)_L^2 = (u, L^*v)_L^2$. Let G_{A^*} the solution operator corresponding the form A^* . As G_A before, G_{A^*} can be considered as operators on $L^2(\Omega)$.

Lemma

With the notation above, we have for $f_1, f_2 \in L^2(\Omega)$

$$(G_A f_1, f_2)_{L^2} = (f_1, G_{A^*} f_2)_{L^2}. \quad (418)$$

In other words G_{A^*} is the L^2 -adjoint of G_A :

$$G_{A^*} = G_A^*. \quad (419)$$

Proof

Let $G_A f_j = u_j$, $j = 1, 2$. We just go through the definitions to verify

$$(G_A f_1, f_2)_{L^2} = A(u_2, u_1), \quad (f_1, G_{A^*} f_2) = A(u_2, u_1), \quad (420)$$

which of course implies (418).

If the form is symmetric, i. e. $A^* = A$, then the problem (436) corresponds to finding the minimizer of the functional

$$J(u) = \frac{1}{2}A(u, u) - l(u) \quad (421)$$

in the space X . In this case the operator G_A is self-adjoint. We have seen that it is also compact. Recalling the important fact that every compact self-adjoint operator on a Hilbert space is diagonal in some orthonormal (Hilbert) basis, we arrive at the following conclusion:

Theorem

Assume A is symmetric and coercive. Then there exists a sequence functions ϕ_1, ϕ_2, \dots in X , and real numbers $\lambda_1 \geq \lambda_2 \geq \dots \searrow 0$ such that ϕ_1, ϕ_2, \dots form an orthonormal Hilbert basis in $L^2(\Omega)$ and

$$G_A \phi_j = \lambda_j \phi_j. \quad (422)$$

Proof

Let $f \in L^2(\Omega)$, $f \neq 0$ and $u = G_A f$. Since A is coercive, we know that $u \neq 0$ and

$$(G_A f, f)_{L^2} = A(u, u) \geq \alpha \|u\|_{W^{1,2}}^2 > 0. \quad (423)$$

This shows that the spectrum is non-negative and that 0 is not an eigenvalue. The operator G_A maps $L^2(\Omega)$ into X , and therefore the equation $\lambda_j \phi_j = G_A \phi_j$ shows that all eigenfunctions are in X .

In terms of the differential operator L the equation for eigenfunction is

$$L\phi_j = \frac{1}{\lambda_j} \phi_j, \quad + \quad \text{boundary conditions}, \quad (424)$$

where the equation is satisfied in the sense of distributions and the boundary conditions are determined by the choice of the space X . For instance, in example 1 above, the boundary condition for each $\phi = \phi_j$ is $\phi = 0$ on Γ_0 and $a_{ij} \frac{\partial \phi}{\partial x_j} n_i + (\tilde{b}_j n_j + \gamma)\phi = 0$ on $\partial\Omega \setminus \Gamma_0$. (The last equation can be taken as it is only when ϕ are sufficiently regular which, as we shall see, is the case when the coefficients are smooth. In the general case of non-smooth coefficients when the expression $a_{ij} \frac{\partial \phi}{\partial x_j}$ is not defined point-wise, one has to interpret it in a weak sense.)

If A is not symmetric then G_A will not be self-adjoint. In general, compact operators on a Hilbert space which are not self-adjoint may not have to have any eigenvalues, the spectrum can be just $\{0\}$. However, this cannot happen with the operators G_A above. Even for non-symmetric A one can show that the real spectrum is non-trivial. This is a consequence of the Krein-Rutman theorem. In general we will have complex eigenvalues. ¹¹⁴

There are important cases when the form A is not coercive, and yet the boundary value problem is uniquely solvable. Let us first consider the following toy example:

Consider $-u'' + au = f$ on $(0, \pi)$, with $u(0) = u(\pi) = 0$. The corresponding form $A(u, v) = \int_0^\pi (u'v' + auv)$, which is not coercive for $a \leq -1$, as one can check by using the Fourier series $u = \sum_{k=1}^\infty u_k \sin kx$. From the Fourier series one can also see that the problem is uniquely solvable for each $f \in L^2$ if and only if $a \neq -k^2$, $k = 1, 2, \dots$

The above example is a special case of the following consequence of the spectral theory for the compact operators G_A considered above. If A is coercive and L is the corresponding operator, then the problem $Lu - \lambda u = f$ + natural boundary conditions is uniquely solvable for all $f \in L^2(\Omega)$ when for all $\lambda \in \mathbb{R} \setminus S$, where S is a discrete countable subset of \mathbb{R} . The forms corresponding to $L - \lambda$ are not coercive when λ is large.

In general, the Lax-Milgram form associated with the equations such as

$$a_{ij}(x)u_{ij} = f, \quad u|_{\partial\Omega} = 0$$

(where u_{ij} denotes $\frac{\partial^2 u}{\partial x_i \partial x_j}$) or

$$\Delta u + b_j(x)u_j = f, \quad u|_{\partial\Omega} = 0$$

(where $u_j = \frac{\partial u}{\partial x_j}$) are not coercive, although there are interpretations of those equations¹¹⁵ from which it is more or less clear that they should be uniquely solvable. We will see that one can still use the Direct Method approach to handle these cases, but only with the help of additional arguments (such as the maximum principle).

¹¹⁴Is the space generated by the (possibly complex) eigenfunctions dense in (the complexification of) X ? I am sure this has been investigated and the answer should be in the literature.

¹¹⁵such as the probabilistic interpretation

Lecture 31, 11/19/2010

We now consider problems with non-coercive Lax-Milgram forms. We assume that $A(u, v)$ is of form (403) as in the last lecture (with the same assumptions of the coefficients). As before, $A(u, v)$ is considered on some space X with $H_0^1(\Omega) \subset X \subset W^{1,2}(\Omega)$, which is dictated by the boundary conditions.

For $\lambda \in \mathbb{R}$ we consider the form

$$A_\lambda(u, v) = A(u, v) + \lambda(u, v)_L^2. \quad (425)$$

Lemma

There exists λ_0 such that for $\lambda \geq \lambda_0$ the form A_λ is coercive in $W^{1,2}(\Omega)$, with

$$A_\lambda(u, u) \geq \frac{\nu}{2} \|u\|_{W^{1,2}}^2, \quad u \in W^{1,2}(\Omega), \quad (426)$$

where ν is the constant in the ellipticity condition (455).

Proof: We have

$$A(u, u) \geq \int_\Omega (\nu |\nabla u|^2 - C_1 |\nabla u| |u| - C_2 |u|^2) - C_3 \int_{\partial\Omega} |u|^2. \quad (427)$$

We use

$$|\nabla u| |u| \leq \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{2\varepsilon} |u|^2 \quad (428)$$

and choosing $\varepsilon = \frac{\nu}{2C_1}$ we obtain

$$A(u, u) \geq \int_\Omega \left(\nu |\nabla u|^2 - \frac{\nu}{4} |\nabla u|^2 - C_4 |u|^2 \right) - C_3 \int_{\partial\Omega} |u|^2. \quad (429)$$

If it was not for the boundary term, this would prove the lemma (with $\lambda_0 = C_4 + \nu/2$).

To deal with the boundary term, we use

$$\|u|_{\partial\Omega}\|_{L^2(\partial\Omega)}^2 \leq C \|u\|_{W^{1,2}(\Omega)} \|u\|_{L^2(\Omega)}. \quad (430)$$

To obtain this strengthened version of the trace inequality (321), it is enough to replace the identity

$$u(x', g(x')) = \int_{g(x')}^b -\frac{\partial u(x', s)}{\partial x_n} ds, \quad (431)$$

in the proof of the trace inequality by

$$|u(x', g(x'))|^2 = \int_{g(x')}^b -\frac{\partial |u(x', s)|^2}{\partial x_n} ds = \int_{g(x')}^b -2 \frac{\partial u(x', s)}{\partial x_n} u(x', s) ds, \quad (432)$$

integrate over x' , and use the Cauchy-Schwartz inequality. Once we have (458), we can use

$$\|u\|_{W^{1,2}} \|u\|_{L^2} \leq \frac{\varepsilon}{2} \|u\|_{W^{1,2}}^2 + \frac{1}{2\varepsilon} \|u\|_{L^2}^2 \quad (433)$$

and absorb the first term on the right-hand side into the first term of (457) (at the cost replacing ν with a smaller number). This finishes the proof of the lemma.

Let $l \in X^*$ and consider the problem

$$A(u, v) = l(v), \quad v \in X, \quad (434)$$

corresponding to

$$Lu = f \quad + \quad \text{boundary conditions}, \quad (435)$$

as discussed in the last lecture. Let λ_0 be such that the form A_{λ_0} is coercive and write (434) as

$$A_{\lambda_0}(u, v) = l(v) + \lambda_0 \int_{\Omega} uv, \quad v \in X. \quad (436)$$

Let G_0 be the solution operator of the form A_{λ_0} , defined in the last lecture. We can re-write (436) as

$$u = G_0 l + \lambda_0 G_0 u, \quad (437)$$

or

$$u - \lambda_0 G_0 u = G_0 l. \quad (438)$$

We have seen in the last lecture that the operator G_0 can be considered as a compact operator on $L^2(\Omega)$, and hence (438) can be considered as a Fredholm equation in $L^2(\Omega)$. Note that $G_0 l \in X$ and $G_0 v \in X$ for each $v \in L^2(\Omega)$, so any solution $u \in L^2(\Omega)$ of (438) will automatically be in X . We apply the Fredholm theorems to (438). We denote by G_0^* the L^2 -adjoint of G_0 . As we have seen in the last lecture, this is the solution operator associated with the form $A_{\lambda_0}^*$, or the differential operator $L^* + \lambda_0$ (with appropriate boundary conditions).

Let us first look at the question of unique solvability of (434). By Fredholm theory we know that $I - \lambda_0 G_0$ is invertible (as an operator on $L^2(\Omega)$) if and only if the only solution $u \in L^2(\Omega)$ of $u - \lambda_0 G_0 u = 0$ is $u = 0$. Tracing back our definitions, we see that the condition $u - \lambda_0 G_0 u = 0$ for a $u \in L^2(\Omega)$ means that $u \in X$ and

$$A(u, v) = 0 \quad v \in X. \quad (439)$$

Hence we have

Theorem

The problem

$$A(u, v) = l(v), \quad v \in X \quad (440)$$

is uniquely solvable in X for each $l \in X^*$ if and only if the homogeneous problem

$$A(u, v) = 0, \quad v \in X \quad (441)$$

has only the trivial solution $u = 0$ in X .

Recall that l can be of the form

$$l(v) = \int_{\Omega} (f_0 v + f_j \frac{\partial v}{\partial x_j}) + \int_{\partial\Omega} g v. \quad (442)$$

and assume that $X = W^{1,2}(\Omega)$. Then the boundary problem corresponding to (440) is

$$-\frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j} + b_j \frac{\partial u}{\partial x_j} - \frac{\partial}{\partial x_j} (\tilde{b}_j u) + c u = f_0 - \frac{\partial f_j}{\partial x_j} \quad \text{in } \Omega, \quad (443)$$

$$a_{ij} \frac{\partial u}{\partial x_j} n_i + (\tilde{b}_j n_j + \gamma) u = g + f_j n_j \quad \text{at } \partial\Omega. \quad (444)$$

The theorem above says that this boundary values problem is uniquely solvable in $W^{1,2}(\Omega)$ for each $f_0, f_1, \dots, f_n \in L^2(\Omega)$ and each $g \in L^2(\partial\Omega)$ if and only if the problem with $f_0 = f_1 = \dots = f_n = 0$ and $g = 0$ has only the trivial solution $u = 0$.

It is important to note that our definition of the solution of (443), (444) is exactly (440). At this stage we cannot tell if the solutions are smooth when all the coefficients are smooth. In particular, even in the case of smooth coefficients, to prove unique solvability of (440), we do not yet know if it is enough to verify that the only *smooth* solution of (443), (444) is $u = 0$. The theorem above merely says that we can conclude unique solvability if the variational formulation (441) of the homogeneous problem has no other solution than $u = 0$. If we knew that each such solution has to be smooth, it would be enough to look only at the smooth solutions. We will see later that this is indeed the case, but we will have to work some more to get to that conclusion.

As an example where the above issues are relevant, let us consider

$$-\Delta u + b_j \frac{\partial u}{\partial x_j} = f \quad \text{in } \Omega, \quad (445)$$

$$u = 0 \quad \text{at } \partial\Omega. \quad (446)$$

In this case we work with $X = H_0^1(\Omega)$. Assume the coefficient b_j are smooth and that Lax-Milgram form is not coercive.¹¹⁶ To decide the unique solvability¹¹⁷ we

¹¹⁶As an exercise, find some b_j where this would be the case.

¹¹⁷assuming that A is not coercive

should look at the case $f = 0$. We suspect, based on the maximum principle,¹¹⁸ that the only solution of (445), (446) with $f = 0$ is $u = 0$. However, if we wish to decide that using only the theorem above, we would have to show that every $H_0^1(\Omega)$ -solution of the problem with $f = 0$ is trivial. At the same time, general functions in $H_0^1(\Omega)$ do not have enough smoothness for our proof of the maximum principle to work. We see that we have to develop the theory a little further before we can take the full advantage of the above theorem.

Let us now go back to (438) and consider the situation when the the equation $v - \lambda_0 G_0 v = 0$ does have non-trivial solutions. The Fredholm theory tells us the following: consider the L^2 -adjoint G_0^* of G_0 , which – as we have seen last time – can also be defined as the solution operator for the form $A_{\lambda_0}^*(u, v) = A_{\lambda_0}(v, u)$. Set

$$Y = \ker(I - \lambda_0 G_0^*) = \{v \in L^2(\Omega), v - \lambda_0 G_0^* v = 0\} \subset X \subset W^{1,2}(\Omega). \quad (447)$$

The equation

$$u - \lambda_0 G_0 u = G_0 l \quad (448)$$

is solvable in $L^2(\Omega)$ if and only if $G_0 l$ is perpendicular to the kernel Y :

$$\int_{\Omega} (G_0 l) v = 0 \quad v \in Y. \quad (449)$$

We have, by our definitions,

$$\int_{\Omega} v(G_0 l) = A(G_0 v, G_0 l) = l(G_0^* v). \quad (450)$$

Remark

We have shown earlier, see (420), that G_{A^*} is the L^2 -adjoint of G_A . The same proof shows (just as (450)) that $l_1(G_{A^*} l_2) = l_2(G_A l_1)$ for $l_1, l_2 \in X^*$. We can also use the notation $\langle l, v \rangle = l(v)$. Then we can write $\langle l_1, G_A l_2 \rangle = \langle l_2, G_{A^*} l_1 \rangle$.

For $v \in Y$ we have $G_0 v = \frac{1}{\lambda_0} v$, and hence

$$\int_{\Omega} v(G_0 l) = \frac{1}{\lambda_0} l(v). \quad (451)$$

We have arrived at an important generalization if the previous theorem:

¹¹⁸You can check which of our proofs of the maximum principle for harmonic functions (lecture 6) work for this more general equation when all the involved functions are sufficiently regular. At least one of them does!

Theorem

Let A be a Lax-Milgram form (403) on X , not necessarily coercive, with a_{ij} satisfying the ellipticity condition (455). Then

$$A(u, v) = l(v), \quad v \in X \quad (452)$$

is solvable if and only if $l(v) = 0$ for each v which solves the homogeneous adjoint problem

$$A(w, v) = 0, \quad w \in X. \quad (453)$$

Moreover, the space of the solutions of the adjoint problem is finite-dimensional.

Proof: See above.

Example

Let $X = W^{1,2}(\Omega)$, let the form A be given by $A(u, v) = \int_{\Omega} \nabla u \nabla v$, and let $l(v) = \int_{\Omega} f v + \int_{\partial\Omega} g v$ for some $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$. This corresponds to solving $-\Delta u = f$ in Ω and $\frac{\partial u}{\partial n} = g$ at $\partial\Omega$. The adjoint problem is $A(w, v) = 0$ for each $w \in W^{1,2}(\Omega)$, corresponding to $\Delta v = 0$ in Ω and $\frac{\partial v}{\partial n} = 0$. (In this case the form is self-adjoint.) It is easy to see that the only solutions of the adjoint problem are constant functions. Clearly these are solutions, and using $w = v$ in the definition of the adjoint problem we see that the solutions of the adjoint problem must satisfy $A(v, v) = 0$. Our assumption is that Ω is connected, and hence v must be constant.¹¹⁹ By the theorem above the original problem will be solvable if and only if $l(v) = 0$ for each constant function v . This is exactly the condition $\int_{\Omega} f + \int_{\partial\Omega} g = 0$, the necessity of which one can get by integrating the original equation over Ω .

An alternative set-up for this problem would be to use the space $X = \{u \in W^{1,2}(\Omega), \int_{\partial\Omega} u = 0\}$. Take A as above, and show that this time A is coercive on X . Therefore the problem $A(u, v) = l(v)$ for each $v \in X$ now has a unique solution for each $v \in X$. As an exercise, explain that this conclusion does not contradict the previous one.

Yet another approach would be to minimize $\int_{\Omega} (|\nabla u|^2 - f u) - \int_{\partial\Omega} g u$ directly on the space $W^{1,2}(\Omega)/\text{constants}$. Note that $\int_{\Omega} f v + \int_{\partial\Omega} g v$ is well-defined on this space if and only if $\int_{\Omega} f + \int_{\partial\Omega} g = 0$.

Of course, all the three approaches are just alternative ways of describing the same.

¹¹⁹Note that we are using the result that if the $\nabla u \in L^2$ exist in the sense of distributions and vanishes, then the function is constant.

We will discuss the meaning of the various terms in the equations conditions we have looked at so far, together with the various boundary conditions. Let us start with the equation

$$-\Delta u = f \quad \text{in } \Omega \quad + \quad \text{boundary conditions} \quad (454)$$

We have seen that this can describe the electric potential u . If the environment in which the potential is considered is inhomogeneous, then Δ is replaced by $\frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j}$, where a_{ij} describe the properties of the material. The boundary conditions can depend on how we model the material of the boundary. Rather than going into the details we can refer the reader to the classical text “Electrodynamics of Continuous Media” by Landau and Lifschitz.

Equation (454) can also describe steady distribution of temperature or density of particles subject to diffusion, in an environment where the heat conductivity or the diffusivity are constant in x . The right-hand side f describes the heat sources or the particle sources respectively. The Dirichlet boundary condition $u = g$ at $\partial\Omega$ means that the temperature (resp. particle density) is forced to a given value g by some external devices (heating/cooling of the boundary or adding/removing particles at the boundary). The Neumann boundary condition $\frac{\partial u}{\partial n} = 0$ at $\partial\Omega$ means that the boundary insulates the domain from external influences. The inhomogeneous boundary condition $\frac{\partial u}{\partial n} = g$ at $\partial\Omega$ means that a given steady flux of heat/particles (specified by g) is kept at the boundary by external devices. The boundary condition $\frac{\partial u}{\partial n} + \gamma u = 0$ means that the boundary is partially insulating, with $\gamma = 0$ corresponding to the insulating case and $\gamma = +\infty$ corresponding to the case with no insulation.

If the heat conductivity or the diffusivity of the environment is not constant, we have to introduce variable coefficients into the laplacian. This has to be done differently for the heat equation and the diffusion equations. In both cases the properties of the environment are described by a matrix $a_{ij}(x)$ which locally characterizes the relevant environment properties.

When describing heat conduction, with u being the temperature, the right equation for u (in the absence of sources) is

$$-\frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} = 0, \quad (455)$$

similar to what we have in electrostatics. This is a *divergence form* equation. Note that u satisfies the maximum principle,¹²¹ as one would heuristically expect: at a steady-state, the temperature cannot have a local maximum or minimum inside the domain if no sources are present. The Neumann boundary

¹²⁰To avoid ambiguity, we should perhaps write $-\frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) = 0$.

¹²¹You can check as an exercise that some proofs of the maximum principle we did for harmonic functions work in this more general case.

condition (describing an insulating boundary) will be $a_{ij}(x)\frac{\partial u}{\partial x_j}n_i = 0$ at $\partial\Omega$, reflecting the fact that the heat flux, given by $I_i = a_{ij}\frac{\partial u}{\partial x_j}$, cannot transport heat across the insulating boundary.

For diffusion the correct equation is (in our simple situation) a special case of the *Fokker-Plank* equation, also called the *forward Kolmogorov equation*. In our special case it will be

$$\frac{\partial^2}{\partial x_i \partial x_j} a_{ij}(x)u = 0. \quad (456)$$

Note that this equation will not satisfy the maximum principle.¹²³ This is because particles can get stuck in areas where diffusion is small, and their concentration in such regions will be higher. The natural Neumann-type boundary condition describing an insulating boundary will be $n_i \frac{\partial}{\partial x_j} (a_{ij}(x)u) = 0$, which is still a “natural boundary condition” for our theory, and it is covered by our theorems, at least for sufficiently regular coefficients a_{ij} .

Although (456) does not satisfy the maximum principle, the dual equation

$$a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} = 0 \quad (457)$$

does satisfy it.¹²⁴ This is a *non-divergence form* equation. A natural interpretation of u in (457) is not in terms of some densities, but in terms of mean values and probabilities. For example, for the non-homogeneous Dirichlet condition $u = g$ at $\partial\Omega$, the value $u(x)$ is $\int_{\partial\Omega} g(x)p(x,y)dy$, where $y \rightarrow p(x,y)$ is a function on $\partial\Omega$, with $\int_M p(x,y)dy$ being the probability that a particle starting at x will hit the boundary for the first time at M . In other words, $y \rightarrow p(x,y)$ is a probability density with which particles “launched” at x will be hitting the boundary. With this interpretation the maximum principle is transparent. Note also the similarity between $p(x,y)$ and the Poisson kernel introduced earlier for the Laplace equation.

You can check that our theory based on the Lax-Milgram lemma works very well for the divergence form equations even in the case when a_{ij} are merely bounded and measurable. On the other hand, to cover the non-divergence form equations, we need stronger assumptions on a_{ij} . With the theorems we have proved so far we need that a_{ij} are Lipschitz.¹²⁵

¹²²Here also we should perhaps write $\frac{\partial^2}{\partial x_i \partial x_j} (a_{ij}(x)u) = 0$ to avoid ambiguity.

¹²³Give an example in dimension 1.

¹²⁴Show that at least one proof we did for the harmonic functions works also in this more general case, at least when the coefficients are smooth.

¹²⁵This can be extended to continuous coefficients a_{ij} and slightly further, but not much, if we wish to keep uniqueness. By contrast, we have seen that uniqueness is no problem for divergence-form equations with bounded measurable coefficients.

Deeper properties (such as Hölder continuity, or Harnack inequality) of solutions of divergence-form equations with bounded measurable coefficients in arbitrary dimensions were discovered in the 1950s and 1960s (by E. DeGiorgi, J. Nash, J. Moser, O. A. Ladyzhenskaya, N. Uraltseva and others) For non-divergence form equations with measurable coefficients these

Let us now turn to the terms $b_j \frac{\partial u}{\partial x_j}$ or $\frac{\partial}{\partial x_j}(b_j u)$. The diffusion interpretation provides a simple heuristics for the term $\frac{\partial}{\partial x_j}(b_j u)$. We imagine that the diffusion takes place in a fluid (compressible or incompressible) which moves, and the motion (often called a *drift*) is given by the vector field b . In addition to the random motion, the particles are “carried with the fluid”. For the simple diffusion given by the laplacian, the equation describing the stationary state of this process is (in the absence of sources)

$$-\Delta u + \frac{\partial}{\partial x_j}(b_j u) = 0, \quad (458)$$

which is again a special case of the general Fokker-Planck equation (or forward Kolmogorov equation).¹²⁶ The interpretation of the various boundary conditions is similar to other examples. The Dirichlet boundary condition $u = 0$ at some part of the boundary means that particles disappear once they touch the boundary. The condition that the boundary is impenetrable to the particles (but does not destroy them) is $-\frac{\partial u}{\partial n} + b_j n_j u = 0$. Note that (458) does not satisfy the maximum principle, as the particles can concentrate at places where the drift is “focusing”. (This is not possible when the underlying flow b is incompressible, i. e. satisfies $\operatorname{div} b = 0$.) The adjoint equation to (458) is

$$-\Delta u - b_j \frac{\partial u}{\partial x_j} = 0. \quad (459)$$

As with (457), this equation does have the maximum principle. The natural interpretation of u in (459) is in terms of probability and mean values (and not some particle density), similar to (457). The boundary condition for an “insulating boundary” is $\frac{\partial u}{\partial n} = 0$.

The reader who is interested in these interpretations should consult a book on stochastic processes.¹²⁷ In the stochastic interpretation, it is often easier to work with the dual of the Fokker-Planck equation (often call the *Kolmogorov backward equation*), as its solution can be directly written down in terms probabilities of events for well-defined processes.

Finally, let us look at the term $c(x)u$ in the equations. The interpretation of this term is simple for the Fokker-Planck equation: adding $c(x)u$ means that the particles decay (if $c \geq 0$). Roughly speaking, each particle moving near a point x has a certain probability (proportional to $c(x)$) of disintegrating and disappearing.

were only proved in the late 1970s (by N. Krylov and M. Safonov from our own department), and the lack of uniqueness in this case was only shown in 1990s (by N. Nadirashvili). It turns out that the regularity results obtained for these general classes of linear equations are crucial for studying *smooth* non-linear equations.

¹²⁶ $-\frac{\partial^2}{\partial x_i \partial x_j} a_{ij}(x)u + \frac{\partial}{\partial x_j}(b_j u) + cu = f$

¹²⁷For example, the book “Stochastic Differential equations: An Introduction with Applications” by B.K.Oksendal provides a god introduction to these topics.

If you are in doubt about the meaning of a particular term in the equation or a specific boundary condition, it is usually helpful to consider simple examples in dimension $n = 1$. Often these are good for obtaining some reasonable heuristics for what happens with the solutions even in dimensions $n \geq 2$.

We now turn to regularity theory for the solutions we constructed by the Direct Method. We will use the standard method based on L^2 -estimates of the higher derivatives. The main idea can be illustrated on the example of the Laplace equation. Suppose $u \in W^{1,2}(\Omega)$ satisfies

$$\int_{\Omega} \nabla u \nabla v = 0, \quad v \in H_0^1(\Omega). \quad (460)$$

We will prove that u has to be smooth in the following steps:

Step 1 (Caccioppoli's inequality).¹²⁸

For any concentric balls $B_{R_1} \subset B_{R_0} \subset \Omega$ we have

$$\int_{B_{R_1}} |\nabla u|^2 \leq \frac{C}{(R_0 - R_1)^2} \int_{B_{R_0}} |u|^2. \quad (461)$$

Step 2 (Formal application of (461) to higher derivatives).

At first (461) may not seem to be a very strong conclusion – we already know that $u \in W^{1,2}(\Omega)$. However it is stronger than it looks due to its local nature. We note that any partial derivative of u satisfies again the Laplace equation, at least formally, so we should be able to apply (461) successively to higher and higher derivatives of u and obtain, for any concentric balls $B_{R_k} \subset B_{R_{k-1}} \cdots \subset B_{R_0} \subset \Omega$:

$$\begin{aligned} \int_{B_{R_k}} |\nabla^k u|^2 &\leq \frac{c_k}{(R_{k-1} - R_k)^2} \int_{B_{R_{k-1}}} |\nabla^{k-1} u|^2, \\ \int_{B_{R_{k-1}}} |\nabla^{k-1} u|^2 &\leq \frac{c_{k-1}}{(R_{k-2} - R_{k-1})^2} \int_{B_{R_{k-2}}} |\nabla^{k-2} u|^2, \\ &\dots \\ \int_{B_{R_1}} |\nabla u|^2 &\leq \frac{c_1}{(R_0 - R_1)^2} \int_{B_{R_0}} |u|^2, \end{aligned} \quad (462)$$

which implies (by choosing for example $(R_j - R_{j-1}) \sim 2^{-j-1}(R_0 - R_k)$)

$$\int_{B_{R_k}} |\nabla^k u|^2 \leq \frac{C_k}{(R_0 - R_k)^{2k}} \int_{B_{R_0}} |u|^2 \quad (463)$$

for suitable constants C_k (which can approach infinity as $k \rightarrow \infty$).¹²⁹

Step 3 (Justification of the formal calculations in step 2).

In step 2 we were taking higher derivatives of the equations, but a priori we do not know that these derivatives are integrable functions. We have to show that this is indeed the case. This can be done by first replacing derivatives with

¹²⁸sometimes also called “local energy inequality”

¹²⁹The constants c_k in (462) depend on k as the number of components of $\nabla^k u$ increases with k .

difference quotients, such as $D_j^h u(x) = \frac{u(x+he_j) - u(x)}{h}$, and checking that at each step of establishing (462) we can first apply (461) to the difference quotient and then pass to the limit. For that we will need a characterization of the condition $\frac{\partial u}{\partial x_j} \in L^2$ in terms of bounds on the difference quotients $D_j^h u$ in L^2 which are uniform in $h \in (0, h_0)$.

Step 4 (Sobolev Imbedding Theorem)

Steps 1–3 establish that u , in the sense of distributions, the derivatives of u of any order belong locally to L^2 . We need to show that this implies that u is smooth. The Sobolev Imbedding Theorem says that if a function has more than $n/2$ derivatives in L^2 , it is continuous. This implies that in our situation above, the function u is smooth.

Steps 1–4 establish the following version of the pointwise estimate for harmonic functions (33) from lecture 4

$$\sup_{B_{R_1}} |\nabla^k u|^2 \leq C(k, R_0, R_1) \int_{B_{R_0}} |u|^2. \quad (464)$$

The factor $C(k, R_0, R_1)$ can be made more precise¹³⁰, but (464) is sufficient for our purposes at the moment.¹³¹

Estimate (464) is an example of a *local interior estimate*. Note that boundary conditions play no role in it. There are two other types of estimates used in the regularity theory: *local boundary estimates* and *global estimates*. We will discuss those later.

The above proof of regularity looks more complicated than the method in lecture 4 using the representation formulae, but it has a significant advantage: it can be very easily generalized to equations with lower order terms, variable coefficients, boundary regularity, etc. We will see that in such more general situations the proof remains essentially the same, except that we have to write more terms in the calculations, and go through some additional steps when proving boundary regularity. In fact, the method works for all equations we have been studying so far when the coefficients are sufficiently regular. For boundary regularity we also need that the boundary is sufficiently regular. The method used in lecture 4 can also be generalized to cover this class of equations, but the generalization is more complicated. Regularity theory for equations with not-so-regular coefficients (or in not-so-regular domains) requires additional ideas.

We will now proceed with a justification of steps 1 – 4 above.

Proof of (461)

Let η be a smooth cut-off function with $\eta = 1$ in B_{R_1} , compactly supported in B_{R_0} , and $|\nabla \eta| \leq \frac{2}{R_0 - R_1}$. We use (460) with

¹³⁰The optimal form would be $\frac{C_k}{(R_0 - R_1)^{2k+n}}$.

¹³¹The more precise form can be useful for the proof of the Liouville theorem, for example.

$$v = u\eta^2 \quad (465)$$

and obtain

$$\int_{\Omega} |\nabla u|^2 \eta^2 = \int_{\Omega} -2(\eta \nabla u)(u \nabla \eta) \leq \frac{\varepsilon}{2} \int_{\Omega} |\nabla u|^2 \eta^2 + \frac{1}{2\varepsilon} \int_{\Omega} 4|u|^2 |\nabla \eta|^2, \quad (466)$$

where $\varepsilon > 0$ can be chosen arbitrarily. Choosing $\varepsilon = \frac{1}{2}$, we obtain (461) with $C = 32$.

Difference quotients

We recall that

$$D_j^h u(x) = \frac{u(x + he_j) - u(x)}{h}, \quad (467)$$

where e_j is the unit vector in the direction of the x_j axis, $h \neq 0$, and $x \in \Omega_{|h|} = \{x \in \Omega, \text{dist}(x, \partial\Omega) > |h|\}$.

Lemma

Let $u \in L^2(\Omega)$. The following conditions are equivalent

- (i) There exists $w_j \in L^2(\Omega)$ such that $\frac{\partial u}{\partial x_j} = w_j$ in Ω in the sense of distributions.
- (ii) For some $h_0 > 0$ we have

$$\int_{\Omega_h} |D_j^h u|^2 \leq C < +\infty \quad (468)$$

uniformly for $h \in (0, h_0)$.

Remark

It will be clear from the proof of the lemma that if one the equivalent conditions is satisfied, than in fact $D_j^h u \rightarrow \frac{\partial u}{\partial x_j}$ as $h \rightarrow 0$ strongly in L^2 on compact subsets of Ω , but we will not need this in what follows.

Proof of the lemma:

The implication (i) \implies (ii) is an easy consequence of the identity

$$D_j^h u(x) = \int_0^1 \frac{\partial u}{\partial x_j}(x + he_j t) dt, \quad x \in \Omega_h, \quad (469)$$

and the continuity properties of the translation operator on Lebesgue spaces.¹³²

¹³²It is a good exercise to go through all the details, there are several things working in the background of this argument.

To prove (ii) \implies (i), we note that for any compactly supported smooth function φ we can write

$$\int_{\Omega} -u \frac{\partial \varphi}{\partial x_j} = \lim_{h \rightarrow 0^+} \int_{\Omega} -u D_j^{-h} \varphi = \lim_{h \rightarrow 0^+} \int_{\Omega} D_j^h u \varphi \leq \sqrt{C} \|\varphi\|_{L^2(\Omega)}. \quad (470)$$

We see that the linear functional $\varphi \rightarrow \int_{\Omega} -u \frac{\partial \varphi}{\partial x_j}$ is continuous with respect to the L^2 -norm, and hence can be represented by an L^2 function w_j as

$$\int_{\Omega} -u \frac{\partial \varphi}{\partial x_j} = \int_{\Omega} w_j \varphi, \quad (471)$$

which exactly says that $\frac{\partial u}{\partial x_j} = w_j$ in the sense of distributions.

Remark

The above proof also shows that the following condition is also equivalent to condition (i) and (ii) in the lemma:

(iii) $\liminf_{h \rightarrow 0} \int_{\Omega_h} |D_j^h u|^2 < +\infty$.

We start with discussion of step 4 – the Sobolev Imbedding Theorem – in the scheme for establishing regularity we discussed in the last lecture. Imbedding theorems are a large topic in the theory of function spaces, and at this point we will only cover one particular result which we need for our regularity proofs. We will return to this topic later and study it in a more systematic way.

For step 4 of our regularity theory we only need to establish, roughly speaking, that a function u with sufficiently many derivatives in L^2 is continuous. Applying this to $\nabla^k u$ will also show that if k more derivatives are in L^2 the function is C^k .

Let us consider some easy special cases.

Example

If u is a compactly supported function in R^3 with $\nabla^2 u \in L^2$, then u is continuous.

This is a direct consequence of the representation formula (25) from lecture 3. Assume the support of u is in B_R and let $G_R = G\chi_{B_{2R}}$, where $\chi_{B_{2R}}$ is the characteristic function of the ball B_{2R} . Assume first that u is smooth. Then for $x \in B_R$

$$u(x) = \int G(x-y)\Delta u(y) dy = \int G_R(x-y)\Delta u(y) dy \leq \|G_R\|_{L^2} \|\Delta u\|_{L^2}. \quad (472)$$

For a continuous function v on R^n let us denote

$$\|v\|_C = \sup_x |v(x)|. \quad (473)$$

Identity (472) shows that

$$\|u\|_C \leq C_R \|\nabla^2 u\|_{L^2}, \quad (474)$$

where C_R is a constant depending on R . If u is not smooth, we can use (474) for $u_\varepsilon = u * \phi_\varepsilon$ (where ϕ_ε is a mollifier), and (474) shows that u_ε converge uniformly to u , showing that u is continuous.¹³³

The reader can check that a similar result works also in dimension $n = 2$.

In dimension $n = 4$ a compactly supported function u with $\nabla^2 u \in L^2$ may no longer be continuous.¹³⁴ On the other hand, if u is supported in B_R , $\nabla^3 u \in L^2$ (and $n = 4$), u will be continuous. This can be seen from the representation formula

$$u = G * G * \Delta^2 u = G_{,j} * G * \Delta u_{,j} = (G_{,j} * G) * \Delta u_{,j} = Q_j * \Delta u_{,j} = (Q_j \chi_{B_{2R}}) * \Delta u_{,j}, \quad (475)$$

¹³³Strictly speaking, if we wish our terminology to be completely accurate, we should not say that u is continuous, but rather that u has a continuous representative. This is because originally our function u is only defined as an element of some Lebesgue space, and its point-wise values are defined only almost everywhere. Our proof above then shows that the class of measurable functions which are equal to u almost everywhere contains a continuous representative. Once it exists, such representative is of course unique.

¹³⁴Hint: consider $u(x) = f(|x|)$ for a suitable $f: (0, \infty) \rightarrow R$.

where $G(x) = \frac{1}{2|S^3||x|^2}$, we use the notation $f_{,j}$ for $\frac{\partial f}{\partial x_j}$, and $Q_j = G_{,j} * G$. The function $Q_j \chi_{B_{2R}}$ is easily checked to be in L^2 . (It is $O(\frac{1}{|x|})$ at the origin.)

The above proof also work in dimension $n = 5$, with $G = \frac{1}{3|S^4||x|^3}$.

In dimension $n = 6$ it is not enough to have $\nabla^3 u$ in L^2 to get continuity.¹³⁵ However, if we have $\nabla^4 u \in L^2$, u will be continuous. This is again seen from

$$u = G * G * \Delta^2 u, \quad G = \frac{1}{(n-2)|S^{n-1}||x|^{n-2}}, \quad n = 6 \quad (476)$$

and

$$G * G \chi_{B_{2R}} \in L^2. \quad (477)$$

Continuing this pattern we arrive at the following conclusion: if $u \in L^2(\mathbb{R}^n)$ is compactly supported and $\nabla^m u \in L^2$ for $m > \frac{n}{2}$, then u is continuous.

If u is not compactly supported, we can multiply it by a suitable cut-off function φ so that φu is compactly supported. If $\nabla^j u \in L^2(\Omega)$ for $j = 0, 1, \dots, m$ and φ is supported in Ω , then we can extend φu by 0 outside Ω and $\nabla^m(\varphi u) \in L^2(\mathbb{R}^n)$. Hence we have shown

*If $\Omega \subset \mathbb{R}^n$ and $u \in L^2_{\text{loc}}(\Omega)$ with $\nabla^j u \in L^2_{\text{loc}}(\Omega)$ for $j = 1, \dots, m$ for some $m > n/2$, then u is continuous in Ω .*¹³⁶

Proof: See above.

We now obtain a generalization of this result by using Fourier Transformation.¹³⁷

Let $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ be the space of complex-valued smooth functions f in \mathbb{R}^n satisfying

$$|\nabla^k f(x)|(1 + |x|^2)^{m/2} \leq C_{k,m} \quad x \in \mathbb{R}^n. \quad (478)$$

The space \mathcal{S} is called the Schwartz class (after L. Schwartz). For $f \in \mathcal{S}$ we define $\hat{f}: \mathbb{R}^n \rightarrow \mathbb{C}$, the Fourier transform of f , by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\xi x} dx. \quad (479)$$

Note that the integral is convergent and that \hat{f} is a smooth function of ξ .

Integration by parts gives

$$\widehat{\frac{\partial f}{\partial x_j}}(\xi) = i\xi_j \hat{f}(\xi), \quad (480)$$

and

$$\widehat{-ix_j f} = \frac{\partial f}{\partial \xi_j}. \quad (481)$$

¹³⁵Hint: consider $u(x) = f(|x|)$ for suitable f .

¹³⁶As we already discussed, the precise statement is that u has a continuous representative.

¹³⁷Expressing functions via their Fourier Transformation can also be thought of as a representation formula, and it makes certain properties of functions very transparent.

These two formulae show that when $f \in \mathcal{S}$, then $\hat{f} \in \mathcal{S}$.

A simple application of the Fubini theorem shows that for $f, g \in \mathcal{S}$ we have

$$\int_{\mathbb{R}^n} f \hat{g} = \int_{\mathbb{R}^n} \hat{f} g. \quad (482)$$

We can calculate explicitly

$$\widehat{e^{-\frac{|x|^2}{2}}} = (2\pi)^{\frac{n}{2}} e^{-\frac{|\xi|^2}{2}}. \quad (483)$$

We also have, for $g \in \mathcal{S}$ and $\varepsilon > 0$

$$\widehat{g(\varepsilon x)} = \varepsilon^{-n} \hat{g}\left(\frac{\xi}{\varepsilon}\right). \quad (484)$$

Using (482) with $g(x) = g_\varepsilon(x) = e^{-\frac{|\varepsilon x|^2}{2}}$ and letting $\varepsilon \rightarrow 0$, we obtain, for any $f \in \mathcal{S}$ the formula:

$$f(0) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) d\xi. \quad (485)$$

Combining (485) with the obvious formula

$$\widehat{f(x+a)} = e^{ia\xi} \hat{f}(\xi) \quad (486)$$

we obtain, for any $f \in \mathcal{S}$ the crucial *inversion formula*

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i\xi x} d\xi. \quad (487)$$

It can be thought of as a kind of representation formula for f . It expresses f as a “linear combination” of the simple functions $x \rightarrow e^{i\xi x}$.

Letting $f^*(x) = \overline{\hat{f}(-x)}$, where the bar denotes conjugation of complex numbers, we check easily

$$\widehat{f^*} = \hat{f}. \quad (488)$$

An easy calculation shows that the product fg for $f, g \in \mathcal{S}$ transforms to convolution

$$\widehat{fg} = \frac{1}{(2\pi)^n} \hat{f} * \hat{g}, \quad (489)$$

and the convolution $f * g$ transform to the product:

$$\widehat{f * g} = \hat{f} \hat{g}. \quad (490)$$

Finally, applying the inversion formula (487) to $f * g^*$ (for $f, g \in \mathcal{S}$), we obtain

$$\int_{\mathbb{R}^n} f\bar{g} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}\bar{\hat{g}}, \quad (491)$$

called Parseval's formula. Taking $f = g$ in it, we obtain

$$\int_{\mathbb{R}^n} |f|^2 = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\hat{f}|^2, \quad (492)$$

called the Plancherel's formula.

We can now define the spaces $H^s = H^s(\mathbb{R}^n)$. We first note that for a non-negative integer k and $f \in \mathcal{S}$ we have

$$\int_{\mathbb{R}^n} (|f|^2 + |\nabla f|^2 + \dots + |\nabla^k f|^2) = \quad (493)$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (1 + |\xi|^2 + |\xi|^4 + \dots + |\xi|^{2k}) |\hat{f}(\xi)|^2 d\xi \sim \quad (494)$$

$$\sim \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (1 + |\xi|^2)^k |\hat{f}(\xi)|^2 d\xi, \quad (495)$$

where \sim means that the ratio of the two quantities involved is bounded both from above and below.

It is therefore natural to define for any $s \in \mathbb{R}$ the following norm for a function $f \in \mathcal{S}$:

$$\|f\|_{H^s}^2 = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi. \quad (496)$$

The Sobolev space $H^s(\mathbb{R}^n)$ is defined as the completion of \mathcal{S} in the norm $\|\cdot\|_{H^s}$.

Theorem (Sobolev Imbedding Theorem)¹³⁸

If $s > \frac{n}{2}$, then all functions in H^s are continuous with $f(x) \rightarrow 0$ as $x \rightarrow \infty$, and

$$\|f\|_C \leq C\|f\|_{H^s}. \quad (497)$$

Proof: It is enough to prove (497) for $f \in \mathcal{S}$. This is not hard:

$$|f(x)| \leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\hat{f}(\xi)e^{i\xi x}| d\xi = \quad (498)$$

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (1 + |\xi|^2)^{\frac{s}{2}} |\hat{f}(\xi)| (1 + |\xi|^2)^{-\frac{s}{2}} d\xi \leq \quad (499)$$

$$\frac{1}{(2\pi)^n} \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \frac{1}{(1 + |\xi|^2)^s} d\xi \right)^{\frac{1}{2}} \leq C\|f\|_{H^s}. \quad (500)$$

¹³⁸This terminology is often used also for more general imbedding results.

It is useful to have also the following result

If $f \in L^2(\mathbb{R}^n)$ together with all distributional derivatives of order k , then $f \in H^k(\mathbb{R}^n)$.

Remark

The space of functions with all distributional derivatives up to order k in L^2 is usually denoted by $W^{k,2}(\mathbb{R}^n)$. The above statement can be re-written as $H^k(\mathbb{R}^n) = W^{k,2}(\mathbb{R}^n)$.

Proof: Take a smooth compactly supported function ϕ on \mathbb{R}^n which is $= 1$ in a neighborhood of 0, and set $\phi_\varepsilon(x) = \phi(\varepsilon x)$. The functions $\phi_\varepsilon u$ are compactly supported and can be approximated in H^s by smooth compactly supported functions. Moreover, it is easy to check that as $\varepsilon \rightarrow 0$, the functions $\phi_\varepsilon u$ form a Cauchy sequence in H^k with $\nabla^j(\phi_\varepsilon u) \rightarrow \nabla^j u$ in $L^2(\mathbb{R}^n)$ for $j = 0, 1, 2, \dots, k$.

Lecture 35, 12/1/2010

In preparation for dealing with boundary regularity, we also prove imbedding version of the Sobolev Imbedding Theorem for bounded domains. Let Ω be a bounded domain. For smooth functions in Ω we define the norm

$$\|u\|_{H^k}^2 = \int_{\Omega} (|u|^2 + |\nabla u|^2 + \cdots + |\nabla^k u|^2) dx. \quad (501)$$

Let $\mathcal{E}(\Omega)$ be the space of functions in Ω which are obtained as restrictions to Ω of smooth functions defined in a some neighborhood of $\overline{\Omega}$. We define the space $H^k(\Omega)$ as the closure of $\mathcal{E}(\Omega)$ in the H^k -norm. We also define the space $W^{k,2}(\Omega)$ is the space of L^2 -functions in Ω for which all distributional derivatives up to order k are in $L^2(\Omega)$. The norm in $W^{k,2}(\Omega)$ is defined by the expression on the right-hand side of (501). In lecture 23 we showed that $H^1(\Omega) = W^{1,2}(\Omega)$ for Lipschitz domains Ω . By inspecting the proof, we see that it in fact works for any $k = 1, 2, \dots$. Hence we have

For any bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ we have

$$H^k(\Omega) = W^{k,2}(\Omega). \quad (502)$$

One way to prove continuity of the functions in $H^k(\Omega)$ is to extend them to the whole space \mathbb{R}^n with some control of the norm of the extension: we can apply the Sobolev Imbedding Theorem in \mathbb{R}^n we proved last time to the extended function, showing the continuity up to the boundary of the original function in Ω . Our next topic is the construction of such an extension.

If $u \in H^k(\Omega)$, can it be extended to $\tilde{u} \in H^k(\mathbb{R}^n)$ with

$$\|\tilde{u}\|_{H^k(\mathbb{R}^n)} \leq C \|u\|_{H^k(\Omega)}, \quad (503)$$

then we can conclude that for $k > n/2$ all functions in $H^k(\Omega)$ will be continuous in Ω up to the boundary (which is the same as having a continuous extension to $\overline{\Omega}$).

We are now going to construct an extension operator. Let us first note that we can localize by using partition of unity: if $u = u\phi_1 + \cdots + u\phi_m$ and we can extend each $u\phi_j$ in a controlled way, we can clearly extend u in a controlled way. For the functions $u\phi_j$ which are compactly supported in Ω there is nothing to prove, they can be trivially considered as functions in \mathbb{R}^n without changing their H^k norm. Therefore the only issue is to extend functions which are supported in small balls near the boundary. Letting $x' = (x_1, \dots, x_{n-1})$, as usual, we assume the boundary by

$$x_n = a(x') \quad (504)$$

where a is a function with Lipschitz $(k-1)$ -st derivative. (This is class of functions is usually denoted by $C^{k-1,1}$ or $W^{k,\infty}$.) The domain $\{x_n > a(x')\}$ can be transformed to the upper half-space by $(x', x_n) \rightarrow (x', x_n - a(x'))$. This transformation has enough regularity to map our situation into the equivalent situation in the half-space $\{x_n > 0\}$. When working with $C^{k-1,1}$ domains, our extension problem for $H^k(\Omega)$ therefore reduces to the problem of extending a function from $\mathbb{R}_+^n = \{x_n > 0\}$ to all \mathbb{R}^n , with good control of $\nabla^j u$, $j = 1, \dots, k$. We note that functions which are restrictions of smooth functions in the neighborhood of Ω are dense, and hence it is enough to construct the extension operator for these functions. Note that such functions are already defined in a neighborhood of Ω and therefore it is simple to extend them to \mathbb{R}^n if we do not worry about controlling the norm of the extension. The main point of the extension we construct below¹³⁹ is that we have a good control the norm of the extension.

When extending a function from \mathbb{R}_+^n to \mathbb{R}^n , it is natural to try to extend it along each line $x' = a' \in \mathbb{R}^{n-1}$. In other words, we take the function $x_n \rightarrow u(x', x_n)$ for each x' , and we want to find a good extension of these functions, originally defined for $x_n \geq 0$, to $x_n \in \mathbb{R}$.

Let $f: [0, \infty) \rightarrow \mathbb{R}$ be a smooth function and let us consider the following formula which extends it to \mathbb{R} :

$$\tilde{f}(x) = \sum_{j=1}^k a_j f(-jx), \quad x < 0, \quad \tilde{f}(x) = f(x), \quad x \geq 0, \quad (505)$$

where the coefficients are to be chosen. Differentiating (505), we see that if we wish \tilde{f} to have continuous derivatives up to order $k-1$, the coefficients must satisfy

$$\sum_{j=1}^k j^l a_j = (-1)^l, \quad l = 0, \dots, k-1. \quad (506)$$

The matrix of this linear system of equations for a_1, \dots, a_k is the Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_k^2 \\ \dots & \dots & \dots & \dots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \dots & \lambda_k^{k-1} \end{pmatrix} \quad (507)$$

with $\lambda_j = j$, and therefore is non-singular and a_1, \dots, a_k are uniquely determined by (506). The distributional derivative $\tilde{f}^{(k)}$ of \tilde{f} of order k is given for $x < 0$ by the expected formula (from differentiation of (505)). At $x = 0$ the derivative $\tilde{f}^{(k)}$ may not be well-defined (the left and right derivatives of $\tilde{f}^{(k-1)}$ may not coincide at $x = 0$), but the important point is that the distributional

¹³⁹originally due to S. M. Nikolskii

derivative of the (continuous) function $\tilde{f}^{(k-1)}$ is a bounded function given by the expected formulae everywhere except perhaps at $x = 0$.¹⁴⁰ If we wish that the extended function has continuous derivatives up to order k we simply do the above construction with k replaced by $k + 1$.

Returning to functions in the upper half-space \mathbb{R}_+^n , we consider a smooth function $u: \overline{\mathbb{R}_+^n} \rightarrow \mathbb{R}$ with bounded support and for each $x' = (x_1, \dots, x_{n-1})$ we apply the one-dimensional construction above to the function $x_n \rightarrow u(x', x_n)$, obtaining an extension $\tilde{u}: \mathbb{R}^n \rightarrow \mathbb{R}$. It is now easy to verify that \tilde{u} will also have bounded support, and $\tilde{u} \in W^{k,2}(\mathbb{R}^n) = H^k(\mathbb{R}^n)$ with

$$\|\tilde{u}\|_{H^k(\mathbb{R}^n)} \leq C \|u\|_{H^k(\mathbb{R}_+^n)}, \quad (508)$$

with C independent of u .¹⁴¹

Summarizing the above considerations, we see that we have proved the following result:

Theorem

Let Ω be a bounded domain with $C^{k-1,1}$ boundary. Then we can define an extension operator $u \rightarrow \tilde{u}$ on from $W^{k,2}(\Omega)$ to $H^k(\mathbb{R}^n)$ such that all functions \tilde{u} are supported in a fixed compact set and for some $C > 0$ we have

$$\|\tilde{u}\|_{H^k(\mathbb{R}^n)} \leq C \|u\|_{W^{k,2}(\Omega)}, \quad u \in W^{k,2}(\Omega). \quad (509)$$

Corollary

If Ω is as in the theorem, then each function in $W^{k,2}(\Omega)$ is continuous in Ω up to the boundary, and

$$\|u\|_{C(\overline{\Omega})} \leq C \|u\|_{W^{k,2}(\Omega)}, \quad u \in W^{k,2}(\Omega). \quad (510)$$

Let us now return to the proof of regularity and look in more detail at some of the steps outlined in lecture 33. We start with interior regularity and Caccioppoli's inequality for the equation

$$-\frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} = \frac{\partial f_j}{\partial x_j} \quad (511)$$

¹⁴⁰Note that in general this will not be the case for $f^{(k+1)}$, due to the possible discontinuity of $f^{(k)}$ at $x = 0$. If you have not come across similar consideration before, I recommend that you go through this in detail. It is enough to check to understand what happens with the derivatives of the function $g(x) = |x|$ on the real line. We do have $g'(x) = \text{sign}(x)$ in the sense of distributions. However, it is not the case that $g'' = 0$ in the sense of distributions. It is important to understand this example completely.

¹⁴¹Note that if we use the extension (505) with a given k , then the extended function \tilde{u} may not be in $W^{k+1,2}(\mathbb{R}^n)$. That is a disadvantage of this particular extension technique: it works only up to a given finite order k . We can choose k in advance, but near $x_n = 0$ the extended functions will typically be of the class $C^{k-1,1} = W^{k,\infty}$, and no better. In particular, the method does not give a C^∞ extension of C^∞ functions.

We recall that we really work with the weak formulations, so that this equations really is

$$\int_{\Omega} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} = \int_{\Omega} -f_j \frac{\partial v}{\partial x_j}, \quad v \in H_0^1(\Omega). \quad (512)$$

Since we are looking at the interior regularity at the moment, we do not worry about the boundary conditions, and that is why we take the space $H_0^1(\Omega)$ for the test functions v in (512). Later when we deal with boundary regularity, we the choice of the test functions v will depend on the boundary conditions.

Repeating the arguments (465), (466) in the proof of (461) from lecture 33, we obtain

$$\int_{B_{R_1}} |\nabla u|^2 \leq \frac{C}{(R_0 - R_1)^2} \int_{B_{R_0}} |u|^2 + C_1 \int_{B_{R_0}} |f|^2, \quad (513)$$

where the constants C, C_1 now depend on the constant ν in the ellipticity condition $a_{ij} \xi_j \xi_i \geq \nu |\xi|^2$.

We now take the derivative of equation (511) in a given direction, say $\frac{\partial}{\partial x_1}$, and denote $u' = \frac{\partial u}{\partial x_1}$. At this stage this is formal, we just wish to check what we would get if we could proceed that way. For the rigorous proof we will replace the derivative by the difference quotient D_j^h defined in lecture 33, see (467). The calculation with the difference quotients will be almost the same as with the derivative. After differentiation, (511) becomes

$$-\frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u'}{\partial x_j} = \frac{\partial}{\partial x_j} (f_j + a'_{ij}(x) \frac{\partial u}{\partial x_j}). \quad (514)$$

It is of the same form as before, except the right-hand side has an additional term coming from the differentiation of a_{ij} . Assuming that a'_{ij} are Lipschitz, we can again apply Caccioppoli's inequality to the solution u' and the new right-hand side. (If the equation has more terms with variable coefficients, the terms with the derivatives of those coefficients would also appear on the right-hand side. (Assuming that all the coefficients are Lipschitz, the extra terms from the lower order terms would be “weaker”, the term coming from $a_{ij}(x)$ is the “most dangerous” term.)

$$\int_{B_{R_2}} |\nabla u'|^2 \leq \frac{C}{(R_1 - R_2)^2} \int_{B_{R_1}} |u'|^2 + C_1 \int_{B_{R_1}} (|f'|^2 + |a'_{ij} \nabla u|^2). \quad (515)$$

We do this for $u' = \frac{\partial u}{\partial x_j}$, $j = 1, \dots, n$.

Assuming we can justify this so far formal procedure, it is clear how we continue: we take a derivative of (514) in each direction, and repeat the procedure. Then we take the next derivative, etc, eventually obtaining (under the assumption that the derivatives of order $k - 1$ of a_{ij} are bounded)

$$\|u\|_{H^{k+1}(B_R)} \leq C_1(R, R_0) \|u\|_{L^2(B_{R_0})} + C_2(R, R_0) \|f\|_{H^k(B_{R_0})} \quad (516)$$

for $R < R_0$. (We choose $R_0 > R_1 > \dots R_k = R$, so that we get to radius R in step k .) This is a *local interior regularity estimate*. It shows that at the

level of H^k spaces, the solution is as good as it can be. (It is clear from (511) that the solution u cannot gain more than one derivative over f .) Also note that both terms in on right-hand side of the estimate are important. The first term is necessary to get control of the non-trivial solutions with $f = 0$, and the importance of the second term is self-evident.

The dependence of the constants C_1, C_2 on R, R_0, ν and the bounds of the derivatives of a_{ij} can be made more explicit, if necessary, but the form (516) is often sufficient.

It remain to justify the procedure of taking derivatives. The key is the last lemma in lecture 33. At each step we first replace the derivative with the difference quotient, and obtain estimates for the difference quotient first. Then we take $h \rightarrow 0$ in the difference quotient and use the lemma to establish the existence of the derivative. The only issue we have to deal with in this process is that the Leibnitz rule we used in differentiating the equation has to be slightly modified for the difference quotient. Namely, for any two functions f, g we have

$$D_j^h(fg) = D_j^h(f)(\tau_h g) + f(D_j^h g), \quad (517)$$

where $\tau_h g(x) = g(x + h)$. It is easy to see that this modification of the Leibnitz rule makes no difference in the proofs, and hence the proof of the local interior estimate (516) is finished.

Today we will consider local boundary regularity. The scheme is quite similar to the local interior regularity, with some adjustments. Our set-up is the same as in lecture 30. We will proceed in the following steps:

Step 1 (Caccioppoli's inequality near the boundary)

It is similar to the interior case. Let us assume for simplicity that the weak form of the equation is

$$\int_{\Omega} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \int_{\partial\Omega} \gamma uv = \int_{\Omega} f_j \frac{\partial v}{\partial x_j} + \int_{\partial\Omega} gv, \quad v \in X, \quad (518)$$

where the space X with $H_0^1(\Omega) \subset X \subset W^{1,2}(\Omega)$ is chosen according to the boundary condition. We will work in a ball centered at the boundary. We assume that the boundary has already been (locally) straightened up by a suitable coordinate change. (Note that the form (518) of the equation is invariant under diffeomorphisms.) We will use the notation

$$B_R^+ = \{x, |x| < R, x_n > 0\}. \quad (519)$$

Assume that in the area we are interested in the boundary $\partial\Omega$ is given by $x_n = 0$, and assume $B_{R_0}^+$ is contained in Ω . Consider $R_1 < R_0$. We consider a cut-off function η defined in $B_{R_0}^+$, obtained as a restriction to $B_{R_0}^+$ of a function which is $= 1$ in $B_{R_1}^+$ and is compactly supported in $B_{R_0}^+$, with $|\nabla\eta| \leq 2/(R_0 - R_1)$. We will assume that the space X has the property $u \in X \implies u\eta^2 \in X$, not only for η , but also for the restriction to $B_{R_0}^+$ of any other smooth function $\tilde{\eta}: B_{R_0}^+ \rightarrow R$ compactly supported in B_{R_0} , i. e. $\tilde{\eta} \in \mathcal{D}(B_{R_0})$. In particular, these test functions may not vanish at the flat part of the boundary of $B_{R_0}^+$. The condition is satisfied for example for $X = W^{1,2}(\Omega)$ (corresponding to the Neumann or mixed boundary conditions), but it is not satisfied in Examples 2 and 3 in lecture 3. In each case there is an easy "fix". In Example 2 we subtract from u a smooth function with the same boundary condition as u (which adds a new smooth term to the right-hand side). For Example 3 we use the alternative formulation presented at the end of lecture 31.

We now use (518) with $v = u\eta^2$ and can proceed as in (465), (466), except we get the boundary terms

$$\int_{\partial\Omega} \gamma u^2 \eta^2, \quad \int_{\partial\Omega} gu\eta^2. \quad (520)$$

These can be dealt with in the same way as in the proof of the coercivity of the form A_λ for large λ in lecture 31. There we used (458) to prove (403), and the same method will work again in the situation at hand. For example, we have, assuming that γ is bounded,

$$\int_{\partial\Omega} \gamma u^2 \eta^2 \leq C \|\gamma\|_{L^\infty} \|u\eta\|_{L^2} \|u\eta\|_{W^{1,2}} \leq \int_{B_{R_0}^+} \varepsilon |\nabla u|^2 \eta^2 + \tilde{C}(\varepsilon, R_0, R_1, \gamma) \int_{B_{R_0}^+} |u|^2. \quad (521)$$

The second integral in (520) is estimated similarly. The end result of our calculation is the boundary version of the Caccioppoli inequality:

$$\int_{B_{R_0}^+} |\nabla u|^2 \eta^2 \leq \frac{C}{(R_0 - R_1)^2} \int_{B_{R_0}^+} |u|^2 + C_1 \int_{B_{R_0}^+} |f|^2 + C_2 \int_{\partial' B_{R_0}^+} |g\eta|^2, \quad (522)$$

where the constants depend on ν and $\|\gamma\|_{L^\infty}$, and $\partial' B_{R_0}^+$ denotes the flat part of the boundary of $B_{R_0}^+$.

Step 2 (Taking higher derivatives)

In principle one can take derivatives of the equation and the boundary conditions as in the case of the interior estimates, but one has to be slightly more careful. First of all, it is natural to take derivatives in the directions x_1, \dots, x_{n-1} at first, as this does not interfere with the boundary condition. Let us denote by $'$ the derivative $\frac{\partial}{\partial x_1}$. It is best to go to higher derivatives directly from the variational formulation (518). For that we take as a test function in (518) the function

$$v = -(u'\eta^2)', \quad (523)$$

where η is a cut-off function adapted to the balls $B_{R_1}^+$ and $B_{R_2}^+$. Strictly speaking, we should replace the expression for v by

$$v = -D_1^{-h}((D_1^h u)\eta^2) \quad (524)$$

obtain estimates for $D_1^h \nabla u$, and then let $h \rightarrow 0$. Note also that the expression with the difference quotients belongs to X under quite general assumptions (and certainly in the case of the homogeneous Dirichlet, Neumann, or the mixed conditions). The reason for using D_1^{-h} in (524) is to get good expressions after an integration by parts we will do. We will work directly with the derivative to obtain formal estimates first, and we will see that the calculation with the difference quotient is similar.

Substituting (524) into (518) and integrating by parts, we obtain

$$\int_{\Omega} (a_{ij} u'_{,j})' (u'\eta^2)_{,i} + \int_{\partial\Omega} (\gamma u)' u' \eta^2 = \int_{\Omega} f'_j u'_{,j} \eta^2 + \int_{\partial\Omega} g' u' \eta^2. \quad (525)$$

which is the same as

$$\int_{\Omega} a_{ij} u'_{,j} (u'\eta^2)_{,i} + \int_{\partial\Omega} \gamma (u')^2 \eta^2 = \int_{\Omega} (f'_i - a'_{ij} u_{,j}) u'_{,j} \eta^2 + \int_{\partial\Omega} (g' - \gamma' u) u' \eta^2. \quad (526)$$

This is what we used to get (522), except u, f_i, g are replaced by $u', f'_i - a'_{ij} u_{,j}, g' - \gamma' u$ respectively. We control the last three quantities by assumptions on f', g', γ' ¹⁴² and estimate (522) from the first step.

¹⁴²It is natural to assume $f' \in L^2(B_{R_0}^+)$, $g' \in L^2(B_{R_0}^+ \cap \partial\Omega)$ and $\gamma' \in L^\infty(B_{R_0}^+ \cap \partial\Omega)$ at this step.

Therefore we have estimated

$$\int_{B_{R_1}^+} |\nabla u'|^2. \quad (527)$$

We can continue this procedure, by taking in the k^{th} step the test function

$$v = (-1)^k (u^{(k)} \eta^2)^{(k)}, \quad (528)$$

where (k) denotes the k -the derivative generated by $'$.

We can do this in all tangential directions x_1, \dots, x_{n-1} , and this gives an estimate of

$$\int_{B_{R_k}^+} |\nabla (\nabla')^k u|^2. \quad (529)$$

To estimate the full gradient $\nabla^{k+1} u$, we can use the equation. We already know that in the interior of Ω the solution is as smooth as the coefficients allow, and therefore we can use the equation freely. In particular, denoting $u_{,ij}, u_{,i}$ the partial derivatives $\frac{\partial^2 u}{\partial x_i \partial x_j}, \frac{\partial u}{\partial x_i}$, we have

$$a_{nn} u_{,nn} = -a_{ij,i} u_{,j} + f_{j,j} - \sum_{(i,j) \neq (n,n)} a_{ij} u_{,ij}. \quad (530)$$

The ellipticity condition $a_{ij} \xi_i \xi_j \geq \nu |\xi|^2$ implies that $a_{nn} \geq \nu$. Therefore (530) shows that the number of derivatives $\frac{\partial}{\partial x_n}$ in any expression $\frac{\partial^m}{\partial x_n^m} (\nabla')^l u$ can be lowered as long as $m \geq 2$, at the expense of bringing in linear combinations of the tangential derivatives. The end result of this procedure of lowering of the degree of $\frac{\partial}{\partial x_n}$ will be expressions estimated by (529) and

$$\int_{B_{R_l}^+} |(\nabla')^l u|^2. \quad (531)$$

In the end we obtain the following estimate

$$\|u\|_{H^{k+1}(B_R^+)} \leq C_1 \|f\|_{H^k(B_{R_0}^+)} + C_2 \|g\|_{H^k(\partial' B_{R_0}^+)} + C_3 \|u\|_{L^2(B_{R_0}^+)}, \quad (532)$$

where the constants C_1, C_2, C_3 depend on k, R_0, R, ν and the bounds on the coefficients of the equation and their derivatives up to order k .

Step 3 (Difference quotients)

As we already indicated, to obtain the bound of (530), one should start at each step with a difference quotient of the preceding derivative, obtain estimates for the difference quotient, and then pass to the limit using the last lemma in lecture 33. This presents no difficulty, and the details are left to the reader.

Step 4 (Sobolev Imbedding Theorem)

We have already proved the Sobolev Imbedding theorem “up to the boundary” in lecture 34, so we know that estimate (??) implies the continuity of $(\nabla)^l u$ up to the boundary whenever $l + n/2 < k$.

This completes our proof of boundary regularity. We did not work with the most general Lax-Milgram form (403), but the form we considered contains the most difficult terms, and covers many boundary conditions encountered in practice. The terms in the forms (403) which we did not include in the regularity proof easier to handle than those we did, and hence we can say that we proved the local boundary estimate (532) for the general Lax-Milgram form and for the boundary conditions which allow our construction of the test functions (529). This includes the Dirichlet and Neumann boundary conditions, together with the “mixed” boundary condition $a_{ij}u_{,j}n_i + \gamma u = g$.

We have established local interior estimates and local boundary estimates for the problem (436) with the Lax-Milgram form (403). We emphasize that these estimates are valid regardless of the coercivity of the form $A(u, v)$, as long as the leading coefficients are elliptic. If we combine the local estimates with the Fredholm alternative results obtained in lecture 31, we can easily obtain the global estimates mentioned in lecture 31. In what follows we assume that the coefficient of the equation are as regular as necessary for the estimates we will consider.

We consider global estimates of the form

$$\|u\|_{H^{k+1}(\Omega)} \leq C_1 \|f\|_{H^k(\Omega)} + C_2 \|g\|_{H^k(\partial\Omega)} \quad (533)$$

for the problem

$$A(u, v) = l(v) = - \int_{\Omega} f_j \frac{\partial v}{\partial x_j} + \int_{\partial\Omega} gv, \quad v \in X \quad (534)$$

where A is given by (403) and X is a subspace of $W^{k,2}(\Omega)$ depending on the boundary conditions. We assume that X contains $H_0^1(\Omega)$ and satisfies the conditions needed for the construction of test functions in the last lecture. For example, when $X = W^{1,2}(\Omega)$, then the conditions are satisfied.

We should point out that there are simple situations when (534) is not satisfied for $k \geq 1$. A typical case is when we “switch” from Neumann to Dirichlet boundary conditions in the middle of a connected component of the boundary. Consider the following example. Let Ω be the unit disc $\{|x| < 1\}$ in \mathbb{R}^2 , and let $\Gamma_1 = \partial\Omega \cap \{x_1 \geq 0\}$ and $\Gamma_2 = \partial\Omega \cap \{x_1 < 0\}$. Let us consider the space $X = \{u \in W^{1,2}(\Omega), u|_{\Gamma_1} = 0\}$ and the Lax-Milgram form $A(u, v) = \int_{\Omega} \nabla u \nabla v$. Consider (534) with $g = 0$. Our boundary value problem then is

$$-\Delta u = \frac{\partial f_j}{\partial x_j} \quad \text{in } \Omega, \quad (535)$$

$$u|_{\Gamma_1} = 0, \quad (536)$$

$$\frac{\partial u}{\partial n}|_{\Gamma_2} = 0. \quad (537)$$

We do not expect that the local boundary estimate (521) will be satisfied near points $a = (0, 1)$ and $b = (0, -1)$ for $k \geq 1$, as we are passing from one boundary condition to a completely different one at those points.¹⁴³

For our global estimate (533) our problem must admit local estimates in a neighborhood of any interior point and also any boundary point, so we must avoid situations similar to the example above.

¹⁴³The problem of investigating the exact regularity of solutions at such points is interesting. To see that it can be non-trivial, try to decide if all solutions with smooth f are continuous at the points a and b .

Theorem

The following conditions are equivalent

- (i) When $f = 0$ and $g = 0$, the problem (534) only has the trivial solution $u = 0$, and all solutions satisfy the local interior and boundary estimates near each point of $\bar{\Omega}$.
- (ii) The problem is uniquely solvable for each $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$ and the global estimates (533) are satisfied.

Proof:

The implication (ii) \implies (i) is trivial. To prove (i) \implies (ii), it is enough to prove the global estimate. Assuming the global estimate fails, for some $k \geq 0$, we can produce a sequence of functions u^m, f^m, g^m solving the problem such that $\|u^m\|_{H^{k+1}(\Omega)} = 1$ and $\|f^m\|_{H^k(\Omega)} \rightarrow 0, \|g^m\|_{H^k(\partial\Omega)} \rightarrow 0$. We can assume without loss of generality that u^m converge weakly in $H^{k+1}(\Omega)$ to some function $u \in H^{k+1}(\Omega)$. This implies that the gradients ∇u^m converge weakly in $L^2(\Omega)$ to ∇u and the functions u^m converge strongly in $L^2(\Omega)$ to u (by Rellich's compactness theorem). Now the local estimates imply

$$\|u^m\|_{H^{k+1}(\Omega)} \leq C_1 \|f^m\|_{H^k(\Omega)} + C_2 \|g^m\|_{H^k(\partial\Omega)} + C_3 \|u^m\|_{L^2(\Omega)}, \quad (538)$$

for suitable constants $C_1, C_2, C_3 \geq 0$. This gives a non-trivial bound from below of $\|u^m\|_{L^2}$, and hence the limit function u cannot vanish. On the other hand, one sees easily that u satisfies (534) with $f = 0$ and $g = 0$. This contradicts the uniqueness assumption in (i), and the proof is finished.¹⁴⁴

We now discuss the assumptions under which we derived the regularity of the solutions. So far we have measured the regularity of our solutions only in one way: we check how many derivatives of the solution are in L^2 . To derive that $k + 1$ derivatives of u are in L^2 , we need to assume that we can take k derivatives of the equation, and that the derivatives of the coefficients $a_{ij}, b_j, \tilde{b}_j, c$ which produce new terms in the derived equations are bounded. Let us look at some simple examples:

Example 1 (divergence-form equation)

$$\frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} = 0. \quad (539)$$

If we only assume that a_{ij} are bounded measurable (and satisfy the ellipticity condition, of course), the only information we have about the solution from our theory is $u \in H_0^1(\Omega)$. Even in dimension $n = 2$ this does not give continuity of

¹⁴⁴We note that the global estimate is non-trivial even for $k = 0$, as we do not assume the Lax-Milgram form is coercive. In fact, the technique of local estimates can be used for an alternative approach to the Fredholm theory in lecture 31.

u .¹⁴⁵ To prove continuity by the method we have used so far, we need to take k derivatives of the equation, with $k + 1 > n/2$, and $\nabla^k a_{ij}$ has to be bounded. If we wish to prove that the solution is once continuously differentiable, we need to take $k > n/2$ and $\nabla^{k+1} a_{ij}$ should be bounded. These requirements are way stronger than what is really necessary. For example, it can be shown that for bounded measurable a_{ij} satisfying the ellipticity assumptions the solutions are *always* Hölder continuous. This is a famous result of E. De Giorgi and J. Nash proved in 1950s. The proof is not easy and we will probably not cover it in this course. In the special case of dimension $n = 2$ the proof is much easier, and we will at least show how to prove the continuity of the solutions in that situation in one of the next lectures. It is clear that the proof cannot rely on taking derivatives of the equation and a different idea has to be used.

Let us now assume that a_{ij} are Hölder continuous.¹⁴⁶ What can we say about the regularity of the solution? The method of taking derivatives we have used so far cannot be applied, as the coefficients a_{ij} still may not be differentiated. One can get a hint from the easy case of dimension $n = 1$. In this case the equation is

$$(a(x)u')' = 0. \quad (541)$$

The general solution of this ODE is

$$u(x) = \int \frac{A}{a(x)} dx + B, \quad (542)$$

where A, B are constants. We see that if a is Hölder continuous, then the first derivatives of u are Hölder continuous (with the same Hölder exponent), but in general we do not have any additional regularity for the second derivatives. In higher dimension the situation is the same, although the proof is more difficult. For equation

$$-\frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} = \frac{\partial f_j}{\partial x_j} \quad (543)$$

We have a local estimate (for $R < R_0$)

$$\|u\|_{C^{1,\alpha}(B_R)} \leq C_1 \|f\|_{C^{0,\alpha}(B_{R_0})} + C_2 \|u\|_{L^2(B_{R_0})}, \quad (544)$$

¹⁴⁵We will see later that in this two-dimension case the continuity of u can be proved by a simple argument, which nevertheless requires a new idea.

¹⁴⁶We recall that the Hölder norm $\|f\|_{C^{0,\alpha}}$ of a function on Ω is usually defined for $\alpha \in (0, 1)$ by

$$\|f\|_{C^{0,\alpha}} = \sup_{x,y \in \Omega, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} + \sup_{\Omega} |f|. \quad (540)$$

The Hölder norm $C^{k,\alpha}$ can be defined as the sum of the Hölder norms of derivatives up to order k . (There various equivalent definitions, depending on which lower-order terms we include.) The definition makes sense also for $\alpha = 1$, in which case we are dealing with Lipschitz functions. However, estimates with $C^{0,1}$ usually fail in the context of PDEs we are considering.

where C_1, C_2 depend on R_0, R, ν , the dimension n and the $C^{0,\alpha}$ Hölder norm of the coefficients in B_{R_0} . The dependence can be made more explicit, if necessary.¹⁴⁷ If the coefficients are in $C^{k,\alpha}$, one has

$$\|u\|_{C^{k+1,\alpha}(B_R)} \leq C_1 \|f\|_{C^{k,\alpha}(B_{R_0})} + C_2 \|u\|_{L^2(B_{R_0})}, \quad (545)$$

where the constants will now also depend on k .

There are several techniques for establishing (543), but none of them is an obvious generalization of the “energy estimates” technique which we used in the previous lectures. Sometimes such estimates are called Schauder estimates, although some authors may use the term in the slightly different situation of the non-divergence form equations, which we are going to consider next.

The non-divergence form equations are usually written as

$$a_{ij}(x)u_{,ij} = f. \quad (546)$$

This equation is covered by our “direct method” only if a_{ij} are Lipschitz, or, equivalently, if ∇a_{ij} is bounded. (This could be relaxed to $\nabla a_{ij} \in L^n$, but not much more.) The reason for this regularity requirement is that the Lax-Milgram form is

$$A(u, v) = \int_{\Omega} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + a_{ij,i} \frac{\partial u}{\partial x_j} v. \quad (547)$$

On the other hand, the natural classes of coefficients a_{ij} for which the solutions of (546) should be considered (and are well-behaved) include the case when a_{ij} are merely Hölder continuous, without any assumptions on their derivatives. Such situation is not covered by our techniques so far. We will probably cover it later, but for now we should at least mention the *Schauder estimate* for (546). When a_{ij} are in $C^{0,\alpha}$, one has for any solution of (546) the local interior estimate

$$\|u\|_{C^{2,\alpha}(B_R)} \leq C_1 \|f\|_{C^{0,\alpha}(B_{R_0})} + C_2 \|u\|_{C(B_{R_0})}, \quad (548)$$

where $\|\cdot\|_C$ denoted the usual sup-norm in the space of continuous functions. Using the one-dimensional example

$$au'' = f, \quad (549)$$

one sees easily that estimate (548) is more or less optimal. The techniques used to prove the Schauder estimate (549) are quite different from the energy techniques we have been using. One way to establish (548) is to return to the representation of solutions via potential we used in the beginning of the course. Another approach can be found in the book “Lectures on Elliptic and Parabolic Equations in Hölder Spaces” by Professor N. Krylov of our own department.

¹⁴⁷See for example the paper “The inverse function theorem of Nash and Moser” by R. Hamilton, Bulletin AMS, Vol. 7, No. 1, July 1982, Section 3.3, p. 155.

Homework assignment 3

due December 22, 2010

Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and let $b = (b_1, \dots, b_n)$ be a bounded smooth vector field in Ω . Let us consider the boundary value problem

$$\begin{aligned} -\Delta u + \frac{\partial}{\partial x_j}(b_j u) &= 0 \quad \text{in } \Omega, \\ -\frac{\partial u}{\partial n} + b_j n_j u &= 0 \quad \text{at } \partial\Omega. \end{aligned} \tag{550}$$

Show that the problem has at least one non-trivial solution. (Hint: use Fredholm theory.)

Additional optional problems:

1. Show that the space of solutions of (550) is one-dimensional.
2. Show that if $b = \nabla\varphi$ for some smooth function φ , then u is given by $u = Ce^\varphi$.
3. Show that the solutions of (550) do not change sign.
4. Study the same problems for the equation $\frac{\partial^2}{\partial x_i \partial x_j} a_{ij}(x)u = 0$ in Ω , with the boundary condition $n_i \frac{\partial}{\partial x_j}(a_{ij}u) = 0$.

In our proofs of regularity so far we have been measuring regularity of functions in a particular way: we were looking at how many derivatives are in the space L^2 . This leads to the hierarchy of spaces H^k , $k = 0, 1, \dots$. These spaces work quite well in some situations, but when we work with coefficients with limited regularity, these spaces are often not a good match with the problem at hand. We have seen examples of this last time, when we introduced another hierarchy of spaces - the Hölder spaces $C^{k,\alpha}$. There is still another scale of spaces, which is a generalization of the H^k -scale, which gives us yet more flexibility when dealing with regularity. These are the spaces $W^{k,p}$, which measure the regularity of functions in terms of how many derivatives are in L^p . In comparison with the $H^k = W^{k,2}$ scale, we have an additional parameter p to play with. Let us illustrate how this can be used on a simple example. Consider the equation

$$\Delta u + b(x)\nabla u = 0, \tag{551}$$

where $b\nabla u = b_j u_{,j}$ and the vector field $b = (b_1, \dots, b_n)$ is in L^∞ , but not better. Assume that $u \in W^{1,2}(\Omega)$ is a solution. We note that the term $b\nabla u$ belongs to $L^2(\Omega)$. Considering the term as the right-hand side in

$$\Delta u = -b\nabla u = f \tag{552}$$

we see that we can take one derivative of the equation. Writing $u' = \frac{\partial u}{\partial x_j}$, we have

$$\Delta u' = \frac{\partial f}{\partial x_j}, \tag{553}$$

which can be used to estimate $\nabla u'$ in L^2 , see lecture 35, (515). Therefore we can conclude that $\nabla^2 u \in L^2$.¹⁴⁸

If we only use the H^k -estimates, this is as far as we can get. We cannot take any more derivatives of the equation, and if we only work with L^2 -type spaces, there is not much more we can do. However, the situation becomes different if we use L^p spaces for general p to measure the regularity of the derivatives. In other words, we measure the regularity of a function u by the quantities $\|\nabla^l u\|_{L^p}$, $p \geq 1, l = 0, 1, \dots$. This gives us more flexibility: instead of taking higher derivatives, we can try to increase p . We define the spaces $W^{k,p}(\Omega)$ in the same way as the spaces $W^{k,2}$ except that the condition $u, \nabla u, \dots, \nabla^k u \in L^2(\Omega)$

¹⁴⁸In general, we expect that for the operators L of the form (458) with smooth coefficients, the second derivatives $\nabla^2 u$ of the solutions of $Lu = f$ should have the same regularity as f , i. e. we should have $\nabla^2 u \in L^2$ in the case at hand. We did not formulate our regularity estimates in a way which would directly estimate $\nabla^2 u$ in terms of f , because in the general case $\frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j} = f$ with $f \in L^2$ we cannot conclude that $\nabla^2 u \in L^2$ unless we have additional regularity for the coefficients. It is therefore more natural to have the right-hand-side in the divergence form, i. e. $\frac{\partial f_j}{\partial x_j}$ rather than f , if we work irregular coefficients. However, if the coefficients have some regularity, it is also natural to consider estimates comparing $\nabla^2 u$ with f . In our setting we can obtain these by taking one derivative of the equation.

is replaced by $u, \nabla u, \dots, \nabla^k u \in L^p(\Omega)$. The corresponding norm is denoted by $\|\cdot\|_{W^{k,p}}$.

The Sobolev Imbedding Theorem and the H^k estimates can be generalized as follows.

- (i) Imbedding Theorems for $W^{k,p}$ spaces. We will discuss these in some detail later in the course. Roughly speaking, there are two fundamental results (for sufficiently regular bounded domains Ω and $p \in [1, \infty)$) also called Sobolev Imbedding Theorems:

(a) $\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$ for $\frac{1}{q} = \frac{1}{p} - \frac{1}{n} > 0$ and

(b) $\|u\|_{C^{0,\alpha}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$ for $\frac{1}{p} - \frac{1}{n} = -\frac{\alpha}{n} < 0$.

- (ii) L^p -estimates for elliptic operators. In the simplest case of the laplacian the estimates roughly says that if $\Delta u = f$ with $f \in L^p$, then the second gradient $\nabla^2 u$ will also be in L^p .¹⁴⁹ (Later we will formulate this result more precisely.) In the general, the regularity requirements on the leading coefficients $a_{ij}(x)$ are stronger than for the H^k estimates, but if the leading term in the equation is the laplacian, we do not have to worry about this point.

Let illustrate how one can apply these results to (551). We have already seen that $\nabla^2 u$ is in L^2 . By the imbedding theorem this means that ∇u is in L^q with $1/q = 1/2 - 1/n$. This means that the right-hand side in (552) is in L^q , and by the L^p estimates this implies that $\nabla^2 u$ is in L^q . This still further improves the regularity of ∇u , and we can run the argument again.¹⁵⁰ After a few steps we will obtain that ∇u is Hölder continuous. In this case (552) gives that $\nabla^2 u$ is in L^p for any $p \geq 1$ and ∇u is in $C^{0,\alpha}$ for each $\alpha \in (0, 1)$, which is more or less optimal regularity for (551) with bounded measurable $b(x)$. (Clearly $\nabla^2 u$ cannot be better than L^∞ , as the Δu may be discontinuous if b is. Elliptic estimates typically do not work in L^∞ (and in L^1). Instead of L^∞ we get L^p for each $p < \infty$.)

Remark

We should note that in the above argument one does not really have to use the L^p estimates (item (ii) above), as one can directly look at what (552) implies for ∇u . This is more or less equivalent to item (i) above. So one does not need the L^p estimates to get optimal regularity for (551). Moreover, one does not need the optimal imbedding theorem in (i) either: we can run the whole argument if we get *some* gain at each step, it does not have to be the optimal gain. For example, proving (i)(a) with $1/p - 1/n > 1/q$ is much easier than with $1/p - 1/n = 1/q$, and can be done with using only Young's inequality for convolutions. So the whole argument that the solutions of (551) are $C^{1,\alpha}$ can be quite elementary. However, one does have to go beyond H^k estimates.

¹⁴⁹However, these estimates cannot be accomplished by a simple integration by parts which works well for the L^2 -estimates.

¹⁵⁰This is known as "bootstrapping".

We have introduced the main spaces used in elliptic PDEs - the spaces $W^{k,p}$ and $C^{k,\alpha}$. The regularity proofs using these spaces - such as our simple example (551)- often use only the fact that inverting elliptic operator of order m gains m derivatives in these spaces. If a proof is based on this result, it usually works well not only for scalar equations of second order, but also for general systems of equations, the method is quite robust.

Some of the deepest results about scalar elliptic equations of the second order use additional properties of solutions of these equations, such as those related to the maximum principle. The proof of the famous result of De Giorgi and Nash that the solutions of (539) with measurable a_{ij} are Hölder continuous is a good example. It uses in a crucial way properties of solutions of *scalar* second order equations. It cannot be generalized to system of equations, because the result simply fails for systems. We will not cover the proof of the DeGiorgi - Nash result at this point, but we will illustrate in a simpler situation how the maximum principle can be used in regularity theory.

Let us consider the equation

$$-\frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} = 0 \quad (554)$$

in dimension $n = 2$. We assume that the coefficients a_{ij} are bounded measurable and satisfy the ellipticity condition $a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2$. We assume that $u \in W^{1,2}(\Omega)$ is a weak solution of (554), i. e. we have

$$\int_{\Omega} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} = 0, \quad v \in H_0^1(\Omega). \quad (555)$$

The solutions of this equation satisfy a maximum principle. As general functions in $W^{1,2}(\Omega)$ are not defined point-wise, we have to be somewhat careful in the formulation of the maximum principle. We will use the following notation: $x_+ = \max(x, 0)$. An important property of $W^{1,2}(\Omega)$ functions is that if $u \in W^{1,2}(\Omega)$, $a \in \mathbb{R}$, then the function $x \rightarrow (u(x) - a)_+$, which will be denoted by $(u - a)_+$ again belongs to $W^{1,2}(\Omega)$. The proof of this statement is left to the reader as an exercise.¹⁵¹ We will say that a function $u \in W^{1,2}(\Omega)$ satisfies a maximum principle, if it has the following property: for any open set $\mathcal{O} \subset \Omega$ and any $a \in \mathbb{R}$, if $(u - a)_+ \in H_0^1(\mathcal{O})$, then $(u - a)_+ = 0$ in \mathcal{O} . We note that if \mathcal{O} is a ball $B_{x,r}$ compactly contained in Ω , the condition is equivalent to saying $u|_{\partial B_{x,r}} \leq a$ a. e. implies $u \leq a$ a. e. in $B_{x,r}$. Note that $u|_{\partial B_{x,r}}$ is well-defined in $L^2(\partial B_{x,r})$, by the Trace theorem we proved in lecture 24.

We will say that u is *monotone*¹⁵² if both u and $-u$ satisfy the maximum principle. We aim to show the following:

- (1) The solutions of (555) are monotone.
- (2) In dimension $n = 2$, monotone functions in $W^{1,2}$ are continuous.¹⁵³

¹⁵¹It is a good exercise, I recommend that you do it.

¹⁵²Sometimes the term *monotone in the sense of Lebesgue* is used.

¹⁵³This statement is sometimes called the Lebesgue-Courant lemma.

Lecture 39, 12/10/2010

We continue with investigating the monotonicity in (the sense of Lebesgue) of equation (555) and its consequences. Let us first prove that any weak solution is monotone. This is quite easy: Let $\mathcal{O} \subset \Omega$ be open. Assuming $(u-a)_+ \in H_0^1(\mathcal{O})$, we have, by definition

$$\int_{\mathcal{O}} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial (u-a)_+}{\partial x_i} = 0. \quad (556)$$

Hence

$$0 = \int_{\mathcal{O}} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial (u-a)_+}{\partial x_i} = \int_{\mathcal{O}} a_{ij} \frac{\partial (u-a)_+}{\partial x_j} \frac{\partial (u-a)_+}{\partial x_i} \geq \nu \int_{\mathcal{O}} |\nabla (u-a)_+|^2, \quad (557)$$

and we see that $(u-a)_+$ must vanish in \mathcal{O} . The same argument applies to $-u$, and we see that u is monotone (in the sense of Lebesgue).

Our next goes is to show the following result, often called the Lebesgue-Courant lemma:

Theorem

In dimension $n = 2$, a function $u \in W^{1,2}(\Omega)$ which is monotone (in the sense of Lebesgue) is continuous.

As preparation for the proof of the theorem, let us start with looking more closely at properties of general functions in $W^{1,2}(\Omega)$. For simplicity we assume that Ω is the cube $[0, 1]^n$. Let us denote $x = (x', x_n)$. Consider any $u \in W^{1,2}(Q)$. The values of the function $u(x)$ are not defined at all points. In fact, except in dimension $n = 2$, the function $x_n \rightarrow u(x', x_n)$ is not well-defined for all x' . However, even with merely assuming $u \in L^2(Q)$ this function is well defined, as an element of $L^2(0, 1)$, for almost every $x' \in Q' = [0, 1]^{n-1}$.

Lemma

For any $u \in W^{1,2}(Q)$ the function $x_n \rightarrow u(x', x_n)$ belongs to $W^{1,2}((0, 1))$ for almost every $x' \in Q'$. Moreover,

$$\int_{Q'} \|u(x', \cdot)\|_{W^{1,2}((0,1))}^2 dx' \leq \|u\|_{W^{1,2}(Q)}^2. \quad (558)$$

Proof:

For $h > 0$ consider the difference quotient $D_n^h u(x) = \frac{u(x+he_n) - u(x)}{h}$. Set

$$F_h(x') = \int_0^{1-h} |D_n^h u(x', x_n)|^2 dx_n. \quad (559)$$

We have (see lecture 33, (469))

$$\int_{Q'} F_h(x') dx' \leq \int_Q \left| \frac{\partial u}{\partial x_n} \right|^2 dx. \quad (560)$$

Let

$$F(x') = \liminf_{h \rightarrow 0_+} F_h(x'). \quad (561)$$

By Fatou's lemma

$$\int_{Q'} F(x') dx' \leq \int_Q \left| \frac{\partial u}{\partial x_n} \right|^2 dx. \quad (562)$$

If $x_n \rightarrow u(x', x_n)$ is well-defined as an L^2 function and $F(x') < +\infty$, we can see from the proof of the last lemma in lecture 33 ((470) and (471) with \lim replaced by \liminf) that $u(x', \cdot) \in W^{1,2}((0, 1))$ and $\|u(x', \cdot)\|_{W^{1,2}}^2 \leq F(x')$. This finishes the proof of the lemma.

A similar proof and a simple (local) change of coordinates gives the following:

If $u \in W^{1,2}(\Omega)$ and $B_{x,r_0} \subset \Omega$, then $u|_{\partial B_{x,r}} \in W^{1,2}(\partial B_{x,r})$ for almost every $r \in (0, r_0)$.

If $n = 2$ and $u \in W^{1,2}(\partial B_{x,r})$, then by a simple application of the Cauchy-Schwartz inequality we have, for each $y, z \in \partial B_{x,r}$,

$$|u(y) - u(z)| \leq \left(\int_{\partial B_{x,r}} |\nabla u|^2 \right)^{\frac{1}{2}} (\pi r)^{\frac{1}{2}}. \quad (563)$$

(Strictly speaking, we should say that u has a representative for which it is true, as the functions in $W^{1,2}$ are well-defined only module a set of measure 0.)

Let us denote by

$$\operatorname{osc}_X u = \sup_{x,y \in X} |u(x) - u(y)|. \quad (564)$$

and also (for a measurable set X)

$$\operatorname{ess\,osc}_X u = \inf\{\varepsilon > 0, |u(x) - u(y)| < \varepsilon \text{ for almost every } x, y \in X\}. \quad (565)$$

We aim to show that under the assumptions of the theorem we have

$$\operatorname{ess\,osc}_{B_{x,r}} u \rightarrow 0 \quad r \rightarrow 0_+, \quad x \in \Omega. \quad (566)$$

This implies that u has a continuous representative.¹⁵⁴

Let, for a given $x \in \Omega$ with $B_{x,r_0} \subset \Omega$ and $r \in (0, r_0)$

$$\omega(r) = \operatorname{ess\,osc}_{B_{x,r}} u. \quad (567)$$

¹⁵⁴It is a good exercise to work out the details of this claim.

We note that Ω is a non-decreasing function of r . For a u which is monotone (in the sense of Lebesgue) we clearly have

$$\omega(r) = \text{ess osc}_{\partial B_{x,r}} u . \quad (568)$$

(Note that by the theorem about traces in lecture 24 the function $u|_{\partial B_{x,r}}$ is defined for every $r \in (0, r_0)$.)

By (563) we have

$$\frac{\omega^2(r)}{r} \leq \pi \int_{\partial B_{x,r}} |\nabla u|^2 . \quad (569)$$

Integrating (569) between over (r_1, r_0) , and using that ω is non-decreasing, we obtain

$$\omega^2(r_1) \log \frac{r_0}{r_1} \leq \pi \int_{B_{x,r_0}} |\nabla u|^2 . \quad (570)$$

We see that $\omega(r) \rightarrow 0$ as $r \rightarrow 0$, uniformly on compact subsets of Ω , with the rate depending only on $\int_{\Omega} |\nabla u|^2$. This finishes the proof of the theorem.

We next turn to the maximum principle for the equation

$$-\frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j} + b_j \frac{\partial u}{\partial x_j} = 0 . \quad (571)$$

It is still possible to prove the following result, which is also called *the weak maximum principle*.

Theorem Any weak solution of (571) is monotone (in the sense of Lebesgue).

This is true without any smoothness assumptions on a_{ij}, b_j (we only assume that a_{ij}, b_j are bounded measurable and a_{ij} satisfy the ellipticity condition $a_{ij}\xi_i\xi_j \geq \nu|\xi|^2$), however it is not as easy to prove as in the case when $(b_1, \dots, b_n) = 0$. Let us first give the classical proof in the case when the coefficients are smooth. Then the solution is also smooth, and we can use derivatives freely. Assume that for some domain $\mathcal{O} \subset \Omega$ the maximum of the solution u over $\overline{\mathcal{O}}$ is attained at $x_0 \in \mathcal{O}$, with $(u-a)_+ \in H_0^1(\mathcal{O})$ for some $a < u(x_0)$. At the point x_0 the gradient ∇u vanishes. Our differential operator can be written as $-a_{ij}u_{,ij} + \tilde{b}_j u_{,j}$, where $\tilde{b}_j = b_j - a_{ij,i}$. As $u_{ij} = u_{ji}$, we can assume $a_{ij} = a_{ji}$ without loss of generality (as long as we allow general \tilde{b}_j). We can choose coordinates so that $a_{ij}(x_0)$ is diagonal, with $\lambda_1 > 0, \dots, \lambda_n > 0$ on the diagonal. At x_0 we will have

$$-\lambda_1 u_{,11}(x_0) - \dots - \lambda_n u_{,nn}(x_0) = 0 . \quad (572)$$

As $u_{11}(x_0) \leq 0, \dots, u_{,nn}(x_0) \leq 0$, this is close to a contradiction, but the equation can still be satisfied if $\nabla^2 u(x_0) = 0$. We use a trick which we already mentioned in the proof of the maximum principle for harmonic functions. If we had some strictly negative number on the right-hand side of (572), we would have

a contradiction at the point of the maximum x_0 . We can achieve this situation (with x_0 slightly shifted) by adding to u a function εv where v satisfies

$$-\frac{\partial}{\partial x_i} a_{ij} \frac{\partial v}{\partial x_i} + b_j \frac{\partial v}{\partial x_j} < 0. \quad (573)$$

and $\varepsilon > 0$ is small. This will clearly give a real contradiction (for small ε). The only remaining issue to construct a v satisfying (573). Such a v can be written down explicitly. For example, $v(x) = e^{\mu x_1}$ work well, if we take $\mu > 0$ sufficiently large. This finished the proof when the coefficients are smooth.

When the coefficients are not smooth, the proof is more difficult. One way to proceed is as follows. We can use $(u - a)_+$ in the weak formulation as a test function, and we obtain

$$\int_{\mathcal{O}} a_{ij} u_{,j} [(u - a)_+]_{,i} + b_j u_j (u - a)_+ = 0. \quad (574)$$

This gives

$$\int_{\mathcal{O}} |\nabla(u - a)_+|^2 \leq C[(u - a)_+]^2. \quad (575)$$

This inequality, used for a close to the maximum of u , leads to a contradiction. We will not complete the proof at this point, as it would lead us in a different direction,¹⁵⁵ but it is hoped that this example illustrates well some of the issues which arise in connection with our equations when the coefficients are not smooth.

¹⁵⁵Nevertheless, it is probably worth mentioning the main idea of the proof. One can proceed for example as follows. Let $E_a = \{x \in \mathcal{O}, u \geq a\}$, and let $1/p = 1/2 - 1/n$ for $n \geq 3$, and $p > 2$ for $n = 2$. Using imbeddings mentioned in lecture 38, we have

$$\begin{aligned} \int (u - a)_+^2 &\leq \left\{ \int (u - a)_+^p \right\}^{2/p} |E_a|^{1-2/p} \leq C_1 \int |\nabla(u - a)_+|^2 |E_a|^{1-2/p} \leq \\ &C_2 |E_a|^{1-2/p} \int (u - a)_+^2. \end{aligned}$$

This shows that u is bounded. Let $M = \text{ess sup}_{\mathcal{O}} u$. If $|E_a| \rightarrow 0$ as $a \rightarrow M_-$ (which is the same as $|E_M| = 0$), we see from the above inequality that $M = 0$. If $|E_M| > 0$ we let $v_a = (u - a)_+ / (M - a)$. The functions v_a again satisfy (575), and hence they are bounded in $W^{1,2}$, and stay in a compact subset of $L^2(\mathcal{O})$. We can assume that $v_a \rightarrow v$ weakly in $W^{1,2}$ as $a \rightarrow M_-$. It is not hard to see that v must be the characteristic function of E_M , which however cannot be in $W^{1,2}$ when $|E_M| \geq 0$. We see that $M \leq 0$.

We would like to explain some of the motivation behind studying the problems with low-regularity coefficients. One reason is “intrinsic”: the class of equations which can be treated by our existence theory includes equations with bounded measurable coefficients in such a natural way that one could be compelled to try extend as many “smooth case” results as possible to this case. Historically, one of the main reasons for the study of the case of low-regularity coefficients was their appearance in connection with regularity theory for *nonlinear equations*. The regularity theory for non-linear equations goes back to Hilbert’s problems no. XIX and XX, formulated around 1900. Let us look at a special case of these problems. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded open set. We have seen in lecture 21 that the solution of the problem

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega, \\ u|_{\partial\Omega} &= g && \text{at } \partial\Omega, \end{aligned}$$

can be obtained by minimizing the functional

$$\int_{\Omega} |\nabla u|^2 \tag{576}$$

over the set $W_g^{1,2}(\Omega) = \{u \in W^{1,2}(\Omega), u|_{\partial\Omega} = g\}$.

Assume now that (576) is replaced by

$$I(u) = \int_{\Omega} f(\nabla u(x)) \, dx \tag{577}$$

for some function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ in the above problem. Can we minimize I in $W_g^{1,2}(\Omega)$? Historically, one of the first cases people looked at for $n = 2$ is

$$f(\xi) = \sqrt{1 + |\xi|^2}. \tag{578}$$

In this case the functional $I(u)$ gives the area of the graph of u , and the problem of minimization of I over $W_g^{1,2}(\Omega)$ arises if we wish to find minimal surfaces with the boundary curve given by the graph of the function g . It turns out that the functional I with (579) is quite difficult to handle in general, as the minimal surfaces can sometimes prefer not to be graphs of functions. In terms of f this can be seen as a result of the slow growth and lack of uniform convexity as $\xi \rightarrow \infty$. Let us consider an easier problem when the function f has bounded second derivatives and is uniformly convex:

$$\left| \frac{\partial^2 f(\bar{\xi})}{\partial \xi_i \partial \xi_j} \right| \leq C \quad \bar{\xi} \in \mathbb{R}^n, \tag{579}$$

and

$$\frac{\partial^2 f(\bar{\xi})}{\partial \xi_i \partial \xi_j} \xi_i \xi_j \geq \nu |\xi|^2, \quad \bar{\xi} \in \mathbb{R}^n, \xi \in \mathbb{R}^n. \tag{580}$$

Can we then minimize I over $W_g^{1,2}(\Omega)$ and if so, what is the smoothness of the minimizing functions? This is an important special case of Hilbert's XIXth and XXth problems.

The proof of existence and uniqueness of the minimizer is quite similar to the quadratic case (576), and does not present a difficulty, once the theory of $W^{1,2}$ -spaces is developed. One has to find a suitable replacement for the explicit expressions we used when I was quadratic, so one cannot copy the proof line-by-line but it is a good exercise to show that – similarly to the quadratic case we treated in lecture 21 – any minimizing sequence is Cauchy in $W^{1,2}(\Omega)$ and the problem has a unique minimizer. (These are consequences of the uniform convexity of f and the geometric picture is the same as in the quadratic case.)

At the minimizer u , the derivative of I in any direction $v \in H_0^1(\Omega)$ will vanish:

$$I'(u)v = \int_{\Omega} f_{\xi_i}(\nabla u) v_{,i} = 0, \quad v \in H_0^1(\Omega), \quad (581)$$

where $f_{\xi_i} = \frac{\partial f}{\partial \xi_i}$. This is the weak form of the *Euler-Lagrange equation* for the functional I :

$$-\frac{\partial}{\partial x_i} f_{\xi_i}(\nabla u) = 0. \quad (582)$$

The vector field $f_{\xi_i}(\nabla u)$ can be thought of as a flux induced by the gradient ∇u , but the dependence is now non-linear.

It is not hard to see that the minimizers u have to be monotone (in the sense of Lebesgue). Therefore in dimension $n = 2$ we know right away that the minimizers are continuous, based on the result of last lecture.¹⁵⁶

The hard part of Hilbert's problem is the following question:

Are the minimizers smooth?

Recall that f is assumed to be smooth. At the boundary the minimizer u can be only as smooth as allowed by the boundary condition g , of course.

The problem was solved (with positive answer) in dimension $n = 2$ in 1930s by Ch. B. Morrey, and in general dimension in 1956 by DeGiorgi and Nash. The general case $\int_{\Omega} f(x, u, \nabla u)$ was solved by Ladyzhenskaya and Uraltseva a few years later.¹⁵⁷

Let us now explain how linear equations with low regularity coefficients appear in these problems. A simple observation is that (582) can be written as

$$\frac{\partial}{\partial x_i} \tilde{a}_{ij}(x) u_{,j} = 0 \quad (583)$$

¹⁵⁶In fact, it is easy to see that the minimizing sequence can be taken to be monotone (in the sense of Lebesgue). In dimension $n = 2$ such minimizing sequences converge uniformly.

¹⁵⁷Many of these developments are explained in the book "Linear and Quasilinear Elliptic Equations" by Ladyzhenskaya and Uraltseva.

with

$$\tilde{a}_{ij}(x) = \int_0^1 f_{\xi_i \xi_j}(t \nabla u(x)) dt. \quad (584)$$

In other words u solves a linear elliptic equation with low-regularity coefficients. Another linear equation one formally has is

$$a_{ij}(x)u_{,ij} = 0 \quad a_{ij}(x) = f_{\xi_i \xi_j}(\nabla u(x)), \quad (585)$$

but it has the disadvantage of not being well-defined for $u \in W^{1,2}$.

Although (584) or (585) can be sometimes of interest, they do not get us very far with the regularity problem. The key idea leading to regularity results is to take derivatives of the Euler-Lagrange equation (582). Let us first do that formally, postponing the justification of the calculation till later. Let us set $u' = \frac{\partial u}{\partial x_1}$ and take $\frac{\partial}{\partial x_1}$ of (582). We obtain

$$-\frac{\partial}{\partial x_i} a_{ij}(x)u'_{,j} = 0, \quad a_{ij}(x) = f_{\xi_i \xi_j}(\nabla u(x)). \quad (586)$$

The assumptions on f guarantee that the coefficients $a_{ij}(x)$ are bounded, and satisfy the ellipticity condition $a_{ij}\xi_i\xi_j \geq \nu|\xi|^2$. It is also clear that we cannot say much about the smoothness of the coefficients at this point, as we only know that $\nabla u \in L^2$. Therefore we have to treat $a_{ij}(x)$ as general L^∞ functions (satisfying the ellipticity conditions), until we obtain more information about ∇u .

We now recall Caccioppoli's inequality from lecture 33, and see that (586) gives us a new estimate for u :

$$\int_{B_R} |\nabla u'|^2 \leq \frac{C}{(R_0 - R)^2} \int_{B_{R_0}} |u'|^2. \quad (587)$$

The right-hand side is controlled by the known quantity $\int_\Omega |\nabla u|^2$. In this calculation u' can be the derivative in any direction, and therefore we have obtained an L^2 -estimate for $\nabla^2 u$ in L^2 . It remains to justify the formal calculation. However, this is not hard to do by using the difference quotients technique. To apply it in the non-linear case, one can use the formula

$$\frac{1}{h} (f_{\xi_i}(\nabla u(x + he_1)) - f_{\xi_i}(\nabla u(x))) = a_{ij}^h(x)(u^h)_{,j}, \quad (588)$$

where

$$a_{ij}^h(x) = \int_0^1 f_{\xi_i \xi_j}(t \nabla u(x + h) + (1-t)\nabla u(x)) dt \quad (589)$$

and

$$u^h(x) = D_1^h u(x). \quad (590)$$

We do the calculation with a_{ij} and u' replaced by a_{ij}^h and u^h , respectively, and obtain and estimate for ∇u^h in B_R which is uniform in h . We can then take the

limit $h \rightarrow 0$ using the last lemma in lecture 33 to obtain $\nabla u' \in L^2(B_R)$. This also justifies (586).

We have proved in the last lecture that in dimension $n = 2$ the solutions of (586) are continuous. Therefore we can conclude (still for $n = 2$) that u' is continuous. The same is true for any other partial derivative, and we see that ∇u is continuous. We used the linear equation satisfied by a quantity from a non-linear equation to improve our information about the regularity of the solution. Now that we know that ∇u is continuous, we see that the coefficients a_{ij} in (586) are continuous. This enables us still further increase the regularity of u' , which further increases the regularity of the coefficients, which still further increases the regularity of u' , etc. The optimal argument (which is technically slightly different even in the two-dimensional case from what we used so far) works as follows:

1. Prove that any solution u' of (586) is Hölder continuous. This is true in any dimension for any solution of (586) if a_{ij} are bounded, measurable, and satisfy the ellipticity condition $a_{ij}\xi_i\xi_j \geq \nu|\xi|^2$. The proof of this statement is the most difficult step, completed by DeGiorgi and Nash in 1956 for general n . In dimension $n = 2$ it is easier to show. (We have seen that when $n = 2$ we get relatively easily that the coefficients a_{ij} are continuous, which simplifies things substantially.)
2. If ∇u is of class $C^{k,\alpha}$ for some k (including $k = 0$), then $a_{ij}(x)$ are of class $C^{k,\alpha}$. Using linear regularity for (586), we obtain that ∇u is of class $C^{k+1,\alpha}$. This way we increase the regularity of u step-by-step, going up by one derivative in each step. This procedure is often called “bootstrapping”.

We see that practically every step in the proof is based on some quite precise estimates for linear equations, which is what we aimed to illustrate in this lecture. In the theory of PDE this is quite common, it is not limited to the type of equations we have been studying.¹⁵⁸

¹⁵⁸On the other hand, not all important estimates are “linear” (although this was the case in our example). One can encounter important estimates which are of non-linear nature.

Lecture 41, 12/15/2010

Today we will consider the *strong maximum principle* for the solutions of the equations

$$-\frac{\partial}{\partial x_i} a_{ij} u_{,j} + b_j u_{,j} = 0. \quad (591)$$

Let us first start by making a few remarks about the weak maximum principle we considered in lecture 39. There we showed that solutions of (591) are monotone (in the sense of Lebesgue). The same proof can be used to show that the solutions of

$$-\frac{\partial}{\partial x_i} a_{ij} u_{,j} + b_j u_{,j} \leq 0, \quad (592)$$

satisfy the maximum principle (as defined in lecture 39), and *non-negative* solutions of

$$-\frac{\partial}{\partial x_i} a_{ij} u_{,j} + b_j u_{,j} + cu \leq 0, \quad (593)$$

with $c = c(x) \geq 0$ satisfy the maximum principle.¹⁵⁹ These generalizations are simple, but they can be useful.

The weak maximum principle says, roughly speaking, that u cannot attain a strict local maximum at a compact set contained in the interior of a domain.¹⁶⁰ The strong maximum principle describes what happens when we leave out the condition that the maximum be “strict”. Clearly, constant functions solve (591) and do attain a (non-strict) maximum at interior points. The strong maximum principle says that these constant solutions are the only case when this can happen. More precisely, we have the following results

Theorem (Strong Maximum Principle)

Let u be a solution of (591) in a domain Ω and assume that for some $x_0 \in \Omega$ $u \leq u(x_0)$ in Ω . Then $u \equiv u(x_0)$ in Ω .

Remarks:

1. We did not explicitly state any smoothness assumptions on the coefficients. This is because the result is true for general L^∞ coefficients, as long as a_{ij} satisfy the ellipticity condition $a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2$. However, we will do the proof only when the coefficients are “sufficiently regular”. Essentially we need that the weak maximum principle is true and that the solution u is differentiable at each point. This will certainly be the case of the coefficients are smooth. If we use

¹⁵⁹Our proof based in the evaluation of the second derivatives needs that u is of class C^2 , which dictates some assumptions about c . However, the statement is true for any bounded measurable c (and the boundedness can be replaced by still weaker assumptions, if necessary).

¹⁶⁰Typically we think of the set where the local maximum is attained as a point, but in principle it could be for example a curve, or a more complicated object.

some of the facts mentioned in lectures 37 and 38, we see that our proof will work when a_{ij} are Lipschitz and b_j are bounded. The proof for the general L^∞ coefficients requires more advanced techniques.

2. It is easy to see that the statement remains true for inequality (592) (i. e. when u is a *subsolution* rather than a solution), and also for inequality (593) under the assumptions that $c \geq 0$ and $u \geq 0$.

3. If we replace the assumption that $u \leq u(x_0)$ in Ω by the weaker condition that $u \leq u(x_0)$ in a neighborhood of x_0 , we have by the above that u is constant in a neighborhood of x_0 . If the leading coefficients a_{ij} are Lipschitz, the theory of *unique continuation* implies that u has to be constant in Ω . However, this result is quite subtle and requires new techniques. For harmonic functions we can obtain it by using analyticity of the solutions. When the coefficients of the equations are analytic, it can be shown that the solutions are also analytic, and the same argument as for the harmonic functions can be used. If the coefficients are not analytic, another argument is needed.

The proof of the strong maximum principle in the case of sufficiently regular coefficients is traditionally based on the following lemma of E. Hopf, which is of independent interest.

Hopf's Lemma

Assume that the coefficients in (591) are sufficiently regular¹⁶¹ and let u be a solution with in the ball B_R , with $u \in C^1(\overline{B}_R)$. Let $x_0 \in \partial B_R$ be such that $u(x) < u(x_0)$ for each $x \in B_R$. Let $\nu(x_0) = (\nu_1(x_0), \dots, \nu_n(x_0))$ be the outward unit normal to B_R at x_0 . Then

$$\frac{\partial u}{\partial \nu}(x_0) > 0. \quad (594)$$

Proof:

We note that on $\partial B_{R/2}$ we have $u \leq u(x_0) - \delta$ for some $\delta > 0$. Assume that we construct a smooth function v in the closure of the set $\mathcal{O} = B_R \setminus B_{R/2}$ such that

$$(i) \quad v|_{\partial B_R} = u(x_0), \quad v|_{\partial B_{R/2}} = u(x_0) - \delta,$$

$$(ii) \quad -\frac{\partial}{\partial x_i} a_{ij} v_{,j} + b_j v_{,j} \geq 0 \text{ in } \mathcal{O}, \text{ and}$$

$$(iii) \quad \frac{\partial v}{\partial \nu}(x_0) > 0.$$

By the weak maximum principle we then have $u \leq v$ in \mathcal{O} , and since $u(x_0) = v(x_0)$ we see from (iii) that $\frac{\partial u}{\partial \nu}(x_0) > 0$.

We will seek v in the form

$$v(x) = -\varepsilon e^{-\lambda \frac{|x|^2}{2}} + c, \quad (595)$$

¹⁶¹Our proof will require Lipschitz a_{ij} and bounded b_j

where $\lambda > 0$, $\varepsilon > 0$ and $c \in \mathbb{R}$ are chosen so that boundary conditions (i) are satisfied.

It is now enough to verify that for $w = e^{-\lambda \frac{|x|^2}{2}}$ with a suitable $\lambda > 0$ we have

$$-\frac{\partial}{\partial x_i} a_{ij} w_{,j} + b_j w_{,j} \leq 0, \quad \text{in } \mathcal{O}. \quad (596)$$

If a_{ij} are Lipschitz and b_j are bounded, we only need to look at

$$-a_{ij} w_{,ij} + \tilde{b}_j w_{,j} \quad (597)$$

with some bounded \tilde{b}_j . We calculate

$$w_{,i} = -\lambda x_i w, \quad w_{,ij} = \lambda^2 x_i x_j w - \lambda \delta_{ij} w, \quad (598)$$

and substituting this into the expression (597) it is not hard to see that the expression (597) is ≤ 0 a. e. in \mathcal{O} when λ is large enough, as (597) will be

$$-\lambda^2 a_{ij} x_i x_j + \lambda \gamma(x) \leq -\lambda^2 \nu |R/2|^2 + \lambda M, \quad (599)$$

where $\gamma(x)$ is an L^∞ function in \mathcal{O} , $M = \text{ess sup}_{\mathcal{O}} \gamma$, and $\nu > 0$ is the constant in the ellipticity condition $a_{ij} \xi_i \xi_j \geq \nu |\xi|^2$.¹⁶² The proof of the lemma is finished.

It is now easy to prove the theorem. Let $K = \{x \in \Omega, u(x) = u(x_0)\}$ and assume that both K and $\mathcal{O} = \Omega \setminus K$ are non-empty. As Ω is connected by our assumptions, this means that $\partial \mathcal{O} \cap \Omega \neq \emptyset$. Let us take $x_1 \in \partial \mathcal{O} \cap \Omega$. Set $r_1 = \text{dist}(x_1, \partial \Omega)$, and choose $x_2 \in \mathcal{O}$ with $|x_1 - x_2| < r_1/2$. Let $R = \text{dist}(x_2, \partial \mathcal{O})$ and let $x_3 \in \partial B_{x_2, R} \cap K$. By Hopf's lemma, $\frac{\partial u}{\partial \nu}(x_3) > 0$, which is in contradiction with the condition that $u \leq u(x_3) = u(x_0)$ in Ω . This finishes the proof of the theorem.

¹⁶²We also use ν for the normal, but hopefully there is no danger of confusion here.

Last semester our focus was on elliptic equations. This semester we will study more general classes of linear equations, including parabolic equations and (examples of) dispersive and hyperbolic equations. We will start with an introduction into the theory of distributions, which can be considered as an ideal language for formulating many results about linear differential operators and Fourier transformation. To introduce this language, we will need to go through a certain number of definitions. As we will see, the flexibility the language of the distribution provides is well worth the modest effort needed to introduce it.

We will denote by $\mathcal{D}(\Omega)$ the space of all smooth, compactly supported complex-valued functions in an open set $\Omega \subset \mathbb{R}^n$. For $\varphi \in \mathcal{D}(\Omega)$ and $\alpha = (\alpha_1, \dots, \alpha_n)$ where α_j are non-negative integers, we will denote by $\partial^\alpha \varphi$ the derivative $\frac{\partial^{|\alpha|} \varphi}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$. (We use the usual notation $|\alpha| = \alpha_1 + \dots + \alpha_n$.)

For $\varphi \in \mathcal{D}(\Omega)$ we set

$$\|\varphi\|_m = \sup_{x \in \Omega, |\alpha| \leq m} |\partial^\alpha \varphi(x)|. \quad (600)$$

Each $\|\cdot\|_m$ obviously is a norm on $\mathcal{D}(\Omega)$. We note that $\mathcal{D}(\Omega)$ is not complete in those norms.¹⁶³

Let $K \subset \Omega$ be a compact set. We define $\mathcal{D}_K(\Omega)$ as the set of all functions in $\mathcal{D}(\Omega)$ which are supported in K . On the space $\mathcal{D}_K(\Omega)$ we consider the topology given by the following notion of convergence: $\varphi_j \rightarrow \varphi$ when for each m we have $\|\varphi_j - \varphi\|_m \rightarrow 0$ as $j \rightarrow \infty$. This topology is defined by a metric, which can be taken for example as

$$\text{dist}(\varphi_1, \varphi_2) = \sum_m 2^{-m} \frac{\|\varphi_1 - \varphi_2\|_m}{1 + \|\varphi_1 - \varphi_2\|_m}. \quad (601)$$

The space $\mathcal{D}_K(\Omega)$ is easily seen to be complete in this metric. In the terminology used in Functional Analysis, the space $\mathcal{D}_K(\Omega)$ is a *Fréchet space*. Fréchet spaces are a natural generalization of Banach spaces to the situation where convergence is defined not just by one norm, but by a countable family norm (or semi-norms), and the resulting metric space is complete. Many classical consequences of completeness known in the theory of Banach spaces¹⁶⁴ can be more or less directly generalized to Fréchet spaces.¹⁶⁵

The following result, despite its elementary nature, is very useful. Its main point is that although the definition of convergence in $\mathcal{D}_K(\Omega)$ involves the derivatives

¹⁶³As an exercise you can characterize the completion of $\mathcal{D}(\Omega)$ in the norm $\|\cdot\|_m$.

¹⁶⁴Such as the Open Mapping Theorem, Closed Graph Theorem, Uniform Boundedness Principle (Banach-Steinhaus Theorem)

¹⁶⁵A good reference for these results is for example W. Rudin's textbook Functional Analysis.

of arbitrary high orders, the values of any given continuous *linear* functional on $\mathcal{D}_K(\Omega)$ converge on sequences $\{\varphi_j\}$ where only finitely many derivatives are convergent.

Lemma

A linear functional l on $\mathcal{D}_K(\Omega)$ is continuous if and only if we have

$$|l(\varphi)| \leq C \|\varphi\|_m, \quad \varphi \in \mathcal{D}_K(\Omega) \quad (602)$$

for some $C > 0$ and some non-negative integer m .

We leave the proof to the reader as an exercise.

We now define convergence in $\mathcal{D}(\Omega)$ in the following way. By definition, a sequence $\varphi_j \in \mathcal{D}(\Omega)$ converges to $\varphi \in \mathcal{D}(\Omega)$ when

(a) There exists a compact $K \subset \Omega$ such that all functions φ_j are supported in K .

(b) $\varphi_j \rightarrow \varphi$ in $\mathcal{D}_K(\Omega)$, i. e. $\partial^\alpha \varphi_j$ converges uniformly to $\partial^\alpha \varphi$ for each multi-index α .

The above convergence in $\mathcal{D}(\Omega)$ defines a topology on $\mathcal{D}(\Omega)$, which however is not metrizable. As we will see, this does not cause problems, since one can always work with the simple definition of convergence.

By definition, a *distribution* in Ω is a linear functional on $\mathcal{D}(\Omega)$ which is continuous with respect to the above defined convergence. The set of all distributions in Ω is denoted by $\mathcal{D}'(\Omega)$. For $u \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$ we will denote by $\langle u, \varphi \rangle$ or $u(\varphi)$ the value of u at the function φ .

Examples

1. Each locally integrable function u in Ω defines a distribution by

$$\langle u, \varphi \rangle = \int_{\Omega} u \varphi. \quad (603)$$

2. Given $u \in \mathcal{D}'(\Omega)$, we define its derivatives $\partial^\alpha u \in \mathcal{D}'(\Omega)$ by

$$\langle \partial^\alpha u, \varphi \rangle = \langle u, (-1)^{|\alpha|} \partial^\alpha \varphi \rangle. \quad (604)$$

The verification that $\partial^\alpha u \in \mathcal{D}'(\Omega)$ is straightforward and is left to the reader as an easy exercise. For u represented by a sufficiently regular function u the derivative $\partial^\alpha u$ of the distribution u is represented by the usual derivative $\partial^\alpha u$, as one can see from the integration by parts. In general, the derivatives $\partial^\alpha u$ may not be representable in terms of locally integrable functions.

3. For $a \in \Omega$ we define the distribution δ_a (also called the Dirac mass) by

$$\langle \delta_a, \varphi \rangle = \varphi(a). \quad (605)$$

The derivatives of δ_a are given by

$$\langle \partial^\alpha \delta_a, \varphi \rangle = (-1)^{|\alpha|} \partial^\alpha \varphi(a). \quad (606)$$

As a consequence of the above lemma, one can see easily the following statement:
A linear functional on $\mathcal{D}(\Omega)$ is a distribution if and only if for each compact set $K \subset \Omega$ there exist a non-negative integer $m = m(K, u)$ and a constant $C = C(K, u) \geq 0$ such that

$$|\langle u, \varphi \rangle| \leq C \|\varphi\|_m, \quad \varphi \in \mathcal{D}_K(\Omega). \quad (607)$$

We say that a distribution $u \in \mathcal{D}'(\Omega)$ vanishes in an open subset $\mathcal{O} \subset \Omega$ if $\langle u, \varphi \rangle = 0$ for each $\varphi \in \mathcal{D}(\mathcal{O})$.

The *support* of a distribution $u \in \mathcal{D}'(\Omega)$ is defined as the complement of the union of all open sets on which u vanishes. The support of u will be denoted by $\text{supp } u$. If $\text{supp } u \subset K$ for some compact set $K \subset \Omega$, we say that u is supported at K . If u is given by a locally integrable function then the definition coincides with the usual definition. To prove that, one needs the following statement:

If u is a locally integrable function in an open set \mathcal{O} and $\int u\varphi = 0$, for each $\varphi \in \mathcal{D}(\mathcal{O})$, then $u(x) = 0$ for almost every $x \in \mathcal{O}$.

This is a standard (non-trivial) fact from the theory of Lebesgue integration, which can be seen from $u * \phi_\varepsilon \rightarrow u$ (locally in L^1) as $\varepsilon \rightarrow 0$, where ϕ_ε is the standard mollifier introduced in lecture 2, see (21).

The support of the distributions $\partial^\alpha \delta_a$ introduced above is the one-point set $\{a\}$. The following result is important in that it shows that our definition of distributions does not allow any unexpected “parasitic” objects.

Proposition

Let $a \in \Omega$ and assume that $u \in \mathcal{D}'(\Omega)$ is supported at a . Then there exists a non-negative integer m such that

$$u = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha \delta_a \quad (608)$$

for some coefficients $c_\alpha \in \mathbf{C}$.

Lecture 43, 1/20/2011

We proceed with the proof of the last proposition in the previous lecture. We can assume without loss of generality that $a = 0$, and that a closed ball $K = \{x, |x| \leq r_1\}$ is contained in Ω . By the first lemma in the last lecture, see (602), we know that for some $C > 0$ and a non-negative integer m we have

$$\langle u, \varphi \rangle \leq C \|\varphi\|_m, \quad \varphi \in \mathcal{D}_K(\Omega). \quad (609)$$

We claim that

$$\langle u, \varphi \rangle = 0 \quad \text{whenever } \varphi \in \mathcal{D}(\Omega) \text{ and } \partial^\alpha \varphi(0) = 0 \text{ for } |\alpha| \leq m. \quad (610)$$

To prove the claim, let us consider a smooth function ψ supported in the unit ball of \mathbb{R}^n with $\psi = 1$ in $\{x, |x| \leq \frac{1}{2}\}$ and for $\varepsilon > 0$ let us set

$$\psi_\varepsilon(x) = \psi\left(\frac{x}{\varepsilon}\right). \quad (611)$$

For each $\varphi \in \mathcal{D}(\Omega)$ the value $\langle u, \varphi \psi_\varepsilon \rangle$ is independent of ε . To see this, we note that for each $0 < \varepsilon_1 \leq \varepsilon_2$ and each $\varphi \in \mathcal{D}(\Omega)$ the support of the function $\varphi \psi_{\varepsilon_1} - \varphi \psi_{\varepsilon_2}$ is contained in $\Omega \setminus \{0\}$, and by the assumption $\text{supp } u \subset \{0\}$ we must have $\langle u, \varphi \psi_{\varepsilon_1} - \varphi \psi_{\varepsilon_2} \rangle = 0$. In view of (609), the proof of (610) will be finished if we show that

$$\lim_{\varepsilon \rightarrow 0^+} \|\varphi \psi_\varepsilon\|_m = 0 \quad \text{whenever } \varphi \in \mathcal{D}(\Omega) \text{ and } \partial^\alpha \varphi(0) = 0 \text{ for } |\alpha| \leq m. \quad (612)$$

Assuming $\partial^\alpha \varphi(0) = 0$ for $|\alpha| \leq m$, let us estimate the derivatives $\partial^\alpha(\varphi \psi_\varepsilon)$ for $|\alpha| \leq m$. By Leibnitz rule, these derivatives are a linear combination of the terms

$$(\partial^\beta \varphi)(\partial^\gamma \psi_\varepsilon), \quad |\beta| + |\gamma| \leq m. \quad (613)$$

Under our assumptions on φ we have

$$|\partial^\beta \varphi(x)| = O(|x|^{m+1-|\beta|}), \quad x \rightarrow 0 \quad (614)$$

and

$$|\partial^\gamma \psi_\varepsilon| \leq \frac{C_\gamma}{\varepsilon^{|\gamma|}} \quad \text{in } \text{supp } \psi_\varepsilon \subset B_\varepsilon = \{x, |x| < \varepsilon\}. \quad (615)$$

We see that for $|\beta| + |\gamma| \leq m$ we have

$$\langle (\partial^\beta \varphi)(\partial^\gamma \psi_\varepsilon), 0 \rangle = O(\varepsilon), \quad \varepsilon \rightarrow 0 \text{ in } \Omega, \quad (616)$$

which implies (612) and hence also (610) (due to (609)).

With (610) established, it is easy to finish the proof. Let N be the number of multi-indices α with $|\alpha| \leq m$, and let

$$\pi: \mathcal{D}(\Omega) \rightarrow \mathbf{C}^N \quad (617)$$

be the map defined by

$$\varphi \rightarrow \{\partial^\alpha \varphi(0)\}_{|\alpha| \leq m}. \quad (618)$$

We have established that

$$\pi(\varphi) = 0 \implies \langle u, \varphi \rangle = 0. \quad (619)$$

In other words, the kernel of the linear map π is contained in the kernel of the linear map u . This means that the map $\varphi \rightarrow \langle u, \varphi \rangle$ can be factored through π . That is, there exists a linear map $l: \mathbf{C}^N \rightarrow \mathbf{C}$ such that

$$\langle u, \varphi \rangle = l(\pi(\varphi)). \quad (620)$$

Every map $l \circ \pi$ as above is clearly of the form

$$\varphi \rightarrow \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha \varphi(0), \quad (621)$$

and the proof of the proposition is finished.

If we replace the one-point $\{a\}$ in the proposition by a compact set $K \subset \Omega$, the following result can be obtained.

Let $u \in \mathcal{D}'(\Omega)$ and $\text{supp } u \subset K$, where $K \subset \Omega$ is compact. Then there exists a non-negative integer m and $L^2(\Omega)$ -functions g_α , $|\alpha| \leq m$, such that

$$\langle u, \varphi \rangle = \int_{\Omega} \sum_{|\alpha| \leq m} g_\alpha(x) \partial^\alpha \varphi(x) dx. \quad (622)$$

Note that it is not claimed that the functions g_α are supported in K . However, once the existence of g_α is known, they can be easily modified so that they have support in any given neighborhood of K .

For general compact sets K it is not always possible to achieve a representation $\langle u, \varphi \rangle = \int_K \sum_{|\alpha| \leq m} \partial^\alpha \varphi d\mu_\alpha$, where μ_α are (signed) measures supported in K .¹⁶⁶

¹⁶⁶See for example the book "The Analysis of Linear Partial Differential Operators I" by L. Hörmander, Section 2.3.

Lecture 44, 1/24/2011

We prove the representation (622) from the last statement of the previous lecture. Let $\psi \in \mathcal{D}(\Omega)$ be a function which is $\equiv 1$ in a neighborhood of K . Since $\text{supp } u \subset K$, we have $\langle u, \varphi \rangle = \langle u, \varphi\psi \rangle$ for each $\psi \in \mathcal{D}(\Omega)$. In view of the first lemma in lecture 42, see (602), we have for some $C \geq 0$ and a non-negative integer m

$$\langle u, \varphi\psi \rangle \leq C \|\varphi\psi\|_m \quad \varphi \in \mathcal{D}(\Omega). \quad (623)$$

As ψ is fixed, we can write (after possibly changing C)

$$\langle u, \varphi \rangle \leq C \|\varphi\|_m. \quad (624)$$

From lecture 34 we recall the definition of the Sobolev spaces $H^s(\mathbb{R}^n)$ and the Sobolev Imbedding Theorem, see (497). We note that $\mathcal{D}(\Omega)$ can be obviously considered as a subspace of $H^s(\mathbb{R}^n)$. In view of (497), when $s > m + n/2$ we can write

$$\langle u, \varphi \rangle \leq C \|\varphi\|_{H^s}, \quad \varphi \in \mathcal{D}(\Omega) \quad (625)$$

for some $C \geq 0$. Let us take s to be an integer and let N be the number of multi-indices α with $|\alpha| \leq s$. We consider the mapping

$$\iota: \mathcal{D}(\Omega) \rightarrow [L^2(\Omega)]^N \quad (626)$$

defined by

$$\iota(\varphi) = \{\partial^\alpha \varphi\}_{|\alpha| \leq s}. \quad (627)$$

The image of $\mathcal{D}(\Omega)$ under the map ι is some linear subspace Y of $[L^2(\Omega)]^N$. The map $\varphi \rightarrow \langle u, \varphi \rangle$ defines a linear functional l on Y by

$$l: \{\partial^\alpha \varphi\}_{|\alpha| \leq s} \rightarrow \langle u, \varphi \rangle. \quad (628)$$

By (625) this map is continuous on Y (taken with the $[L^2(\Omega)]^N$ -norm). We can now extend the linear functional l to the whole space $[L^2(\Omega)]^N$ so that the extension (still denoted by l) satisfies

$$l(v) \leq C \|v\|_{[L^2(\Omega)]^N}, \quad v = \{v_\alpha\}_{|\alpha| \leq s} \in [L^2(\Omega)]^N. \quad (629)$$

To construct the extension, we can either directly apply the Hahn-Banach Theorem, or extend by $l(v) = l(Pv)$, where P is the orthogonal projection onto \bar{Y} . By the Riesz representation theorem for linear functionals on Hilbert spaces, there exist function $g_\alpha \in L^2(\Omega)$, $|\alpha| \leq s$ such that

$$l(v) = \int_{\Omega} \sum_{|\alpha| \leq s} g_\alpha v_\alpha, \quad v = \{v_\alpha\}_{|\alpha| \leq s} \in [L^2(\Omega)]^N. \quad (630)$$

In particular,

$$\langle u, \varphi \rangle = \int_{\Omega} \sum_{|\alpha| \leq s} g_\alpha \partial^\alpha \varphi, \quad \varphi \in \mathcal{D}(\Omega), \quad (631)$$

which finishes the proof.

Remarks

1. Using $\langle u, \varphi \rangle = \langle u, \varphi \psi \rangle$ where ψ is as above, we see that the functions g_α can be taken so that their support is in any given neighborhood of K .

2. By the construction, the regularity of the functions g_α is not very high - they are just L^2 -functions. However, we can increase the regularity of the representing functions g_α at the cost of taking a higher s in the following way. Let $\mathcal{G} = \Delta^{-1}$ be the inversion of the Laplace operator by the Newton potential we studied in lecture 2. We can write

$$\int_{\Omega} g_\alpha \partial^\alpha \varphi = \int_{\Omega} g_\alpha \mathcal{G}(\Delta \partial^\alpha \varphi) = \int_{\Omega} (\mathcal{G} g_\alpha) \Delta \partial^\alpha \varphi. \quad (632)$$

This improves the regularity of the functions g_α , although for the new functions α does not refer to the index in $\partial^\alpha \varphi$ in the integral (631). We can repeat this step as many times as we wish. In some sense, the procedure moves regularity from φ (where regularity is always “available”) to g_α .

3. Any distribution $u \in \mathcal{D}'(\Omega)$ can be written as a locally finite sum of compactly supported distributions. For this we can consider a suitable partition of unity $\phi_1 + \phi_2 + \dots = 1$ in Ω , where $\phi_j \in \mathcal{D}(\Omega)$ and only finitely many functions are non-zero in any compact subset of Ω . We define $\langle u_j, \varphi \rangle = \langle u, \phi_j \varphi \rangle$. Then $u = \sum_j u_j$ in Ω , and each u_j has a representation as in the above proved theorem (with $m = m_j$ now depending on j , and possibly approaching ∞ as $j \rightarrow \infty$). Moreover, the construction can be done so in such a way that at most finitely many representing functions $g_{j,\alpha}$ are non-zero on any given compact subset of Ω . The details are left to the reader as an exercise.

It is easy to construct examples of u in which the “degree” m_j of the localized distributions u_j in the above construction (i. e. the minimal possible number m in (624)) is unbounded as $j \rightarrow \infty$. For example one can take $\sum_{j=1}^{\infty} c_j \partial^{\alpha_j} \delta_{a_j}$, where $a_j \in \Omega$ is a sequence with no accumulation points in Ω , $c_j \neq 0$, and $|\alpha_j| \rightarrow \infty$ as $j \rightarrow \infty$.

We now investigate convergence of distributions. Let $u_j \in \mathcal{D}'(\Omega)$, $j = 1, 2, \dots$. We say that u_j converge to $u \in \mathcal{D}'(\Omega)$ if

$$\langle u_j, \varphi \rangle \rightarrow \langle u, \varphi \rangle, \quad \varphi \in \mathcal{D}(\Omega). \quad (633)$$

If $u_j \rightarrow u$ in $\mathcal{D}'(\Omega)$, then clearly for each $\varphi \in \mathcal{D}(\Omega)$ the sequence $\langle u_j, \varphi \rangle$ is bounded. We prove the following result.

Theorem

For any sequence of distributions $u_j \in \mathcal{D}'(\Omega)$ with the property that the sequence $\langle u_j, \varphi \rangle$ is bounded for each $\varphi \in \mathcal{D}(\Omega)$, we can choose a subsequence u_{j_k} which converges in $\mathcal{D}'(\Omega)$ to some $u \in \mathcal{D}'(\Omega)$.

The theorem can be thought of as a certain compactness property of suitable subsets of $\mathcal{D}'(\Omega)$.

The theorem is a consequence of two important principles of Functional Analysis: the Uniform Boundedness Principle¹⁶⁷ and the Cantor diagonal argument.

Recall that in the context of Banach spaces the Banach-Steinhaus Theorem says the following: If X is a Banach space, and $M \subset X^*$ is a subset of its dual space such that the set $\{l(x), l \in M\}$ is bounded for each $x \in X$ (with the bound possibly dependent on x), then M is bounded in norm in X^* , i. e. the bound on the sets $\{l(x), l \in M\}$ can be taken to be independent of x . The statement is one of the classical consequences of completeness of X . The standard proof works also for Fréchet spaces, without much change.¹⁶⁸ Applying this result in our situation (using that $\mathcal{D}_K(\Omega)$ are Fréchet spaces), we obtain the following statement:

(S) *If $M \subset \mathcal{D}'(\Omega)$ is a set of distributions such that $\{\langle u, \varphi \rangle, u \in M\}$ is bounded for each $\varphi \in \mathcal{D}(\Omega)$, then M is uniformly bounded in the following sense: for each compact $K \subset \Omega$ there exists $C \geq 0$ and a non-negative integer m such that*

$$|\langle u, \varphi \rangle| \leq C \|\varphi\|_m, \quad \varphi \in \mathcal{D}_K(\Omega), u \in M. \quad (634)$$

With the statement (S) the proof of the theorem is easy: first, under the assumptions of the theorem we see from (S) that the sequence $\{u_j\}_{j=1}^\infty$ is uniformly bounded in $\mathcal{D}'(\Omega)$ in the sense of (634). Second, we choose a countable dense subset X in $\mathcal{D}(\Omega)$ and use the Cantor diagonal argument to produce a subsequence u_{j_k} such that $\langle u_{j_k}, \varphi \rangle$ converges to a finite limit for each $\varphi \in X$. From the bound (634) (applied with $M = \{u_j\}$) we conclude that $\langle u_{j_k}, \varphi \rangle$ converges to a finite $l(\varphi)$ limit for each $\varphi \in \mathcal{D}(\Omega)$, with the bound (634) still satisfied for $\varphi \rightarrow l(\varphi)$. Hence $l(\varphi) = \langle u, \varphi \rangle$ for some $u \in \mathcal{D}'(\Omega)$.

We remark that the above proof also shows the following results.

If u_j is a sequence of distributions such that $l(\varphi) = \lim_{j \rightarrow \infty} \langle u_j, \varphi \rangle$ exists and is finite for each $\varphi \in \mathcal{D}(\Omega)$, then l defines a distribution, i. e. there exists $u \in \mathcal{D}'(\Omega)$ such that $l(\varphi) = \langle u, \varphi \rangle$ for each $\varphi \in \mathcal{D}(\Omega)$.

¹⁶⁷In the context of Banach spaces it is also known as the Banach-Steinhaus Theorem.

¹⁶⁸See for example W. Rudin's "Functional Analysis", Section 2.3.

In lecture 2 we introduced mollification as a procedure to approximate a given locally integrable function by a smooth function. Let $\phi_\varepsilon = \frac{1}{\varepsilon^n} \phi(\frac{x}{\varepsilon})$ be mollifiers as in lecture 2. For $\varepsilon > 0$ we will denote by Ω_ε the set $\{x \in \Omega, \text{dist}(x, \partial\Omega) > \varepsilon\}$. For a locally integrable function in Ω the function $u * \phi_\varepsilon$ (defined in Ω_ε) can be interpreted as a “linear combination” of shifts of u . More precisely, let us define the shift of u by $y \in \mathbb{R}^n$ by

$$\tau_y u(x) = u(x - y). \quad (635)$$

Then

$$u * \phi_\varepsilon = \int \tau_y u \phi_\varepsilon(y) dy \quad \text{in } \Omega_\varepsilon. \quad (636)$$

Formula (636) can be naturally applied to distributions. For $u \in \mathcal{D}'(\Omega)$ and $y \in \mathbb{R}^n$ we define $\tau_y u$ by

$$\langle \tau_y u, \varphi \rangle = \langle u, \tau_{-y} \varphi \rangle, \quad \text{when } \tau_{-y} \varphi \in \mathcal{D}(\Omega). \quad (637)$$

For $u \in \mathcal{D}'(\Omega)$ and $\varepsilon > 0$ we define

$$u_\varepsilon = u * \phi_\varepsilon = \int \tau_y u \phi_\varepsilon(y) dy. \quad (638)$$

We have $u_\varepsilon \in \mathcal{D}'(\Omega_\varepsilon)$ and

$$\langle u_\varepsilon, \varphi \rangle = \int \langle u, \tau_{-y} \varphi \rangle \phi_\varepsilon(y) dy. \quad (639)$$

Denoting $\tilde{\phi}_\varepsilon(y) = \phi_\varepsilon(-y)$, we can write

$$\langle u_\varepsilon, \varphi \rangle = \langle u, \varphi * \tilde{\phi}_\varepsilon \rangle. \quad (640)$$

We can also write

$$\langle u_\varepsilon, \varphi \rangle = \int \langle u, \tau_y \phi_\varepsilon \rangle \varphi(y) dy, \quad (641)$$

which shows that the distribution u_ε is given by the function

$$u_\varepsilon(x) = \langle u, \tau_x \tilde{\phi}_\varepsilon \rangle, \quad (642)$$

which can be also formally obtained as $u_\varepsilon(x) = \langle u_\varepsilon, \delta_x \rangle$.¹⁶⁹ From the definitions it is easy to see that the function $x \rightarrow u_\varepsilon(x)$ is smooth in Ω_ε and $u_\varepsilon \rightarrow u$ as $\varepsilon \rightarrow 0_+$ in $\mathcal{D}'(\Omega_{\varepsilon_1})$ for every $\varepsilon_1 > 0$. Although we do not have $u_\varepsilon \in \mathcal{D}'(\Omega)$, we notice that for any given $\varphi \in \mathcal{D}(\Omega)$, the value $\langle u_\varepsilon, \varphi \rangle$ is well defined for sufficiently small ε .

¹⁶⁹The expression $\langle u, \delta_x \rangle$ is of course not defined for general distributions u . It can be defined as the limit of $\langle u, \varphi_j \rangle$ where $\varphi_j \rightarrow \delta_x$ gives an approximation of the Dirac mass δ_x if the limit exists in a suitable sense.

With the definitions above the usual formulae

$$\partial^\alpha(u * \phi_\varepsilon) = (\partial^\alpha u) * \phi_\varepsilon = u * (\partial^\alpha \phi_\varepsilon) \quad (643)$$

are valid, as can be easily checked.

The convolution $u*v$ can in fact be defined when both u and v are distributions, provided some natural conditions concerning supports of u, v are satisfied.¹⁷⁰ For now we will not study this aspect in more detail, but we may return to it in the future.

As an application of the approximation procedure, we give a prove of the important result that any distribution with vanishing derivatives has to be constant. We use the following notation: for $u \in \mathcal{D}'(\Omega)$ we denote by ∇u the n -tuple of distributions $(\partial_1 u, \dots, \partial_n u)$. Also, we will say that a distribution $u \in \mathcal{D}'(\Omega)$ is constant if it is given by $\langle u, \varphi \rangle = \int_\Omega c \varphi$ for some $c \in \mathbf{C}$.

Theorem

Assume Ω is connected and $u \in \mathcal{D}'(\Omega)$ with $\nabla u = 0$. Then u is constant in Ω .

Proof: Consider the approximating distributions $u_\varepsilon = u * \phi_\varepsilon$. By (643) we see that $\nabla u_\varepsilon = 0$ in Ω_ε . Therefore u_ε is a locally constant function in Ω_ε . For any given $\varphi \in \mathcal{D}(\Omega)$ the function u_ε will be constant on the support of φ when ε is sufficiently small. Let c_ε be the value of the constant. As $\langle u_\varepsilon, \varphi \rangle$ converges when $\varepsilon \rightarrow 0$, we see that the constants c_ε have to converge as $\varepsilon \rightarrow 0$ to some $c \in \mathbf{C}$. It remains to show that c is independent of the choices made in this process, and that $u = c$ in Ω . We leave this to the reader as an easy exercise.

Remarks

1. The above theorem is important in that it shows that we have the right definition of the derivative. The result that a function with zero derivative should be constant is a good test whether a given generalized notion of a derivative is reasonable. If we could have non-constant functions with vanishing generalized derivative, then the definition of the generalized derivative would not be suitable for PDE purposes.

2. The above proof of the theorem via approximations may not look like the most elegant proof of the result. Probably the most natural proof is the following: if $\varphi \in \mathcal{D}(\Omega)$ with $\int_\Omega \varphi = 0$, then there exist functions $\psi_1, \dots, \psi_n \in \mathcal{D}(\Omega)$ such that $\varphi = \partial_j \psi_j$ (summation convention is understood). When $u \in \mathcal{D}'(\Omega)$ with $\nabla u = 0$, then by definition $\langle u, \partial_j \psi_j \rangle = 0$, and hence $\langle u, \varphi \rangle = 0$ for each $\varphi \in \mathcal{D}(\Omega)$ with $\int_\Omega \varphi = 0$. This shows that $\langle u, \varphi \rangle = c \int_\Omega \varphi$ for each $\varphi \in \mathcal{D}(\Omega)$. However, the proof of the existence of ψ_j requires some work. If we look at the proof via approximations which we used above in more detailed, we see that it

¹⁷⁰See for example the book "The Analysis of Linear Partial Differential Operators I" by L. Hörmander.

is not so far away from the proof just mentioned, and to some degree it also suggests how to construct the ψ_j .

Examples

1. For $n = 1$ we consider the distribution u given by the characteristic function of the interval $(0, \infty)$,

$$\langle u, \varphi \rangle = \int_0^\infty \varphi. \quad (644)$$

This distribution (often called the Heaviside function) is sometimes denoted by H and also θ . The derivative of u is given by

$$\langle u', \varphi \rangle = -\langle u, \varphi' \rangle = \int_0^\infty -\varphi' = \varphi(0) = \langle \delta, \varphi \rangle, \quad (645)$$

where, as usual, δ denotes the Dirac mass at 0. In short,

$$u' = \delta. \quad (646)$$

In spite of its simplicity, this example is important in that it shows a difference between the point-wise notions of the derivative and the distributional derivative. In this case the distributional derivative is a measure.

2. Let us consider the characteristic function u of the half-space $\{x, x_1 > 0\}$ in \mathbb{R}^n . It can be identified with the distribution

$$\langle u, \varphi \rangle = \int_{\{x_1 > 0\}} \varphi. \quad (647)$$

Let δ_Σ be the natural surface measure on the hyperplane $\Sigma = \{x, x_1 = 0\}$. As an exercise show that

$$\partial_1 u = \delta_\Sigma \quad (648)$$

and

$$\partial_j u = 0, \quad j \geq 2. \quad (649)$$

3. Let u be the characteristic function of the set $\{x \in \mathbb{R}^n; x_j \geq 0, j = 1, 2, \dots, n\}$. As an exercise you can show that

$$\partial_1 \partial_2 \dots \partial_n u = \delta. \quad (650)$$

The calculation consist of repeating n times the calculation from example 1.

4. Let us consider the distribution given in \mathbb{R}^3 by the locally integrable function

$$u(x) = -\frac{1}{4\pi|x|} \quad (651)$$

introduced in lecture 2, where we showed that

$$\Delta(u * \varphi) = \varphi \quad \varphi \in \mathcal{D}(\mathbb{R}^n). \quad (652)$$

Show that this is equivalent to

$$\Delta u = \delta. \quad (653)$$

If K is the kernel of the Newton potential in general dimension n (see lecture 11), we have

$$\Delta K = \delta \quad (654)$$

in distributions.

Lecture 46, 1/28/2011

We will continue with some more examples of distributions. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$. In lecture 14 we defined the integral

$$\text{p.v.} \int_{\mathbb{R}} \frac{\varphi(x)}{x} dx. \quad (655)$$

The functional u in $\mathcal{D}'(\mathbb{R}^n)$ defined by

$$\langle u, \varphi \rangle = \text{p.v.} \int_{\mathbb{R}} \frac{\varphi(x)}{x} dx \quad (656)$$

is clearly linear. We show that it in fact defines a distribution, e. i. $u \in \mathcal{D}'(\mathbb{R})$. Note first that $|\langle u, \varphi \rangle| \leq C(a) \|\varphi\|_1$ for φ with $\varphi(0) = 0$ and $\text{supp } \varphi \subset [-a, a]$, where $a > 0$. For general φ write $\varphi = \varphi - \varphi(0)\psi + \varphi(0)\psi$, where $\psi \in \mathcal{D}(\mathbb{R})$ with $\psi = 1$ in a neighborhood $(-h, h)$ of $x = 0$. Note that $\langle u, \psi \rangle = \int_{\mathbb{R} \setminus (-h, h)} \frac{\psi(x)}{x} dx$. Note that when ψ is even, the last integral vanishes and in this case we have for any $\varphi \in \mathcal{D}'(\mathbb{R})$ with $\text{supp } \varphi \cup \text{supp } \psi \subset [-a, a]$

$$|\langle u, \varphi \rangle| \leq C(a) \|\varphi - \varphi(0)\psi\|_1 \leq C(a) \|\varphi\|_1 + C(a) \|\psi\|_1 \|\varphi\|_0. \quad (657)$$

We can also keep ψ “general” not necessarily even,¹⁷¹ and add an additional term $C_\psi \|\varphi\|_0$ to (657).

Another way to establish that $u \in \mathcal{D}'(\mathbb{R})$ is to show that

$$u = \frac{d}{dx} \log |x|, \quad (658)$$

where we differentiate in the sense of distributions. The function $\log |x|$ is locally integrable, and hence its derivative is well-defined as a distribution. We need to check (658), which is an easy calculation left to the reader as an exercise. (Note that the main issue is to show that (658) is valid across 0.)

Heuristically we expect that the distribution u satisfies

$$x u = 1. \quad (659)$$

To make this more precise we first define formally the multiplication

$$\psi v \quad (660)$$

for any $v \in \mathcal{D}(\Omega)$ and any $\psi \in \mathcal{E}(\Omega)$, where $\mathcal{E}(\Omega)$ denote the space of all smooth functions in Ω .¹⁷² The definition of course is

$$\langle \psi v, \varphi \rangle = \langle v, \psi \varphi \rangle, \quad (661)$$

¹⁷¹Still with $\psi \equiv 1$ in a neighborhood of 0, of course.

¹⁷²Note that the functions in $\mathcal{E}(\Omega)$ can become unbounded and oscillatory as we approach the boundary of Ω . This is not dangerous as the functions in $\mathcal{D}'(\Omega)$ have compact support.

and the reader can easily check that with this definition we have $\psi v \in \mathcal{D}'(\Omega)$ for each $v \in \mathcal{D}'(\Omega)$ and each $\psi \in \mathcal{E}(\Omega)$. With these definition the relation (659) is easily verified.

The equation (659) can be thought of as an equation for an unknown $u \in \mathcal{D}'(\mathbb{R})$. When (659) is considered in distributions, it is somewhat more complicated than when it is considered only as an equations for functions, although the difficulties are only at $x = 0$. In general, if we have the equation $\psi u = v$ where $\psi \in \mathcal{E}(\Omega)$, $v \in \mathcal{D}'(\Omega)$ and $u \in \mathcal{D}'(\Omega)$ is the unknown, one can check easily from the definitions that in any open set where ψ does not have zeros we have $u = \psi^{-1}v$. (In particular, it is clear that any solution of (659) is given by $u = 1/x$ away from the origin.) So the equation determines u uniquely in regions where ψ does not have zeroes. This is no longer case across the zeroes of ψ . For example, equation (659) does not determine u uniquely. It determines u only up to the solutions of

$$x v = 0, \quad (662)$$

where $v \in \mathcal{D}'(\mathbb{R})$. To determine the solutions of (662), we first note that

$$\text{supp } v \subset \{0\} \quad (663)$$

for any solution. Therefore

$$v = \sum_{j=0}^m c_j \partial_x^j \delta, \quad (664)$$

and the reader can check that the only distribution of this form satisfying (662) is

$$v = c \delta. \quad (665)$$

We can also check what happens with the equation

$$x^k u = 1, \quad (666)$$

where k is a non-negative integer and the unknown u is a distribution. The solution is again uniquely determined in the set $\{x \neq 0\}$ by $u = x^{-k}$. It is easy to check that the solutions of

$$x^k v = 0 \quad (667)$$

in $\mathcal{D}'(\mathbb{R})$ are exactly the distributions

$$v = \sum_{j=0}^{k-1} c_j \partial_x^j \delta. \quad (668)$$

Therefore the solutions of (666) in $\mathcal{D}'(\mathbb{R})$ are not unique, and are determined only modulo distributions of the form (668).

Does (666) has solutions for general positive integer k ? One can guess that the distribution

$$u_k = \frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}} \left(\text{p.v.} \frac{1}{x} \right) \quad (669)$$

is a solution, and the reader can check this as an exercise.

Let us look in more detail at the distribution u_2 , which is given away from the origin by the function $\frac{1}{x^2}$. First we note that for functions $\varphi \in \mathcal{D}(\mathbb{R})$ with $\varphi(0) = 0$ and $\varphi'(0) = 0$ we have

$$\langle u_2, \varphi \rangle = \int_{\mathbb{R}} \frac{\varphi(x)}{x^2} dx. \quad (670)$$

As above, let $\psi \in \mathcal{D}(\mathbb{R})$ be a function which is $\equiv 1$ in a neighborhood of the origin. We can write a general $\varphi \in \mathcal{D}(\mathbb{R})$ as

$$\varphi(x) = \tilde{\varphi}(x) + \varphi(0)\psi(x) + \varphi'(0)x\psi(x), \quad (671)$$

with

$$\tilde{\varphi}(x) = \varphi(x) - \varphi(0)\psi(x) - \varphi'(0)x\psi(x). \quad (672)$$

The value $\langle u_2, \tilde{\varphi} \rangle$ is well-defined as in (670), so we only need to calculate $\langle u_2, \psi \rangle$ and $\langle u_2, x\psi \rangle$. Using the definitions, we check that

$$\langle u_2, \psi \rangle = \int_{\mathbb{R}} \frac{\psi'(x)}{x} dx, \quad \langle u_2, x\psi \rangle = \text{p.v.} \int_{\mathbb{R}} \frac{\psi(x)}{x} dx. \quad (673)$$

Note that the second integral vanishes if ψ is even, and the first integral tends to zero if we take replace ψ by $\psi_\varepsilon(x) = \psi(\varepsilon x)$ and take $\varepsilon \rightarrow 0$. Therefore, in some sense we can set $\langle u_2, 1 \rangle = 0$ and $\langle u_2, x \rangle = 0$. The evaluation of the integral $\int_{\mathbb{R}} \frac{\varphi}{x^2}$ which formally defines u_2 can then be seen as simply subtracting the infinite value $\int_{\mathbb{R}} \frac{1}{x^2} dx$ and formally taking $\int_{\mathbb{R}} \frac{1}{x} dx = 0$. The distributions u_k can be interpreted in a similar way for any positive integer k .

It is interesting to see what happens when we try to approximate the distributions u_k by the distributions $u_{k,\varepsilon}$ defined by the functions $(x + i\varepsilon)^{-k}$, $\varepsilon \rightarrow 0_+$, i. e.

$$\langle u_{k,\varepsilon}, \varphi \rangle = \int_{\mathbb{R}} \frac{\varphi(x)}{(x + i\varepsilon)^k} dx. \quad (674)$$

For example, one has

$$\lim_{\varepsilon \rightarrow 0_+} u_{1,\varepsilon} = u_1 - i\pi\delta. \quad (675)$$

This formula is related to formula (164) from lecture 14, and the reader can verify it as an exercise. The limits $u_{k,0_+} = \lim_{\varepsilon \rightarrow 0_+} u_{k,\varepsilon}$ are easily seen to satisfy $x^k u_{k,0_+} = 1$, but they differ from u_k , as already seen in (675) for $k = 1$. For general k we can infer from (669)

$$u_{k,0_+} = \frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}} u_{1,0_+} = u_k - i\pi \frac{(-1)^{k-1}}{(k-1)!} \delta^{(k-1)}. \quad (676)$$

Another interesting family of distributions in \mathbb{R} is

$$v_a(x) = \frac{x_+^a}{\Gamma(a+1)}, \quad (677)$$

where a is a complex number, $x_+ = (|x| + x)/2$ and Γ is the standard gamma function of Euler. This family of distributions naturally extends the family

$$v_k(x) = \frac{x_+^k}{k!} \tag{678}$$

defined for positive integers k , which are important due to the identity

$$\frac{d^k}{dx^k} v_k = \delta, \tag{679}$$

which the reader can easily verify. The distributions v_a are transparently well-defined for $a \in \mathbf{C}$ with $\operatorname{Re} a > -1$. However, using the relation

$$\frac{d}{dx} v_a = v_{a-1}, \tag{680}$$

the definition can be extended to all $a \in \mathbf{C}$. The dependence $a \rightarrow v_a$ is analytic in \mathbf{C} , in the sense that $a \rightarrow \langle v_a, \varphi \rangle$ is analytic in a for each $\varphi \in \mathcal{D}(\mathbb{R})$. The reader can check L. Hörmander's book "The Analysis of Linear Partial Differential Operators I", section 3.2, for further details.

In lecture 34 we introduced the Fourier transformation and some of the basic results concerning it, see (479) and (487). The Fourier transform is one of the most effective tools for studying linear PDEs (and their also some of their non-linear perturbations). One reason for this is the following. Recall that for each $y \in \mathbb{R}^n$ we have the translation operator τ_y on $L^2(\mathbb{R}^n)$ given by

$$\tau_y f(x) = f(x - y). \quad (681)$$

Partial derivatives are related to “infinitesimal translations” $\tau_{(\delta y)}$ (where $(\delta y) \in \mathbb{R}^n$ is “infinitesimally small”) by

$$\tau_{(\delta y)} f(x) = -\frac{\partial f}{\partial x_i}(x)(\delta y)_i. \quad (682)$$

The operators τ_y , $y \in \mathbb{R}^n$, are clearly unitary and commute with each other. Therefore, by general principles of the Spectral Theory, they should be simultaneously diagonalizable. The Fourier Transformation accomplishes this diagonalization. In view of (682), it should also diagonalize the operators $f \rightarrow \frac{\partial f}{\partial x_i}$, and this is indeed the case, as one can see from (480).

The integrals

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\xi x} dx. \quad (683)$$

and

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i\xi x} d\xi. \quad (684)$$

are both well-defined in the usual sense of the Lebesgue theory if both f and \hat{f} are in $L^1(\mathbb{R}^n)$. This condition is too restrictive for many applications. One try to circumvent this difficulty by working with integrals which are not absolutely convergent, introducing various regularizations, etc. However, the most effective way to overcome the difficulty seems to be to extend the definition of the Fourier transformation to a certain class of distributions, called “tempered distributions” which will be introduced below. The idea is due to L. Schwartz. The motivation for the definition of \hat{u} for distributions comes from formula (482), which we reproduce here for convenience:

$$\int_{\mathbb{R}^n} f \hat{g} = \int_{\mathbb{R}^n} \hat{f} g. \quad (685)$$

By analogy with (685), we define \hat{u} for a suitable class of distributions by

$$\langle \hat{u}, \varphi \rangle = \langle u, \hat{\varphi} \rangle. \quad (686)$$

If the map $\varphi \rightarrow \hat{\varphi}$ was continuous from $\mathcal{D}(\mathbb{R}^n)$ to $\mathcal{D}(\mathbb{R}^n)$, formula (686) would define \hat{u} as a distribution. However, we note that for $\varphi \in \mathcal{D}(\mathbb{R}^n)$ the function $\hat{\varphi}$ is never in $\mathcal{D}(\mathbb{R}^n)$, unless $\varphi \equiv 0$. This can be seen from the fact that for

any $f \in L^1(\mathbb{R}^n)$ which is compactly supported, the function \hat{f} is analytic. The analyticity is clear from (683), as the formula can be clearly used not only for $\xi \in \mathbb{R}^n$, but also for $\zeta \in \mathbb{C}^n$, and the resulting $\hat{f}(\zeta)$ satisfies the Cauchy-Riemann conditions in \mathbb{C}^n . We now recall that an analytic function in \mathbb{R}^n cannot vanish on any open subset of \mathbb{R}^n unless it vanishes identically.

The function $\hat{\varphi}$ is still quite nice at ∞ : any derivative of $\partial_\xi^\alpha \hat{\varphi}(\xi)$ approaches zero faster than any inverse power of $|\xi|$ as $\xi \rightarrow \infty$. This can be seen from formulae (480) and (481). This motivates the definition of the *Schwartz class*

$$\mathcal{S}(\mathbb{R}^n) = \{ \varphi: \mathbb{R}^n \rightarrow \mathbb{R}; \varphi \text{ is smooth and } |x|^m |\partial^\alpha \varphi(x)| \rightarrow 0 \text{ as } x \rightarrow \infty \text{ for any } m \geq 0 \text{ and any } \alpha \}. \quad (687)$$

Note in particular that there is no condition on the support of φ . The space $\mathcal{S}(\mathbb{R}^n)$ can be equipped with a natural countable family of norms which turns it into a Fréchet space:

$$\|\varphi\|_k = \sup_{x \in \mathbb{R}^n, |\alpha| \leq 2k} (1 + |x|^2)^k |\partial^\alpha \varphi(x)| \quad (688)$$

As an exercise the reader can prove that the space $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$.¹⁷³ Using (683), (684), (480), and (481), it is not hard to show that the Fourier transformation $\varphi \rightarrow \hat{\varphi}$ is a bijection of $\mathcal{S}(\mathbb{R}^n)$, continuous in both directions. In other words, the map $\varphi \rightarrow \hat{\varphi}$ is an isomorphism of the Fréchet space $\mathcal{S}(\mathbb{R}^n)$. We note that $\varphi_j \rightarrow \varphi$ in $\mathcal{D}(\mathbb{R}^n)$ can be easily seen to imply $\varphi_j \rightarrow \varphi$ in $\mathcal{S}(\mathbb{R}^n)$, while the opposite implication fails.

We will denote by $\mathcal{S}'(\mathbb{R}^n)$ the space of linear continuous functionals on $\mathcal{S}(\mathbb{R}^n)$. By the previous remark we see that $\mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$. The distributions in $\mathcal{S}'(\mathbb{R}^n)$ are called *tempered distributions*.

Examples

Consider $u_1, u_2, u_3 \in \mathcal{D}'(\mathbb{R})$ defined by

$$u_1 = \sum_k 2^k \delta_k, \quad u_2 = \sum_k \partial^k \delta_k, \quad u_3 = \sum_k \delta_{1+1/2+\dots+1/k} \quad (689)$$

As an exercise, you can show that all u_j are well defined distributions in $\mathcal{D}'(\mathbb{R}^n)$, but they do not belong to $\mathcal{S}'(\mathbb{R}^n)$. On the other hand, every function u which is locally integrable and has at most polynomial growth defines an element of $\mathcal{S}'(\mathbb{R}^n)$ by the usual integration

$$\langle u, \varphi \rangle = \int_{\mathbb{R}^n} u \varphi. \quad (690)$$

The formula (686) can now be used to define Fourier transformation for each $u \in \mathcal{S}'(\mathbb{R}^n)$. As we shall see, this definition removes most of the difficulties related to the non-convergent integrals one has to face when dealing with the “point-wise” formulae (683) and (684).

¹⁷³In other words, for each $\varphi \in \mathcal{S}(\mathbb{R}^n)$ there exists a sequence $\varphi_j \in \mathcal{D}(\mathbb{R}^n)$ such that $\|\varphi_j - \varphi\|_k \rightarrow 0$ for each k when $j \rightarrow \infty$.

We start with a few elementary examples illustrating some of the features of the Fourier transformation.

Example 1.

Consider $a > 0$ and let $u = u_a = \chi_{(-a,a)}$ be the characteristic function of the interval $(-a, a)$. The function is in $L^1(\mathbb{R})$, and therefore we can calculate the Fourier transform directly:

$$\hat{u}(\xi) = \int_{-a}^a e^{-ix\xi} dx = \frac{2 \sin a\xi}{\xi}. \quad (691)$$

We note the following simple facts:

(i) The function $\hat{u}(\xi)$ is smooth and, in fact, it is an analytic function of ξ , defined globally for $\xi \in \mathbf{C}$. We can see that already from the fact that u has compact support. The integral (691) transparently defines a holomorphic function of $\xi \in \mathbf{C}$.

(ii) The decay of $\hat{u}(\xi)$ as $\xi \rightarrow \infty$ is relatively slow. In particular, \hat{u} is not in $L^1(\mathbb{R})$. This can also be inferred from general principles, without calculating \hat{u} explicitly: if \hat{u} were in $L^1(\mathbb{R})$, the inversion formula (684) would imply that u is continuous, which is not the case. One can also see without calculation that a jump discontinuity should produce some $O(1/|\xi|)$ behavior of the Fourier transform at infinity: the decay of $\hat{u}(\xi)$ as $\xi \rightarrow \infty$ is due to the cancellations in the integral (691), resulting from the oscillations of $e^{ix\xi}$. However, with the jump discontinuity present, the “remainder” of the cancellation process can be of the order of the period of the function $e^{ix\xi}$, which is of order $1/\xi$.

(iii) The function \hat{u} is in $L^2(\mathbb{R})$. This can again be seen without calculation: it is a consequence of the obvious fact that $u \in L^2(\mathbb{R})$ and Plancherel’s equality (493).

(iv) In the limit $a \rightarrow \infty$ the distributions u_a converge to a distribution given by the constant function 1, and hence \hat{u}_a must converge to $\hat{1}$ in distributions. We will determine $\hat{1}$ momentarily, however we can see from (691) that $\hat{u}_a \rightarrow 0$ in distributions in any open set not containing 0. Hence we expect that $\hat{1}$ will be supported at 0.

(v) The functions $\frac{u_a}{2a}$ converge to δ in distributions as $a \rightarrow 0$. The functions $\frac{\hat{u}_a}{2a}$ converge locally uniformly to 1 as $a \rightarrow 0$. Therefore we have $\hat{\delta} = 1$, which can also be seen directly from (686).

(vi) The inversion formula formally is

$$u_a(x) = \int_{\mathbb{R}} \frac{2 \sin a\xi}{\xi} e^{ix\xi} d\xi. \quad (692)$$

If we do not wish to resort to (686) in interpreting this formula, we can “regularize” the integral in (692) (which is not of course absolutely convergent) in various ways. For example, we can try to calculate

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} \frac{2 \sin a\xi}{\xi} e^{ix\xi} e^{-\frac{|\varepsilon\xi|^2}{2}} d\xi. \quad (693)$$

As $\varepsilon \rightarrow 0$, the distributions given by the integrand of in (693) clearly converge to the distribution given by the integrand of (692), so the limit in (693) should exist in some sense. In fact, the integrands also converge in $L^2(\mathbb{R})$ and hence their Fourier transforms must also converge in $L^2(\mathbb{R})$, by (491). In general, from the convergence in distributions (or even the L^2 convergence) one cannot really infer the existence of the “point-wise” limit (693), but in our example the point-wise limit happens to exist. The integrals can be in fact calculated explicitly in various ways, for example by using formulae (483), (484), and (490). We set

$$g_\varepsilon(x) = \frac{1}{(2\pi\varepsilon^2)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2\varepsilon^2}} \quad (694)$$

and use (483), (484) to infer

$$\hat{g}_\varepsilon(\xi) = e^{-\frac{|\varepsilon\xi|^2}{2}}. \quad (695)$$

By $f \hat{*} g = \hat{f} \hat{g}$, we see that $\frac{2 \sin a\xi}{\xi} e^{-\frac{|\varepsilon\xi|^2}{2}}$ is the Fourier transform of $u_a * g_\varepsilon$, and hence

$$\int_{\mathbb{R}} \frac{2 \sin a\xi}{\xi} e^{ix\xi} e^{-\frac{|\varepsilon\xi|^2}{2}} d\xi = (u_a * g_\varepsilon)(x). \quad (696)$$

From this we see that the limit (693) exists point-wise (for each x), and is equal to $u_a(x)$ except at $x = \pm a$, where it is equal to $1/2$.

Instead of (693), we could use the regularization

$$\int_{-L}^L \frac{2 \sin a\xi}{\xi} e^{ix\xi} d\xi \quad (697)$$

and let $L \rightarrow \infty$. Integral (697) can also be evaluated explicitly, by a similar method as above. We write it as

$$\int_{\mathbb{R}} \frac{2 \sin a\xi}{\xi} e^{ix\xi} \chi_{(-L,L)}(\xi) d\xi \quad (698)$$

and note that $\chi_{(-L,L)}$ is the Fourier transformation of

$$g_L(x) = \frac{1}{2\pi} \frac{2 \sin Lx}{x}, \quad (699)$$

thanks to inversion formula (692). Therefore, formally at least, the function $\frac{2 \sin a\xi}{\xi} \chi_{(-L,L)}(\xi)$ is the Fourier transformation of the function $u_a * g_L$, which is also an approximation of u_a , but of a different nature: it oscillates¹⁷⁴ much more and has slow decay at infinity. At the same time, by (481) the functions $u_a * g_L$ must converge to u_a in L^2 as $L \rightarrow \infty$, and in fact the convergence is point-wise (with limit $1/2$ at $x = \pm a$). The slow decay at ∞ is unavoidable, due to the discontinuities of $\frac{2 \sin a\xi}{\xi} \chi_{(-L,L)}(\xi)$.¹⁷⁵

¹⁷⁴The oscillations will be most pronounced at the points of discontinuity of u_a , where their appearance is often referred to as the Gibbs phenomenon.

¹⁷⁵Note that for some exceptional values of L the function will be continuous, but for a typical L it will be discontinuous at $x = \pm L$.

As you could notice, in the calculations above we benefited from knowing what the result should be. In situations where we do not know what to expect the calculations may not be so easy, but the above examples still give a good idea about certain effects which we may encounter.

Example 2

The function $u(x) = 1$ represents a tempered distribution. The direct calculation of \hat{u} from definition (686) is straightforward:

$$\langle \hat{u}, \varphi \rangle = \langle u, \hat{\varphi} \rangle = \int_{\mathbb{R}} \hat{\varphi} = (2\pi)^n \varphi(0) \quad (700)$$

where we have used the inversion formula (684). One can arrive at the same result from approximating 1 by $e^{-\frac{|\varepsilon x|^2}{2}}$ and using (483), (484):

$$\hat{1} = \lim_{\varepsilon \rightarrow 0} \widehat{e^{-\frac{|\varepsilon x|^2}{2}}} = \lim_{\varepsilon \rightarrow 0} (2\pi)^n \frac{1}{(2\pi\varepsilon^2)^{n/2}} e^{-\frac{|x|^2}{2\varepsilon^2}} = (2\pi)^n \delta \quad (701)$$

It is easy to check that formulae (480) and (481) remain true for distributions. (The multiplication of a tempered distribution u by a function ψ is defined as expected: $\langle \psi u, \varphi \rangle = \langle u, \psi \varphi \rangle$. The function ψ has to satisfy some assumptions so that the map $\varphi \rightarrow \psi \varphi$ is a continuous map from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$. The assumptions are trivially satisfied by any polynomial.) Hence we have

$$\widehat{-ix_j} = (2\pi)^n \frac{\partial}{\partial \xi_j} \delta \quad (702)$$

and hence also

$$\widehat{P(-ix)} = (2\pi)^n P\left(\frac{\partial}{\partial \xi}\right) \delta \quad (703)$$

for any polynomial P .

Similarly, we can take derivatives of the equation

$$\hat{\delta} = 1 \quad (704)$$

to obtain

$$P\left(\frac{\partial}{\partial x}\right) \delta = P(i\xi) \quad (705)$$

Example 3

It is also interesting to take derivatives in the formula

$$\widehat{e^{-\frac{|x|^2}{2}}} = (2\pi)^{\frac{n}{2}} e^{-\frac{|\xi|^2}{2}}. \quad (706)$$

If we take derivatives of order m on the left-hand side, we get a function of the form

$$P(x) e^{-\frac{|x|^2}{2}}. \quad (707)$$

From (480) we see that Fourier transform of such function is of the same form. Therefore functions of the form (707) with degree of P at most m form a subspace invariant under the map $\varphi \rightarrow \hat{\varphi}$

If we use formula (686) with $\varphi = g_\varepsilon$ defined by (694) and assume that $u \in L^1(\mathbb{R})$, we can let $\varepsilon \rightarrow 0$ to recover the classical definition (683) for $\xi = 0$. The general ξ is then obtained by suitably using the shift operators. We leave the details to the reader as an exercise.

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We continue with some more examples.

Example

Let $u = \chi_{[0, \infty)}$ be the Heaviside function from lecture 45, see (644). The Fourier transform of u can be calculated for example as follows: we approximate u by

$$u_\varepsilon(x) = \chi_{[0, \infty)} e^{-\varepsilon x}, \quad \varepsilon \rightarrow 0_+. \quad (708)$$

The distributions u_ε clearly converge to u in $\mathcal{S}'(\mathbb{R})$ as $\varepsilon \rightarrow 0_+$.

We calculate

$$\hat{u}_\varepsilon(\xi) = \int_0^\infty e^{-ix\xi - \varepsilon x} = \frac{1}{i\xi + \varepsilon} \rightarrow -i \text{ p.v. } \frac{1}{\xi} + \pi\delta, \quad \varepsilon \rightarrow 0_+, \quad (709)$$

where we have used (675).

Using the last calculation together with (701) we see that

$$\widehat{u - \frac{1}{2}} = \widehat{\frac{1}{2} \text{sign } x} = -i \text{ p.v. } \frac{1}{\xi}. \quad (710)$$

The last identity can be used to calculate the Fourier transform of $\text{p.v. } \frac{1}{x}$. For that it is useful to introduce notation for the inverse Fourier transform. For $f \in L^1(\mathbb{R})$ we let

$$\check{f}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(\xi) e^{ix\xi} d\xi. \quad (711)$$

For (tempered) distributions we set

$$\langle \check{v}, \varphi \rangle = \langle v, \check{\varphi} \rangle. \quad (712)$$

It is clear that for functions we have

$$\hat{f}(\xi) = (2\pi)^n \check{f}(-\xi). \quad (713)$$

The same formula applied to distributions, with the understanding that we define, with a slight abuse of notation,

$$\langle v(-x), \varphi \rangle = \langle v, \varphi(-x) \rangle, \quad (714)$$

A more formal notation would be to define the operator $v \rightarrow v \circ (-I)$ of $\mathcal{S}'(\mathbb{R}^n)$ by $\langle v \circ (-I), \varphi \rangle = \langle v, \varphi \circ (-I) \rangle$, with $-I$ denoting the map $x \rightarrow -x$. As we have

$$\check{\check{v}} = v, \quad (715)$$

we see from (714) that

$$\hat{\hat{v}} = (2\pi)^n v(-x). \quad (716)$$

Applying this to (710), we get

$$\widehat{\text{p.v.} \frac{1}{x}} = -\pi i \text{sign } \xi. \quad (717)$$

Letting

$$K(x) = \frac{1}{\pi x}, \quad (718)$$

we see that

$$\widehat{K} = -i \text{sign } \xi. \quad (719)$$

This formula is important in connection with the Hilbert transformation we discussed in lecture 14, see (163). Formula (163) can be rewritten as

$$Hg = K * g. \quad (720)$$

Recalling the formula $\widehat{f * g} = \hat{f}\hat{g}$, we see that

$$\widehat{Hg}(\xi) = -i(\text{sign } \xi) \hat{g}(\xi). \quad (721)$$

We see that the Fourier transformation “diagonalizes” the Hilbert transformation (and, in fact, any other convolution operator $f \rightarrow K * f$). It is also transparent from (721) and the Plancherel formula (493) that the Hilbert transform is an L^2 isometry.¹⁷⁶

We now give examples of how the Fourier transform can be used for solving PDEs. We start with a simple 1d example. Assume we wish to find a solution of the equation

$$\frac{d^k}{dx^k} u = \delta, \quad (722)$$

where δ is, as usual, the Dirac mass at $x = 0$. In other words, we wish to determine a fundamental solution of the operator d^k/dx^k . We have already come across a solution of (??)15 in lecture 46, see (679), but this time we will look at things from the Fourier transformation angle.

After taking the Fourier transformation of (722), we obtain

$$(i\xi)^k \hat{u} = 1. \quad (723)$$

As we have seen in lecture 16, equation (723) does not determine u uniquely. This is to be expected, as the same is true for (722): it only determines u up to a polynomial of order $k - 1$. We choose a solution by setting

$$\hat{u}_+ = \lim_{\varepsilon \rightarrow 0_+} \frac{1}{(i\xi + \varepsilon)^k}. \quad (724)$$

¹⁷⁶A number of properties of the Hilbert transform are made transparent by (721). We encourage the reader to revisit lecture 14 and take a look at some of the formulae there from the Fourier transform perspective.

We have seen in lecture 46 that \hat{u}_+ is a well-defined distribution. We need to calculate its inverse Fourier transform. Formally we have

$$u_+(x) = \frac{1}{2\pi} \lim_{\varepsilon \rightarrow 0_+} \int_{\mathbb{R}} \frac{e^{i\xi x}}{(i\xi + \varepsilon)^k} d\xi. \quad (725)$$

The last limit can be easily evaluated via the residue theorem. For simplicity we will assume that $k \geq 2$. (The case $k = 1$ is in fact covered by our first example today.) Let us first assume that $x > 0$. The integral in (725) is well-defined for any $\varepsilon > 0$. We note the the integrand decays exponentially as we move ξ to complex numbers with large positive imaginary part. (Here the assumption $x > 0$ is crucial.) The integrand has a single pole in the upper half-plane, at $\xi = i\varepsilon$. By standard application of the residue theorem we see that the integral is equal to the $2\pi i$ multiple of the residue of the integrand at $\xi = i\varepsilon$. In the limit of $\varepsilon \rightarrow 0_+$, we obtain the $2\pi i$ multiple of the residue of the function $e^{i\xi x}/(i\xi)^k$ at $\xi = 0$. Evaluating the residue, we obtain

$$u_+(x) = \frac{x^{k-1}}{(k-1)!}, \quad x > 0. \quad (726)$$

For $x < 0$ the exponential decays exponentially in the lower half-plane, while the residue stays in the upper half-plane and we see from the residue theorem (or the basic Cauchy formula) that $u_+(x) = 0$.

It remains to justify the somewhat formal calculation. We note that the integral in (688) gives a well-defined function $u_\varepsilon(x)$ for each $\varepsilon > 0$. We have shown that for $x \neq 0$ the point-wise limit of these functions is given by $u_+(x)$. We wish to check that in fact $u_\varepsilon \rightarrow u_+$ in tempered distributions.¹⁷⁷ This can be done for example by estimating the residua we get at $\xi = i\varepsilon$ for $\varepsilon > 0$ and using these bounds to get suitable uniform estimates for u_ε .¹⁷⁸

¹⁷⁷We could al also skip this step and check directly that u_+ solves (722).

¹⁷⁸In more detail, the situation is as follows: Let us denote by v_ε the distribution $(i\xi + \varepsilon)^{-k}$ and by v_+ its limit for $\varepsilon \rightarrow 0_+$. The limit exists in $\mathcal{S}'(\mathbb{R})$. This can be easily checked by using (675) and (676). Our distributions u_ε and u_+ are inverse Fourier transforms of v_ε and v_+ , respectively. The inverse Fourier transform is continuous, and therefore $u_\varepsilon \rightarrow u_+$ in $\mathcal{S}'(\mathbb{R})$. We have shown above by a direct calculation that the functions $u_\varepsilon(x)$ converge point-wise (except possibly for $x = 0$, where we have not investigated the limit) to the expression in (726), which we somewhat prematurely identified with u_+ . It looks very likely that the point-wise limit will be the same as the distributional limit, but, strictly speaking, it is not completely automatic. We should justify the identification of the limits. In many books such technicalities are usually left to the reader and only the formal calculation is made. Often the correctness of the formal calculation is justified by the fact that the result of our calculation transparently solves our problem, as it is also the case in our example above.

When we calculate the Fourier transforms, the use of symmetries can simplify the calculations. Let us first look at the orthogonal symmetries. We will work with the group full orthogonal group $O(n)$. For $n \geq 2$ we can replace $O(n)$ with $SO(n)$ in all the results below. For a transformation R in the orthogonal group $O(n)$ and a function $f: \mathbb{R}^n \rightarrow \mathbf{C}$ we let

$$(R \cdot f)(x) = f(R^{-1}x) \quad (727)$$

We can write

$$\widehat{R \cdot f}(\xi) = \int_{\mathbb{R}^n} f(R^{-1}x) e^{-i\xi x} dx = \int_{\mathbb{R}^n} f(x) e^{-i\xi(Rx)} dx = \int_{\mathbb{R}^n} f(x) e^{-i(R^{-1}\xi)x} dx = R \cdot \hat{f}(\xi). \quad (728)$$

If f is invariant under $O(n)$, i. e. $R \cdot f = f$ for each $R \in O(n)$, we see that \hat{f} is also invariant under $O(n)$ and vice versa.

When dealing with distributions, we need to let the symmetries act on the test functions. For $u \in \mathcal{D}'(\mathbb{R}^n)$ we define $R \cdot u$ by

$$\langle R \cdot u, \varphi \rangle = \langle u, R^{-1} \cdot \varphi \rangle. \quad (729)$$

We check easily from the definitions that

$$\widehat{R \cdot u} = R \cdot \hat{u}. \quad (730)$$

In particular, the Fourier transformation of a tempered distribution invariant under $O(n)$ will again be a tempered distribution invariant under $O(n)$.

Scaling symmetries can also provide useful simplifications. We recall that a function $f: \mathbb{R}^n \rightarrow \mathbf{C}$ is a -homogeneous if $f(\lambda x) = \lambda^a f(x)$. We can extend the definition to distributions by setting

$$\langle u(\lambda x), \varphi \rangle = \langle u, \lambda^{-n} \varphi(\lambda^{-1}x) \rangle, \quad (731)$$

where we slightly abuse notation by writing $u(\lambda x)$ not for the point-wise value of u at the point λx , but for the whole scaled distribution. Hopefully such notation will not cause any problems. The distribution $u(\lambda x)$ will also be denoted by $(u)_\lambda$. In this notation the a -homogeneity of u means

$$(u)_\lambda = \lambda^a u. \quad (732)$$

The operation $u \rightarrow (u)_\lambda$ is defined not only on $\mathcal{D}'(\mathbb{R}^n)$, but also on $\mathcal{D}'(\mathbb{R}^n \setminus \{0\})$.

Lemma 1

If $u \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ is invariant under $O(n)$ and a -homogeneous, it is given by a multiple of the function $|x|^a$. In other words,

$$\langle u, \varphi \rangle = \int_{\mathbb{R}^n \setminus \{0\}} c |x|^a \varphi(x) dx, \quad \varphi \in \mathcal{D}(\mathbb{R}^n \setminus \{0\}) \quad (733)$$

for some $c \in \mathbf{C}$.

Proof

The statement is obvious if we already know that u is represented by a smooth function. The only non-trivial step is to show that u is indeed represented by a smooth function. This can be seen for example as follows. For $R \in O(n)$ and $\lambda > 0$ we set

$$R \cdot [\lambda^{-a}(u)_\lambda] = u_{R,\lambda}. \quad (734)$$

By our assumptions

$$u_{R,\lambda} = u \quad R \in O(n), \lambda > 0. \quad (735)$$

Let $\phi = \phi(\lambda)$ be a smooth function on $(0, \infty)$ which is supported in a small neighborhood of $\lambda = 1$ and satisfies

$$\int_0^\infty \phi(\lambda) d\lambda = 1. \quad (736)$$

We can use the notation $v_{R,\lambda}$ also for a general $v \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$, with the same definition. For such a general v consider the distribution

$$\bar{v} = \int_0^\infty \int_{O(n)} v_{R,\lambda} dR \phi(\lambda) d\lambda. \quad (737)$$

Where dR denotes the invariant measure on $O(n)$. In some sense, \bar{v} is obtained from v by averaging over all possible orthogonal transformations and certain scalings.¹⁷⁹ We claim that \bar{v} is represented by a smooth function invariant under $O(n)$. Heuristically this should be more or less clear. For a formal proof we can write

$$\langle \bar{v}, \varphi \rangle = \langle v, \tilde{\varphi} \rangle, \quad (738)$$

where

$$\tilde{\varphi}(x) = \int_0^\infty \int_{O(n)} \lambda^{-a-n} (R \cdot \varphi)(\lambda^{-1}x) dR \phi(\lambda) d\lambda. \quad (739)$$

If we take formally $\varphi = \delta_y$, the Dirac mass at $y \in \mathbb{R}^n \setminus \{0\}$, the function $\tilde{\varphi}$ will be smooth. More precisely, if φ_j are smooth function approximating δ_y , then $\tilde{\varphi}_j$ converge smoothly to a smooth function ψ_y which is invariant under $O(n)$. The distribution \bar{v} is given by the function

$$\bar{v}(y) = \lim_{j \rightarrow \infty} \langle v, \tilde{\varphi}_j \rangle = \langle v, \psi_y \rangle, \quad (740)$$

¹⁷⁹If you do not know much about the invariant measure on $O(n)$, it is enough to look at the averaging over $O(n)$ heuristically, as a natural procedure to obtain invariant objects. In dimensions $n = 1$ and $n = 2$ it is easy to write the averaging explicitly. The integral over λ should be thought of as a convolution with a smooth function (and hence a smoothing operator), except that we are using the multiplication in the group \mathbb{R}_+ instead of the addition in \mathbb{R} . In the usual convolution we average over certain translations of a given function. Here we average over certain scalings of a given function. If we make a substitution $\lambda = e^y$, our new convolution will become the usual convolution. In this substitution the measure dy corresponds to $\frac{d\lambda}{\lambda}$, which is why the integral (737) is often written with $\frac{d\lambda}{\lambda}$ rather than just $d\lambda$. The normalization condition (736) must then be changed accordingly.

and it is not hard to see that this is a smooth function of y . The smoothness of u now follows from the obvious identity

$$\bar{u} = u, \quad (741)$$

which is a consequence of the invariance of u . This finishes the proof of Lemma 1.

Lemma 1 is no longer valid in full generality if we replace $\mathbb{R}^n \setminus \{0\}$ by \mathbb{R}^n . For example, the Dirac distribution δ is $(-n)$ -homogeneous, but is not of the form $c|x|^{-n}$. The derivatives $\partial^\alpha \delta$ are $(-|\alpha| - n)$ -homogeneous.

However, for $a > -n$ the lemma remains valid in \mathbb{R}^n .

Lemma 2 *Any a -homogeneous distribution invariant under $O(n)$ in \mathbb{R}^n with $a > -n$ is given by a multiple of $|x|^a$.*

This follows easily from Lemma 1. If u is a -homogeneous, we can use the lemma to subtract a multiple of $|x|^a$ from u so that $v = u - c|x|^a$ is supported at 0. Since the only distributions supported at $\{0\}$ are the finite linear combinations of the Dirac mass and its derivatives and $a > -n$, any a -homogeneous distribution supported at $\{0\}$ must vanish.

Lemma 3

Let $u \in \mathcal{S}'(\mathbb{R}^n)$ be a -homogeneous. Then \hat{u} is $(-n - a)$ -homogeneous.

Proof

$$\begin{aligned} \langle \hat{u}(\lambda\xi), \varphi \rangle &= \langle \hat{u}(\xi), \lambda^{-n} \varphi(\lambda^{-1}\xi) \rangle = \langle u, \widehat{\lambda^{-n} \varphi(\lambda^{-1}\xi)} \rangle = \langle u(x), (\hat{\varphi})(\lambda x) \rangle = \\ &= \langle \lambda^{-n} u(\lambda^{-1}x), \hat{\varphi} \rangle = \lambda^{-n-a} \langle u, \hat{\varphi} \rangle = \lambda^{-n-a} \langle \hat{u}, \varphi \rangle. \end{aligned} \quad (742)$$

A simple consequence of Lemmas 1,2,3 is the following statement

Let $a \in (0, n)$. Then

$$\widehat{|x|^{-a}} = c(n, a) |\xi|^{-n+a}. \quad (743)$$

The constant $c(n, a)$ can be evaluated in terms of Euler's Gamma function from the formula

$$\langle \hat{u}, \varphi \rangle = \langle u, \hat{\varphi} \rangle \quad (744)$$

applied with $u = |x|^{-a}$ and $\varphi = e^{-\frac{|x|^2}{2}}$. Recalling that

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx, \quad (745)$$

we can write

$$\langle \hat{u}, \varphi \rangle = c(n, a) \int_{\mathbb{R}^n} |\xi|^{-n+a} e^{-\frac{|\xi|^2}{2}} d\xi = c(n, a) \int_0^\infty |S^{n-1}| r^{-n+a} e^{-\frac{r^2}{2}} r^{n-1} dr, \quad (746)$$

where $|S^{n-1}|$ represents the $(n-1)$ -dimensional volume of the sphere S^{n-1} . Using the substitution $\frac{r^2}{2} = t$, we see that

$$\langle \hat{u}, \varphi \rangle = c(n, a) |S^{n-1}| \int_0^\infty (2t)^{\frac{a-2}{2}} e^{-t} dt = c(n, a) |S^{n-1}| 2^{\frac{a-2}{2}} \Gamma\left(\frac{a}{2}\right). \quad (747)$$

Recalling that for our choice of φ we have $\hat{\varphi} = (2\pi)^{\frac{n}{2}} \varphi$, we obtain by a similar calculation

$$\langle u, \hat{\varphi} \rangle = |S^{n-1}| 2^{\frac{n-a-2}{2}} \Gamma\left(\frac{n-a}{2}\right). \quad (748)$$

We see that

$$c(n, a) = \pi^{\frac{n}{2}} 2^{n-a} \frac{\Gamma\left(\frac{n-a}{2}\right)}{\Gamma\left(\frac{a}{2}\right)}. \quad (749)$$

We can use the above formulae to calculate the fundamental solution of the laplacian Δ . We will consider the equation in the form

$$-\Delta u = \delta \quad (750)$$

We seek u in tempered distributions. After taking the Fourier transformation, we obtain

$$|\xi|^2 \hat{u}(\xi) = 1. \quad (751)$$

For $n \geq 3$ this equation has an obvious solution

$$\hat{u}(\xi) = \frac{1}{|\xi|^2} \quad (752)$$

Applying the inverse Fourier transform, (743) and (748), we obtain

$$u(x) = \frac{c(n, 2)}{(2\pi)^n |x|^{n-2}} = \frac{\Gamma\left(\frac{n-2}{2}\right)}{4\pi^{\frac{n}{2}} |x|^{n-2}} \quad (753)$$

In lecture 11 we saw that

$$u(x) = \frac{1}{(n-2) |S^{n-1}| |x|^{n-2}}. \quad (754)$$

Comparing the two expressions, keeping in mind that $s\Gamma(s) = \Gamma(s+1)$, we recover a well-known formula

$$|S^{n-1}| = \frac{2\pi^{\frac{n}{2}}}{\left(\frac{n-2}{2}\right) \Gamma\left(\frac{n-2}{2}\right)} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}, \quad (755)$$

which can of course be derived more directly.

We note that the solutions of (751) are not unique, as the equation

$$|\xi|^2 \hat{u} = 0 \quad (756)$$

has non-trivial solutions in $\mathcal{S}'(\mathbb{R}^n)$. For example, the Dirac mass δ and its first derivatives are the obvious solutions, but there are many more solutions (all of them supported at $\xi = 0$, of course), reflecting the existence of many harmonic polynomials. Since all solutions of (756) are clearly supported at $\xi = 0$, we can use the last proposition in lecture 42, see (608), to prove the following version of the Liouville theorem.

Any tempered distribution u in \mathbb{R}^n satisfying $\Delta u = 0$ is a harmonic polynomial.

The proof is easy: the Fourier transform \hat{u} of u must be supported at the origin by (756). Hence \hat{u} is a finite linear combination of the Dirac mass at $\xi = 0$ and its derivatives. This means that u is a polynomial.

The above calculations do not immediately give the solution of (751) in dimension $n = 2$. This can be calculated for example as follows. We assume $n = 2$. For $\varepsilon > 0$ we let

$$v_\varepsilon(\xi) = \frac{1}{|\xi|^{2-\varepsilon}} \quad (757)$$

and

$$u_\varepsilon(x) = \check{v}_\varepsilon(x) = \frac{2^\varepsilon}{\Gamma(1-\varepsilon/2)} \frac{\Gamma(\frac{\varepsilon}{2})}{4\pi|x|^\varepsilon}. \quad (758)$$

This does not have a good limit when $\varepsilon \rightarrow 0$, but we note that for the calculation of the fundamental solution we can change u_ε be a constant, or – equivalently – we can add a suitable multiple of the Dirac mass at $\xi = 0$ to v_ε . The Fourier transformation V_ε of

$$U_\varepsilon(x) = \frac{2^\varepsilon}{\Gamma(1-\varepsilon/2)} \frac{\Gamma(\frac{\varepsilon}{2})}{4\pi} (|x|^{-\varepsilon} - 1) \quad (759)$$

still solves the equation

$$|\xi|^{2-\varepsilon} V_\varepsilon = 1. \quad (760)$$

The Γ function has a pole at $s = 0$ with residue 1, and hence we have

$$U_\varepsilon(x) \sim \frac{1}{2\pi} \frac{|x|^{-\varepsilon} - 1}{\varepsilon}, \quad \varepsilon \rightarrow 0_+. \quad (761)$$

Taking the limit as $\varepsilon \rightarrow 0_+$, we obtain a solution of (750) for $n = 2$ in the form

$$u(x) = -\frac{1}{2\pi} \log|x|, \quad (762)$$

which we have of course seen before.

The interesting detail to note in the above calculation is that the limit of the distributions V_ε for $\varepsilon \rightarrow 0_+$, let us call it V solves the equation $|\xi|^2 V(\xi) = 1$, but is not -2 -homogeneous. In fact, the equation does not have a -2 -homogeneous solution in tempered distributions. We refer the reader to the

book “The Analysis of Linear Partial Differential Operators I” by L. Hörmander, Chapter 3.2, for a general discussion of related topics.

There is another way to overcome the complications with solving (751) in dimension $n = 2$. We know that the derivatives of u should satisfy

$$\widehat{\partial_j u} = \frac{i\xi_j}{|\xi|^2}, \quad (763)$$

which is of course a locally integrable function, and its inverse Fourier transform is also a -1 -homogeneous function. We can calculate it by taking derivatives in the inverse Fourier transform version of (743) for a small positive a and then letting $a \rightarrow 0_+$. We know what the result should be: the derivatives of the function (762). The details are left to the reader.

Today we will do some calculations for the heat equation

$$u_t - \Delta u = 0. \quad (764)$$

The unknown function u is a function of $x = (x_1, \dots, x_n) \in \Omega \subset \mathbb{R}^n$ and $t \in (t_1, t_2)$. Typically also needs some boundary conditions, which will be discussed later.

The equation was first studied by J. Fourier around 1812. It models the propagation of heat in a homogeneous body (occupying the region Ω). It is perhaps useful to recall a simple derivation of the equation. The quantity $u(x, t)$ can be thought of as the temperature of the body at site x and time t .

The amount of energy due to heat contained in a domain $\mathcal{O} \subset \Omega$ at time t is $\int_{\mathcal{O}} cu(x, t) dx$, where c is the specific heat per unit volume of the material.¹⁸⁰ We assume that the heat “flows” from the areas of high temperature to the areas of low temperature. Mathematically we model this by introducing the heat flux $q = (q_1, \dots, q_n) = (q_1(x, t), \dots, q_n(x, t))$. If $\mathcal{O} \subset \Omega$ is a domain with smooth boundary, ν the outward unit normal at $\partial\mathcal{O}$, then the quantity $(\int_{\partial\mathcal{O}} q_j(x, t)\nu_j(x) dx) dt$ gives the amount of heat which left \mathcal{O} during the infinitesimal time interval dt . If there are no sources of heat in the body, we clearly must have

$$\frac{d}{dt} \int_{\mathcal{O}} cu(x, t) dx + \int_{\partial\mathcal{O}} q_j(x, t)\nu_j(x) dx = 0 \quad (765)$$

for any (sufficiently smooth) domain $\mathcal{O} \subset \Omega$. Assuming c is independent of t , using the Divergence Theorem¹⁸¹ and the fact that \mathcal{O} can be arbitrary, we obtain

$$cu_t + \operatorname{div} q = 0. \quad (766)$$

To close the system, we need to express q in terms of u . This is done by assuming (following Fourier)

$$q = -\kappa \nabla u, \quad (767)$$

which is known as Fourier’s law. Substituting (767) into (766) we obtain, when κ is constant

$$u_t - \frac{\kappa}{c} \Delta u = 0. \quad (768)$$

Changing the units of time so that $\frac{\kappa t}{c}$ becomes the “new time”, we obtain (764).

To determine $u(x, t)$ from this equation for $x \in \Omega$ and $t \in (t_1, t_2)$, we need to know the field $u(x, t)$ at $t = t_1$ (the initial condition), and the conditions satisfied by u at $\partial\Omega$. These are similar to the elliptic case: we can consider for example the Dirichlet boundary conditions (in which the values of $u(x, t)$ are prescribed

¹⁸⁰In principle c can be considered as depending on x and even on t , but for now we will consider it to be constant.

¹⁸¹ $\int_{\mathcal{O}} \operatorname{div} q = \int_{\partial\mathcal{O}} q \cdot \nu$

for $x \in \partial\Omega$ and $t \in (t_1, t_2)$; the Neumann boundary conditions $\frac{\partial u}{\partial \nu}(x, t) = 0$ for $x \in \Omega$ and $t \in (t_1, t_2)$ (expressing that the boundary is isolated and no heat flows through it); the condition $\frac{\partial u}{\partial \nu} + \gamma u = 0$ for $x \in \partial\Omega$ and $t \in (t_1, t_2)$; we can impose different conditions on different parts of the boundary, etc.

We will often consider the case $\Omega = \mathbb{R}^n$ when no boundary is present, but we have to make some assumptions about the behavior of u at ∞ (e. g. that u is bounded.¹⁸²)

Let us first look at the symmetries of the equation. We have the translational symmetries $u(x, t) \rightarrow u(x - x_0, t - t_0)$ and the rotational symmetries $u(x, t) \rightarrow u(R^{-1}x, t)$, where R is an orthogonal transformation. In addition to these obvious symmetries, we have the scaling symmetry

$$u(x, t) \rightarrow u(\lambda x, \lambda^2 t), \quad (769)$$

where $\lambda > 0$. This symmetry confirms the important practical observation that hot large bodies take a long time to cool.¹⁸³

We now consider the non-homogeneous equation

$$u_t - \Delta u = f(x, t), \quad (770)$$

where f is a given function of x and t . This equation is obtained by adding a source term of the form $\int_{\mathcal{O}} f(x, t) dx$ to the balance law (765). This term models “pumping heat” into our substance at a rate given locally by the function $f(x, t)$. (In areas where $f < 0$ this of course means cooling.)

Let us consider the following situation: we wish to determine u from the equation (770) in $\mathbb{R}^n \times \mathbb{R}$ under the assumption that f is compactly supported in $\mathbb{R}^n \times \mathbb{R}$ and u vanishes for large negative times. Heuristically it seems that in this situation u should be determined by f . This is true if we know that u cannot have a very fast growth as $x \rightarrow \infty$, but it may fail if no additional conditions are imposed on u - we will return to this point later. We will now consider the special case when $f(x, t) = \delta(x, t)$, the Dirac mass at $(x, t) = (0, 0)$. We wish to solve

$$u_t - \Delta u = \delta(x, t) \quad (771)$$

under the assumptions that u vanishes for $t \leq 0$ and behaves “reasonably” at the the spatial infinity for all t . For example, the assumption that u is a tempered distribution is sufficient.¹⁸⁴ The interpretation of u is clear. Assume the whole space is filled with a heat-conducting material. We observe it over the time interval $(-\infty, \infty)$. Nothing happens for $t < 0$, the temperature is identically zero. Then at time $t = 0$ suddenly a unit amount of heat is injected at the origin. After that the system is left of its own, and the solution describes the propagation of the injected heat in the material.

¹⁸²In fact, a much weaker assumption is sufficient.

¹⁸³If $u(x, t)$ describes the cooling process of a body Ω , than a similar cooling process for $\lambda\Omega$ is described by $u(x/\lambda, t/\lambda^2)$. If the cooling took T time units for body Ω , it will take $\lambda^2 T$ time units for body $\lambda\Omega$.

¹⁸⁴It can be relaxed, but that is our concern at the moment.

We will apply the Fourier transformation in the space-time (in both x and t). The “Fourier variable” dual to x will be denoted by ξ and the Fourier variable dual to t will be denoted by τ . We obtain

$$(i\tau + |\xi|^2)\hat{u} = 1 \quad (772)$$

Formally this gives

$$\hat{u}(\xi, \tau) = \frac{1}{i\tau + |\xi|^2}. \quad (773)$$

We have seen that one has to be careful when interpreting such expressions, but in this case there are no difficulties, as the function given by (772) is locally integrable, and decays to 0 as $(\xi, \tau) \rightarrow \infty$. Therefore $\hat{u}(\xi, \tau)$ is a well defined tempered distribution solving (772). It is clear that any other solution of (772) can differ from (773) only by a distribution supported at $(\xi, \tau) = (0, 0)$. This means that the tempered distribution solutions of (771) is uniquely determined modulo polynomials solving the homogeneous heat equation. Such polynomials are called “heat polynomials” or “caloric polynomials”. Their theory is quite analogous to the theory of harmonic polynomials. If we add the requirement that the solution vanishes for $t < 0$, we see that the solution is unique (as any polynomial vanishing on an open set must vanish identically). the Fourier inversion of $\hat{u}(\xi, \tau)$ given by (773) can be formally written as

$$u(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{i\xi x + i\tau t}}{i\tau + |\xi|^2} d\tau d\xi. \quad (774)$$

The integral is not absolutely convergent, which does not surprise us, since we do not expect u to be continuous at the origin $(x, t) = (0, 0)$. The integral over τ can be evaluated by the residue theorem, but we do not really need to do the calculation, as we see from (709) that the inverse Fourier transform of $\frac{1}{i\tau + |\xi|^2}$ in τ is $\chi_{(0, \infty)}(t)e^{-t|\xi|^2}$. We have already seen (see e. g. (483), (484) what the inverse Fourier transform in ξ of this function is. We obtain

$$u(x, t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & t > 0 \\ 0 & t < 0 \end{cases} \quad (775)$$

Although our calculation was completely rigorous (we used Fubini’s theorem for a function which is not in L^1 , and had implicitly chosen a certain regularization of the integral over τ which is not absolutely convergent), one can verify that $u(x, t)$ is the required solution in several ways. For example, we note that u satisfies the heat equation away from $(x, t) = (0, 0)$ and that the functions $x \rightarrow u(x, t)$ converge to $\delta(x)$ as $t \rightarrow 0_+$. Hence for any smooth, compactly supported $\varphi = \varphi(x, t)$ we have

$$\int \int u(x, t)(-\varphi_t - \Delta\varphi) dx dt = \lim_{t_1 \rightarrow 0_+} \int_{\mathbb{R}^n} u(x, t_1)\varphi(x, t_1) dx + \int_{t_1}^{\infty} \int (u_t - \Delta u)\varphi dt dx. \quad (776)$$

The last integral on the right-hand side vanishes as $u_t - \Delta u = 0$ for $t > t_1$, and the first integral on the right-hand side converges to $\varphi(0, 0)$ as $t_1 \rightarrow 0_+$.

If we prefer to do the justification in the Fourier picture, we can verify that u given by (775) satisfies (773) for example as follows. We note that for $\varepsilon > 0$ the function $u_\varepsilon(x, t) = u(x, t)e^{-\varepsilon t}$ is in $L^1(\mathbb{R}^n)$. We can calculate its Fourier transform directly by integrating first over x and then over t . We obtain

$$\hat{u}_\varepsilon(\xi, \tau) = \frac{1}{i\tau + |\xi|^2 + \varepsilon}. \quad (777)$$

This is not surprising, as u_ε satisfies

$$u_{\varepsilon t} - \Delta u_\varepsilon + \varepsilon u_\varepsilon = 0. \quad (778)$$

It is easy to see that $u_\varepsilon \rightarrow u$ in $\mathcal{S}'(\mathbb{R}^n)$ and that \hat{u}_ε converges to $\frac{1}{i\tau + |\xi|^2}$ in $\mathcal{S}'(\mathbb{R}^n)$ as $\varepsilon \rightarrow 0_+$. This shows that the Fourier transform of the tempered distribution u given by (775) is given by the formula (773), which means that the Fourier inversion of (773) must be (775), as our formal calculation suggested.

We note that the behavior of $u(x, t)$ is largely what one would expect it to be from our intuitive picture of the heat conduction, except for the fact that the disturbance at $(0, 0)$ has an effect everywhere in x for any $t > 0$. The effect is very, very small for large x and small t , but it is not zero, and we have to conclude that in this model the disturbances propagate infinitely fast.

Another interesting feature of $u(x, t)$ is that it is smooth everywhere except at $(0, 0)$, but it is not analytic in t at any point $(x, 0)$.

Finally, we note that if we replace (764) by

$$u_t + \Delta u = 0, \quad (779)$$

which is the *backward heat equation*, obtained by reversing the direction of the heat flux q , the problem

$$u_t + \Delta u = \delta(x, t) \quad (780)$$

will not have any solution in tempered distributions vanishing for $t < 0$, while it will have the solution $-u(x, -t)$ (with u given by (775)) which vanishes for $t > 0$. Similarly, problem (771) does not have any solutions in tempered distributions which vanish for $t > 0$. We will return to this when we discuss issues related to well-posedness.

The fundamental solution of the heat equations we calculated last time is often denoted by $\Gamma(x, t)$. We recall that

$$\Gamma(x, t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & t > 0, \\ 0 & t < 0. \end{cases} \quad (781)$$

It is worth pointing out some similarities of this solution with the fundamental solution of the Laplace equation, we studied in the last semester, see e. g. (11). (There are of course also significant differences, but at the moment we focus on the similarities.) To highlight the similarities, it is useful to use *parabolic scaling*. We already saw the parabolic scaling in the last lecture:

$$u(x, t) \rightarrow u(\lambda x, \lambda^2 t). \quad (782)$$

With the scaling $(x, t) \rightarrow (\lambda x, \lambda^2 t) = (x', t')$ the volume element $dV = dx_1 \dots dx_n dt$ scales as

$$dV' = dx'_1 \dots dx'_n dt' = \lambda^{n+2} dV. \quad (783)$$

Based on this scaling we can also consider the parabolic mollification: if $\phi = \phi(x, t)$ is a smooth compactly supported function in $\mathbb{R}^n \times \mathbb{R}$ with $\int \phi(x, t) dx dt = 1$ and $\varepsilon > 0$, we can let

$$\phi_\varepsilon(x, t) = \frac{1}{\varepsilon^{n+2}} \phi(\lambda x, \lambda^2 t). \quad (784)$$

If $f: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbf{C}$ is a continuous function, then obviously $\phi_\varepsilon * f \rightarrow f$ uniformly on compact sets. Also, if $f \in L^p(\mathbb{R}^n \times \mathbb{R})$, then $\phi_\varepsilon * f \rightarrow f$ in L^p as $\varepsilon \rightarrow 0$.¹⁸⁵ If f is locally integrable in $\mathbb{R}^n \times \mathbb{R}$, then $(\phi_\varepsilon * f)(x, t) \rightarrow f(x, t)$ for almost every (x, t) as $\varepsilon \rightarrow 0$.

Because of the scalings (782) and (783), it is sometimes useful to think about the underlying space (x, t) of the heat equation as a space of dimension $(n+2)$, although of course the topological dimension of the space is still $(n+1)$.¹⁸⁶

In what follows we will write $z = (x, t)$ for space-time points, and we will also write for $z = (x, t) \in \mathbb{R}^n \times [0, \infty)$

$$|z|_{\text{par}} = \sqrt{|x|^2 + t}. \quad (785)$$

Let $m = n + 2$ be the parabolic “dimension” of the variable z . Then we have the following:

¹⁸⁵Here and in what follows we assume $\varepsilon > 0$. We should perhaps write $\varepsilon \rightarrow 0_+$, but we’ll just write $\varepsilon \rightarrow 0$, there is no danger of confusion here.

¹⁸⁶There is also another dimension counting in which the variable t is considered as potentially infinite-dimensional, see the famous paper of G. Perelman arXiv:math/0211159, Section 6. We may look at this counting in more detail later.

- Γ is $-(m-2)$ -homogeneous with respect to the parabolic scaling.
- For any non-negative integers k, l the function $\nabla_x^k \partial_t^l \Gamma(z)$ is $-(m-2+k+2l)$ -homogeneous and

$$|\nabla_x^k \partial_t^l \Gamma(z)| \leq \frac{c(m, k, l)}{|z|_{\text{par}}^{m-2+k+2l}}, \quad z = (x, t) \in \mathbb{R}^n \times [0, \infty). \quad (786)$$

We note that these properties are quite analogous to the corresponding properties of the fundamental solution of the laplacian $G(x) = \frac{1}{(n-2)|S^{n-1}||x|^{n-2}}$. Of course, the important difference is that Γ is not as isotropic as G .

Some of the considerations we did for the laplacian apply to the heat operator (and, in fact, to a broader class of operators) without much change. Let us illustrate this by re-proving in the context of the heat operator some of the results we obtained for the laplacian in the first few lectures.

Let us first consider the proof of the inversion formulae (25) for the laplacian. We already implicitly derived the analogous formulae for the heat operator by means of Fourier transformation, but it is useful to look at the situation from a different point of view.

Let K be a smooth function in $\mathbb{R}^n \times \mathbb{R}$ supported in $\{t \geq 0\}$ such that the support of $\Gamma - K$ is compact. We note that

$$\int (\partial_t - \Delta)K(x, t) dx dt = 1. \quad (787)$$

We set

$$K_\varepsilon(x, t) = \frac{1}{\varepsilon^n} K\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right). \quad (788)$$

We note that the function $(\partial_t - \Delta)K_\varepsilon$ can be considered as a parabolic mollifier. Assume now that f is a compactly supported integrable function in $\mathbb{R}^n \times \mathbb{R}$ and set $u = K * f$, $u_\varepsilon = K_\varepsilon * f$. Clearly

$$(\partial_t - \Delta)u_\varepsilon = (\partial_t - \Delta)(K_\varepsilon * f) = [(\partial_t - \Delta)K_\varepsilon] * f \rightarrow f \quad (789)$$

as $\varepsilon \rightarrow 0$. At the same time, $u_\varepsilon \rightarrow u$ as $\varepsilon \rightarrow 0$, and therefore

$$(\partial_t - \Delta)u = f. \quad (790)$$

We see that the operator $f \rightarrow \Gamma * f$ inverts the heat operator, in the sense of the formulae (25) from lecture 2. We could again repeat the whole discussion

¹⁸⁷Hint: Note that we can integrate $\int_0^T \int_{\mathbb{R}^n} \dots dx dt$ as $\Gamma - K$ is compactly supported and the heat operator vanishes on Γ away from $(x, t) = (0, 0)$. The integral $\int_{\mathbb{R}^n} \Delta K(x, t) dx$ vanishes for each t . Now we use $\int_{\mathbb{R}^n} \int_0^T \partial_t K(x, t) dt dx = \int_{\mathbb{R}^n} K(x, T) dx = \int_{\mathbb{R}^n} \Gamma(x, T) dx = 1$ when T is "above the support" of $\Gamma - K$.

from the beginning of lecture 3 addressing the question in which sense is (790) satisfied.

We should note that at this stage we have more technical tools available than we had in lecture 2, and therefore the above calculations can be replaced by referring to general principles of dealing with distributions. We know that $(\partial_t - \Delta)\Gamma = 0$ in distributions, and hence we can just write

$$(\partial_t - \Delta)(\Gamma * f) = [(\partial_t - \Delta)\Gamma] * f = \delta * f = f. \quad (791)$$

The differentiation is taken in the sense of distributions.

Strictly speaking, so far we only defined the convolution $v * w$ for distributions in the case when one of the distributions is a compactly supported smooth function¹⁸⁹, so we should really extend our definition of convolution to cover (791) fully even when, say, f is a compactly supported distribution. This is not hard and we will return to this point later¹⁹⁰.

For now we will look at the consequences of the formula

$$u = \Gamma * [(\partial_t - \Delta) * u] \quad (792)$$

for compactly supported functions u . Here we can follow the reasoning used for the laplacian in lecture 4, with only minor changes

Let u be a solution of the heat equation in an open domain $\mathcal{O} \subset \mathbb{R}^n \times \mathbb{R}$ and let ϕ be a smooth, compactly supported function in \mathcal{O} . Using the formula

$$(\partial_t - \Delta)(\phi u) = u(\phi_t + \Delta\phi) - 2 \operatorname{div}(\nabla\phi u), \quad (793)$$

and (792), we obtain

$$\phi u = \Gamma * [u(\phi_t + \Delta\phi)] + \Gamma * [-2\nabla(\phi u)] = \Gamma * [u(\phi_t + \Delta\phi)] - 2\Gamma_{x_j} * (\phi u)_{x_j}. \quad (794)$$

For $z = (x, t)$ and $R > 0$ we define

$$Q_{z,R} = B_{x,R} \times (t - R^2, t), \quad Q_R = Q_{(0,0),R}. \quad (795)$$

These sets are sometimes called parabolic balls. We consider z as the “parabolic center” of $Q_{z,R}$. The “parabolic boundary” of $Q_{z,R}$ is defined by

$$\partial_{\text{par}} Q_{z,R} = B_{x,R} \times \{t - R^2\} \cup \partial B_{x,R} \times [t - R^2, t] \quad (796)$$

Heuristically, for the solution of the heat equation $u_t - \Delta u = 0$ the value $u(z)$ is uniquely determined by the values of u at $\partial_{\text{par}} Q_{z,R}$. We note that the parabolic center z of $Q_{z,R}$ is not in the topological interior of $Q_{z,R}$. The position of the

¹⁸⁸This is of course the main point behind our interest in the solution of $u_t - \Delta u = \delta$.

¹⁸⁹In(791) this would have to be the function f .

¹⁹⁰The reader can also consult Section 4.2 in “The Analysis of Linear Partial Differential Operators I” by L. Hörmander

parabolic center is natural if we think about how $u(z)$ is determined from the boundary values for the solutions of $u_t - \Delta u = 0$, a topic which we will discuss in some detail later. However, even at this point we can see from (794) and the fact that the fundamental solution Γ is supported in $\{t \geq 0\}$ that $u(z)$ depends only on the values of u in $Q_{z,R} \setminus Q_{z,R'}$ for any $R' < R$. To see that it is enough to take in (794) a function φ which is $\equiv 1$ in $Q_{z,R'}$ and vanishes near $\partial_{\text{par}}Q_{z,R}$. Also, if we think about the propagation of heat, it should be heuristically clear that $u(z)$ should depend only on the values of u at $\partial_{\text{par}}Q_{z,R}$ if no sources of heat are present in $Q_{z,R}$.¹⁹¹

Let us consider a smooth cut-off function ψ in the ball Q_2 which is $\equiv 1$ in $Q_{\frac{3}{2}}$ and vanishes near $\partial_{\text{par}}Q_2$. Let

$$\varphi_R(x, t) = \psi\left(\frac{x}{R}, \frac{t}{R^2}\right). \quad (797)$$

Using (794) with φ_{2R} in the ball Q_{2R} together with (786), we obtain the following analogues of estimates (33). We will use the term *caloric functions* for the solutions of the heat homogeneous heat equation $u_t - \Delta u = 0$.

Let u be a caloric function in the parabolic ball Q_{2R} of radius $2R$. Then for any non-negative integers k, l we have the following estimate for u in the ball Q_R :

$$|\nabla_x^k \partial_t^l u(x, t)| \leq \frac{C_{k,l}}{R^{n+2+k+2l}} \int_{Q_{2R} \setminus Q_R} |u(x, t)| dx dt, \quad (x, t) \in Q_R. \quad (798)$$

where $C_{k,l}$ are constants independent of u and R .

The proof of this statement can be obtained by following the proof of (33) in lecture 4 line-by-line, with obvious adjustments to the parabolic situation.

An obvious consequence of (798) is the following:

For caloric functions in Q_{2R} we have

$$\sup_{Q_R} |\nabla_x^k \partial_t^l u| \leq \frac{C_{k,l}}{R^{k+2l}} \sup_{Q_{2R}} |u|, \quad (799)$$

where the constants $C_{k,l}$ are independent of u and R .

Mimicking the proof of the Liouville theorem for harmonic functions in lecture 4, we obtain the following analogue of the Liouville theorem for harmonic functions:

¹⁹¹Appealing to the physical meaning of the equation is usually helpful, but one has to be somewhat cautious, as the equation may not capture the physical phenomena exactly. For example, we note that the heat equation predicts that heat will propagate with infinite speed, which of course cannot really be true. This wrong prediction is caused by certain idealizations in the model of heat propagation which the heat equation presents. For the propagation of heat in macroscopic bodies the effects of the idealizations are negligible and the heat equation can be used without worries, but the situation is different if we go to the atomic scales.

Liouville Theorem for the heat equation

Let u be a bounded caloric function in $\mathbb{R}^n \times (-\infty, 0)$. Then u is constant.

As an exercise, the reader can formulate and prove an analogue of the Liouville theorem, version 2 from lecture 4.

We also have an analogy of the Weyl's lemma we proved in lecture 4 for harmonic functions:

Weyl's lemma for caloric functions

A weakly caloric function is smooth and caloric.

If we wish to follow the proof of the corresponding statement for harmonic functions from lecture 4, we should define a weakly caloric function as a locally integrable function satisfying the heat equation in the sense of distributions, in analogy to the definition we gave in lecture 4 for harmonic functions.

However, in the context of the theory of distributions, the more natural definition would be that a weakly caloric function is any *distribution* which satisfies the heat equation in the sense of distributions. Weyl's lemma remains true (for caloric functions, harmonic functions, and, in fact, much more general classes of PDE solutions) in this form, but the proof via the representation formulas we used has to be slightly adjusted. The adjustment is not hard: we know that every distribution is locally represented by derivatives of an integrable function and in (794) or in (32) we can move these derivatives on the kernel Γ or G respectively.

In fact, the fact that any distribution satisfying $\Delta u = 0$ in B_R is smooth can be seen easily from the fact that for a radial mollifier ϕ_ε we have $\phi_\varepsilon * u = u$ in $B_{R-\varepsilon}$, as one can see from the discussion at the end of lecture 5.

From the two examples of harmonic functions and caloric functions one can expect that the proof of Weyl's lemma via the representation formulae obtained from a fundamental solution should be adaptable to any PDE operator with constant coefficients with a fundamental solution which is smooth away from the origin. This is indeed the case, although one has to work somewhat harder, as for general operators the formulae are not so explicit. The same is true about the "Harnack theorem" for harmonic function (not to be confused with the Harnack inequality!) which states that a sequence of harmonic functions converging in some weak sense actually converges uniformly with all derivatives. (This is of course a consequence of the estimates on the derivatives.) This statement is also true for the caloric functions, as we also have good estimates on the derivatives, and it is in fact also true for all equations with constant coefficients and a smooth fundamental solution. At the moment we will not pursue these generalizations further, the interested reader can consult for example the book "The Analysis of Linear Partial Differential Operators I" by L. Hörmander, Section 4.4. for more details.

As we mentioned the Harnack inequality, we should say that its generalization to the caloric function is a more subtle issue, and one finds that the if we literally translate the elliptic statement into the parabolic language, it will not be true. It has to be adjusted - we will return to this point later. If we go to even more general operators (e. g. higher order operators, or operators for vector-valued functions), one may not have an analogue of the Harnack inequality.

Last time we have considered the problem

$$u_t - \Delta u = f(x, t) \quad (800)$$

in $\mathbb{R}^n \times \mathbb{R}$ with some compactly supported f . The requirement that f be compactly supported can be significantly relaxed, but that is not our concern for now. We have seen that the requirement that u vanishes for large negative t together with the equation (800) and the “growth condition” $u \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R})$ (which is satisfied when u is bounded, or has at most polynomial growth) determine u uniquely. This set-up models the situation when we have a heat conducting material filling the whole space, for all times before a certain time t_1 “nothing is going on” and the temperature is 0. After time t_1 heat sources of density $f(x, t)$ are “activated”, and the temperature starts changing. We can assume without loss of generality that $t_1 = 0$. The history for $t < 0$ is trivial - nothing was ever going on, so we can just as well disregard it and consider only the problem

$$\begin{aligned} u_t - \Delta u &= f(x, t), & (x, t) &\in \mathbb{R}^n \times (0, \infty), \\ u(x, 0) &= 0, & x &\in \mathbb{R}^n. \end{aligned} \quad (801)$$

The solution is given by $u = \Gamma * f$. To evaluate the convolution we do not need to know f for $t < 0$, as the function Γ is supported in $\{t \geq 0\}$. If we write out the convolution in detail, we have

$$u(x, t) = \int_0^t \int_{\mathbb{R}^n} \Gamma(x - y, t - s) f(y, s) dy ds. \quad (802)$$

Let us now look at the problem

$$\begin{aligned} u_t - \Delta u &= 0, & (x, t) &\in \mathbb{R}^n \times (0, \infty), \\ u(x, 0) &= u_0(x), & x &\in \mathbb{R}^n \end{aligned} \quad (803)$$

It is instructive to solve it by a slightly different application of the Fourier transformation than the one we used in lecture 51 to calculate $\Gamma(x, t)$. This time we only perform the Fourier Transform in the variable x . This approach is somewhat similar to the classical approach to solving an ODE system

$$\dot{x} + Ax = 0 \quad (804)$$

by diagonalizing the matrix A . When A is diagonal in some coordinates (as it is the case with symmetric matrices, for example), the system (804) is easy to solve, and we can then return to the original coordinates to get a formula for our solution. Applying the Fourier transform in x to (803) is an analogous procedure. After taking the Fourier transform in x the problem becomes

$$\begin{aligned} \hat{u}_t + |\xi|^2 \hat{u} &= 0, & (\xi, t) &\in \mathbb{R}^n \times (0, \infty), \\ \hat{u}(\xi, 0) &= \hat{u}_0(\xi), & \xi &\in \mathbb{R}^n, \end{aligned} \quad (805)$$

which can be solved explicitly:

$$\hat{u}(\xi, t) = \hat{u}_0(\xi)e^{-t|\xi|^2}. \quad (806)$$

To find the inverse Fourier transform, we recall formula (490) for the Fourier transform of convolution, and evaluate the inverse Fourier transform of $\hat{u}_0(\xi)$ and $e^{-t|\xi|^2}$. The former is obviously $u_0(x)$, and the latter is¹⁹² again the heat kernel $\Gamma(x, t)$ given by (781). Taking into account the convolution formula, we obtain

$$u(x, t) = \int_{\mathbb{R}^n} \Gamma(x - y, t)u_0(y) dy. \quad (807)$$

Sometimes this formula is written as

$$u(t) = \Gamma(t) * u_0, \quad (808)$$

where $t \rightarrow u(t)$ means the mapping which takes t into the function $x \rightarrow u(x, t)$. This notation emphasized the distinguished role of the t -variable and the function u is viewed as a function from a time interval to a space of functions of x . Similarly for Γ . The convolution in (808) is the spatial convolution in \mathbb{R}^n , in contrast with the formula $u = \Gamma * f$ after (801), where the convolution was taken in space-time.

We see that the fundamental solution $\Gamma(x, t)$ can also be considered as a solution of the problem

$$\begin{aligned} u_t - \Delta u &= 0, & (x, t) &\in \mathbb{R}^n \times (0, \infty), \\ u(x, 0) &= \delta(x), & x &\in \mathbb{R}^n \end{aligned} \quad (809)$$

which at first looks somewhat different than (771). The fact that these two problems have the same solution is a special case of the *Duhamel's principle*. The principle says, loosely speaking, that the solution of an evolution equation

$$u_t = Lu + f(x, t) \quad (810)$$

can be built as a “linear combination”¹⁹³ of the solutions of initial-value problems

$$\begin{aligned} u_t &= Lu, & t &> s \\ u(s) &= u_0^s. \end{aligned} \quad (811)$$

The simplest illustration of the principle can be seen already in ODEs. Let us consider a linear ODE of the form

$$\begin{aligned} \dot{x} + ax &= f(t), & t &\geq 0 \\ x(0) &= 0, \end{aligned} \quad (812)$$

together with the initial-value problems

$$\begin{aligned} \dot{x} + ax &= 0, & t &\geq s, \\ x(s) &= 1. \end{aligned} \quad (813)$$

¹⁹²by (483) and (484)

¹⁹³more precisely, an integral

Let us denote the solution of the last problem (with the convention that it vanishes for $t < s$) by $x^s(t)$. Then the solution of (812) is given by

$$x(t) = \int_0^t x^s(t) f(s) ds. \quad (814)$$

When a is a constant independent of t , then $x^s(t) = e^{-a(t-s)}$ and we obtain the classical formula

$$x(t) = \int_0^t e^{-a(t-s)} f(s) ds \quad (815)$$

for the solution of (812). The case $a \equiv 0$, in spite of its simplicity, is already instructive¹⁹⁴, and contains the basic idea behind Duhamel's principle.

We have shown how to go from (813) to (812). If we wish to go in the opposite direction, we take $f(t) = \delta(t-s)$.

returning back to the heat equation, the shifted fundamental solution $\Gamma(t-s)$ should be thought of as the function $x^s(t) = e^{a(t-s)}$ in the ODE example. The analogue of the formula (815) will be the formula

$$u(t) = \int_0^t \Gamma(t-s) * f(s) ds, \quad (816)$$

where $*$ denotes the convolution in \mathbb{R}^n . It is nothing but formula (802), written in a different notation. Note that we can also write

$$u(t) = \int_{-\infty}^t \Gamma(t-s) * f(s) ds, \quad (817)$$

as we assume that $f(s) = 0$ for $s < 0$. (If we do not make this assumption, then (817) is actually the right formula for the solution of the problem considered in the beginning of the lecture, in which time $t = 0$ does not play a distinguished role.) Using (somewhat formally) (817) with $f(x, t) = u_0(x)\delta(t)$, we recover formula (807).

Let us look a closer look at formula (808). We recall the notation

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon^n} \phi\left(\frac{x}{\varepsilon}\right) \quad (818)$$

we used for mollifiers in the first semester (see lecture 2, (21)). Letting $\phi = \Gamma(1)$, we see that

$$\Gamma(t) = \phi_{\sqrt{t}}, \quad (819)$$

and formula (808) can be thought of in terms of mollification of the initial data $u_0(x)$. The situation is quite similar to the one we studied in lecture 9 for the Poisson kernel, see (72).

¹⁹⁴You can notice that it is just another form of the Fundamental Theorem of Calculus.

The solution of the general Cauchy problem

$$\begin{aligned} u_t - \Delta u &= f(x, t), & (x, t) \in \mathbb{R}^n \times (0, \infty), \\ u(x, 0) &= u_0(x), & x \in \mathbb{R}^n \end{aligned} \quad (820)$$

can be written as a combination of (808) and (816):

$$u(t) = \Gamma(t) * u_0 + \int_0^t \Gamma(t-s) * f(s) ds. \quad (821)$$

Strictly speaking, we should say “a solution” and not “the solution”, since we do not know that the solutions are unique. The representation formula (821) represents our common sense heuristics about how the solution should look like, and we do not expect that there is any other “reasonable” solution.

Due to the linearity of the problem (820), questions about the uniqueness of its solution can be reduced to questions about the uniqueness of the homogeneous equation with the homogeneous boundary conditions, i. e. that case when u_0 and f in (820) vanish. We can see easily that solutions of

$$\begin{aligned} u_t - \Delta u &= 0, & (x, t) \in \mathbb{R}^n \times (0, \infty), \\ u(x, 0) &= 0, & x \in \mathbb{R}^n \end{aligned} \quad (822)$$

with at most polynomial growth at infinity must vanish. For this we extend u as $u = 0$ to negative times and note that the extended function (still denoted by u) is a tempered distribution satisfying $u_t - \Delta u = 0$ in $\mathbb{R}^n \times \mathbb{R}$ in the sense of distributions. Taking the Fourier transform, we see that $(i\tau + |\xi|^2)\hat{u} = 0$, and we see that \hat{u} must be supported at the origin. This means that u is a polynomial. A polynomial vanishing for $t < 0$ must vanish identically, and this finishes the proof.

The proof may be instructive as an application of the Fourier transformation, but it is not optimal. For example, note that it is “global in time”, whereas a uniqueness statement for an evolution equation should really be local in time in the following sense: if we have a solution on $[0, T)$ which vanishes at 0, it should vanish in $[0, T)$. (This is the case when the equation is of the first order in t . If it is of order m in t , the assumption that u vanishes at 0 should of course include also derivatives of order $\leq (m-1)$.) Also, the growth conditions can be much relaxed.

The optimal uniqueness proof is based on the representation of the solution via the fundamental solution. It is not hard and the method is quite general, and therefore we outline the main idea, which is also useful in other situations.

In general if we have a linear PDE operator L in a domain Ω , we can consider its L^2 adjoint defined by

$$\int_{\Omega} Lu v = \int_{\Omega} u L^* v \quad (823)$$

for smooth functions u, v compactly supported in Ω . For this definition the operator L can be a quite general linear operator, not necessarily with constant

coefficients. If the functions u, v are not compactly supported, we can write

$$\int_{\Omega} (uL^*v - Lu v) = \int_{\partial\Omega} \text{boundary terms} . \quad (824)$$

Assume now that $Lu = 0$ and $L^*v = \delta_{x_0}$, the Dirac mass at x_0 . Using (824) for these u, v , we obtain a representation formula

$$u(x_0) = \int_{\partial\Omega} b(x, [u], [v]), \quad (825)$$

where $b(x, [u], [v])$ is a bi-linear form, and the notation $[u], [v]$ indicates that u and v may also appear with derivatives.

We can also replace u by φu , where φ is a cut-off function, and replace the integral over the boundary by an integral over the area where $\nabla\varphi$ does not vanish.

Let us now apply this to the heat operator $L = \partial_t - \Delta$ and the domain $\mathbb{R}^n \times (0, T)$, under the assumption that the decay of u in x is such that we can integrate by parts in x without any boundary terms coming from ∞ . The adjoint operator is $L^*v = -\partial_t - \Delta v$ and the formula (824) becomes

$$\int_0^T \int_{\mathbb{R}^n} u(-v_t - \Delta v) - (u_t - \Delta u)v = \int_{\mathbb{R}^n} -uv \Big|_{t=0}^{t=T} \quad (826)$$

Let us consider $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$. The solution of $-v_t - \Delta v = \delta_{(x_0, t_0)}(x, t)$ is

$$v(x, t) = \Gamma(x_0 - x, t_0 - t). \quad (827)$$

Using (826) with this choice of v , we obtain

$$u(x_0, t_0) = \int_{\mathbb{R}^n} u(x, 0)\Gamma(x_0 - x, t_0) dx. \quad (828)$$

which is the same as (808). To see under which assumptions on u the integration by parts over x in this calculation produces no extra terms so that the above calculation is legitimate, we can either integrate over $B_R \times (0, T)$ and take $R \rightarrow \infty$, or multiply $v(x, t) = \Gamma(x_0 - x, t_0 - t)$ by a suitable cut-off function. Note that v as well as its derivatives have very fast decay as $x \rightarrow \infty$, and therefore u can grow quite fast. It is easy to check that for example the condition

$$|u(x, t)| \leq C e^{\frac{|x|^2}{8T}} \quad (829)$$

for some $C > 0$ is sufficient.

When our aim is to show uniqueness, we assume that $u(x, 0) = 0$ and under the assumption (829) we obtain that $u = 0$ in $\mathbb{R}^n \times (0, T)$. Now we note that T can be taken small, because the conclusion $u = 0$ can be “propagated” in small steps in t . Therefore we obtain the following uniqueness theorem (due to Tikhonov)

If a solution of

$$\begin{aligned} u_t - \Delta u &= 0, & (x, t) &\in \mathbb{R}^n \times (0, \infty), \\ u(x, 0) &= 0, & x &\in \mathbb{R}^n \end{aligned} \quad (830)$$

satisfies the growth condition

$$|u(x, t)| \leq C e^{C|x|^2} \quad (831)$$

for some $C > 0$, then it has to vanish identically.

Proof: See above.

In his well-known paper on the topic, Tikhonov also constructed an example that uniqueness can fail if we do not impose restriction on the growth of the function u . The main idea of the counter-example is the following. Take a non-trivial smooth function F which is compactly supported in (t_1, t_2) for some $0 < t_1 < t_2$ and consider the series

$$u(x, t) = \sum_{k=0}^{\infty} F^{(k)}(t) \frac{x^{2k}}{(2k)!}, \quad (832)$$

where $F^{(k)}$ denotes the k -th derivative. It is easy to check that this function formally satisfies the heat equation, and gives a counter-example to uniqueness of (830). The only issue is to establish the convergence of the series (832). This needs some work. Note that the Taylor series of F cannot be convergent at each point, as the function F cannot be analytic. However, the terms $\frac{x^{2k}}{(2k)!}$ have better decay than the term $\frac{s^k}{k!}$ of the usual Taylor series. For details we refer the reader to the original 1935 paper by Tikhonov, which can be found online.¹⁹⁵

¹⁹⁵See the link on http://en.wikipedia.org/wiki/Andrey_Nikolayevich_Tychonoff

We derived the heat equation as a model for heat conduction, but it arises in other situations. Today we mention one of them – random walks. Consider the following process: we consider a particle moving in the real line according to the following rules. At time $t = 0$ the particle is at $x = 0$. We toss a coin and at time $t = 1$ we move the particle by 1 to the right if we get heads, and to the left if we get tails. So the position of the particle at $t = 1$ is either $x = 1$ or $x = -1$, both with equal probability $\frac{1}{2}$. Now we repeat the procedure: we toss a coin and we move the particle at time $t = 2$ according to the same rule. So at time $t = 2$ the particle can be at $x = -2$ (with probability $1/4$), at $x = 0$ (with probability $1/2$), and at $x = 2$ (with probability $1/4$). We repeat the procedure again. After taking m steps, at time $t = m$, the particle can be at $x = -m$ with probability 2^{-m} , at $x = -m + 2$ with probability $\binom{m}{1}2^{-m}$, at $x = -m + 4$ with probability $\binom{m}{2}2^{-m}$, etc. In general, the probability of the particle being at the position $x = k$ after m steps is non-zero only if $m + k$ is even, $-m \leq k \leq m$, and for such (k, m) the probability is, by elementary combinatorics,

$$p(k, m) = \binom{m}{\frac{m+k}{2}} 2^{-m} = \frac{m!}{\left(\frac{m-k}{2}\right)! \left(\frac{m+k}{2}\right)!} 2^{-m}. \quad (833)$$

In addition to this explicit expression, there is also a “local rule” satisfied by the probabilities $p(k, m)$.

$$p(k, m + 1) = \frac{1}{2}p(k - 1, m) + \frac{1}{2}p(k + 1, m). \quad (834)$$

The justification is immediate from our rules for moving the particle, without using (833). In terms of (833), equation (834) is just the well known identity between the binomial coefficients:

$$\binom{m + 1}{l} = \binom{m}{l} + \binom{m}{l - 1}. \quad (835)$$

Equation (834) can be made look like a PDE with derivatives replaced by difference quotients. Subtracting $p(k, m)$ from both sides we get

$$p(k, m + 1) - p(k, m) = \frac{1}{2}(p(k - 1, m) - 2p(k, m) + p(k + 1, m)) \quad (836)$$

which looks as a finite difference version of

$$p_t = \frac{1}{2}p_{xx}. \quad (837)$$

However, we must remember that the particle can appear in the space-time points with coordinates (k, m) with $k + m$ even. The space-time points involved in (836) cannot all satisfy this parity condition. This is easy to fix: instead of (836), we consider

$$p(k, m + 2) = \frac{1}{4}p(k - 2, m) + \frac{1}{2}p(k, m) + \frac{1}{4}p(k + 2, m), \quad (838)$$

which is an easy consequence of (838). The sums of the coordinates in the arguments of all the terms in (838) all have the same parity. Subtracting $p(k, m)$ from both sides of (838) and dividing by 2, we obtain

$$\frac{p(k, m+2) - p(k, m)}{2} = \frac{1}{2} \left(\frac{p(k-2, m) - 2p(k, m) + p(k+2, m)}{4} \right), \quad (839)$$

which again is a finite-difference version of (837) (with the time step = 2 and the space step also = 2. Equation (839) is satisfied for all (k, m) with $k+m$ even and $m \geq 0$. If we remove the condition $m \geq 0$, still considering only $m+k$ even, we get

$$\frac{p(k, m+2) - p(k, m)}{2} - \frac{1}{2} \left(\frac{p(k-2, m) - 2p(k, m) + p(k+2, m)}{4} \right) = b(k, m), \quad (840)$$

where f is given by

$$\begin{aligned} b(0, -2) &= \frac{1}{2}, \\ b(-1, -1) &= b(1, -1) = \frac{1}{4}, \\ b(k, m) &= 0 \quad \text{elsewhere.} \end{aligned} \quad (841)$$

Our goal is to show that in a suitable scaling limit equation (840) will approach

$$u_t - \frac{1}{2}u_{xx} = \delta(x, t). \quad (842)$$

Since we have an explicit solution (833) to (840), we should have another method for calculating the fundamental solution. It is not the simplest method to arrive at the formula, but it is quite illuminating.

Note that the speed of propagation of our random walk on the grid is finite: after m steps we the position k of the particle cannot be outside $[-m, m]$. On the other hand, for (842) disturbances propagate at infinite speed. This is reconciled by taking the scaling limit.

For the scaling it is useful to consider $p(k, m)$ as some object defined on $\mathbb{R} \times \mathbb{R}$, rather than on a grid. This can be done in many ways. For example, one can consider a piece-wise constant function $\tilde{u}(x, t)$ defined to have value $\frac{1}{2}p(k, m)$ for $k-1 < x < k$ and $m < t < m+1$. We will proceed in a slightly different way: define a distribution u in $\mathbb{R} \times \mathbb{R}$ as

$$u(x, t) = \sum_{\substack{k, m \\ m \geq 0 \\ k+m \text{ even}}} p(k, m) \delta_{k, m}(x, t), \quad (843)$$

where $\delta_{k, m}$ is the Dirac mass located at $(x, t) = (k, m)$. Let us also define

$$f(x, t) = \frac{1}{2}\delta_{0, -2}(x, t) + \frac{1}{4}\delta_{-1, -1}(x, t) + \frac{1}{4}\delta_{1, -1}(x, t) \quad (844)$$

We write equation (840) in terms of u , f and suitable shift operators. The operators are

$$\partial_{ht}u(x, t) = \frac{u(x, t + 2h^2) - u(x, t)}{2h^2}, \quad (845)$$

$$\partial_{ht}^*u(x, t) = \frac{u(x, t - 2h^2) - u(x, t)}{2h^2} \quad (846)$$

$$\Delta_h u(x, t) = \frac{u(x - 2h, t) - 2u(x, t) + u(x + 2h, t)}{4h^2}, \quad (847)$$

$$L_h = \partial_{ht} - \frac{1}{2}\Delta_h, \quad (848)$$

and

$$L_h^* = \partial_{ht}^* - \frac{1}{2}\Delta_h, \quad (849)$$

where $h > 0$. We have written these operators in the notation usually used for functions, but as we have seen in lecture 45, the shifts are also defined for distributions in a natural way.

Equation (840) can be written in terms of the distributions u , f as an equation in distributions in the following way:

$$L_1 u = f, \quad (850)$$

which is the same as

$$\int_{\mathbb{R} \times \mathbb{R}} u L_1^* \varphi \, dx \, dt = \int_{\mathbb{R} \times \mathbb{R}} f \varphi \, dx \, dt, \quad \varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}). \quad (851)$$

If we would prefer to work with the piece-wise functions mentioned above, rather than the linear combinations of Dirac masses, we can replace the distribution u defined by (843) by $u * \chi$, where χ is the characteristic function of a suitable rectangle. (Other approximation schemes can be expressed in terms of different χ .)

The key point now is to introduce the following scaling: for $h > 0$ we set

$$u_h(x, t) = \frac{1}{h} u\left(\frac{x}{h}, \frac{t}{h^2}\right), \quad (852)$$

$$f_h(x, t) = \frac{1}{h^3} f\left(\frac{x}{h}, \frac{t}{h^2}\right). \quad (853)$$

Again, we use the notation for functions, but we have seen in lecture 50 how to make the definitions for distributions, see (731)

If we think of u as some probability density associated to the random walk with step size 1 in t and 1 in x , then u_h represents the probability density associated with random walk with step size h^2 in t and h in x . For example, the density

$x \rightarrow u \sqrt{\frac{t}{m}}(x, t)$ (which can be thought of as a finite combination of Dirac masses in x), represents the probability density obtained by m random steps of size $\sqrt{\frac{t}{m}}$ starting at the origin.

Note that we do not have to talk about random walks at all, and take u_h simply as a function defined on a grid of space step h and time step h^2 , which solves a specific finite difference scheme on the grid. Formula (833) can then be viewed as an explicit solution of the difference scheme.

The function u_h is easily seen to satisfy

$$L_h u_h = f_h, \quad (854)$$

in distribution, or

$$\int_{\mathbb{R} \times \mathbb{R}} u_h L_h^* \varphi \, dx \, dt = \int_{\mathbb{R} \times \mathbb{R}} f_h \varphi \, dx \, dt, \quad \varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}). \quad (855)$$

Since u_h describe some probability densities, we have a simple a-priori bound for u_h which is independent of $h > 0$. Heuristically we expect something like

$$\int_{\mathbb{R}} u_h(x, t) \, dx = 1 \quad (856)$$

but due to the discrete nature of the distribution u_h this is not quite true (and in fact, the integral is formally not really well-defined without additional explanations). It is easier to consider the following bound: for each smooth function $\varphi = \varphi(x, t)$ compactly supported in $\mathbb{R} \times (t_1, t_2)$ we have

$$\left| \int_{\mathbb{R} \times \mathbb{R}} u_h(x, t) \varphi(x, t) \, dx \, dt \right| \leq M(t_1, t_2) \sup |\varphi|, \quad (857)$$

where $M(t_1, t_2)$ is an upper bound on h^2 -multiple of the number of “grid points” $\{mh^2\}_{m=0}^{\infty}$ contained in the interval (t_1, t_2) . When h^2 is small in comparison with $(t_2 - t_1)$ we obviously can take $M(t_1, t_2) \sim (t_2 - t_1)$.

By the last statement in lecture 44, we can choose a subsequence $h_j \rightarrow 0+$ so that the distributions u_h converge to a distribution \bar{u} along the subsequence. The distribution \bar{u} will satisfy bound (857) with $M(t_1, t_2) = t_2 - t_1$. In particular, it will be a tempered distribution. We claim that

$$\bar{u}_t - \frac{1}{2} \bar{u}_{xx} = \delta(x, t) \quad (858)$$

in the sense of distributions. To prove this, it is enough to pass to the limit $h_j \rightarrow 0$ in equation (855) for a given smooth, compactly supported function $\varphi = \varphi(x, t)$. For such a function we can write

$$L_h^* \varphi = (-\varphi_t - \frac{1}{2} \varphi_{xx}) + r(x, t, h), \quad (859)$$

where $r(x, t, h)$ (which of course depends of φ) satisfies

$$|r(x, t, h)| \leq Ch^2, \quad h \in (0, 1), \quad C = C(\varphi). \quad (860)$$

Using the bound (857) together with (862) and (863), and the obvious fact that $f_h \rightarrow \delta$ in distributions as $h \rightarrow 0_+$, we see that the function \bar{u} will satisfy

$$\int_{\mathbb{R} \times \mathbb{R}} \bar{u} \left(-\varphi_t - \frac{1}{2} \varphi_{xx} \right) dx dt = \varphi(0, 0), \quad \varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}). \quad (861)$$

This proves (858). By uniqueness of the fundamental solution in the class of tempered distributions, we see that actually we do not have to pass to a subsequence, and that $u_h \rightarrow \bar{u}$ as $h \rightarrow 0_+$ in distributions.

In the above calculation we have not used formula (833). In fact, since we calculated the fundamental solution \bar{u} by other means, we know that

$$\bar{u}(x, t) = \begin{cases} \frac{1}{(2\pi t)^{\frac{1}{2}}} e^{-\frac{|x|^2}{2t}} & t > 0 \\ 0 & t < 0 \end{cases} \quad (862)$$

(Note that $\bar{u}(x, t) = \Gamma(x, \frac{t}{2})$, with Γ given by (775). The difference is due to the factor $\frac{1}{2}$ is front of \bar{u}_{xx} .)

If we did not calculate \bar{u} before, we could use (833) to calculate it. For that we recall Stirling's formula

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e} \right)^n, \quad (863)$$

which we can use in (833) to obtain

$$p(k, m) \sim 2 \frac{1}{\sqrt{2\pi m}} e^{-\frac{k^2}{2m}}, \quad (864)$$

which is of course the well-known description of the asymptotics of the binomial distribution in terms of the "bell curve". From (864) one can see directly that the distributions u_h defined above must converge to (862), without using any of the "local relations" we investigated above (such as (839), (855), or (858)). Therefore Stirling's formula (863) gives an alternative way of calculating the fundamental solution of the heat equation, without really solving any differential equations. (Vice versa, a more careful analysis of the convergence of $u_h \rightarrow \bar{u}$ should give the approximation (864) without using Stirling's formula.)

It is also possible to find the fundamental solution by solving an ODE. The above considerations show that the solution \bar{u} of (858) will be invariant under the scaling

$$\bar{u}(x, t) \rightarrow \frac{1}{\lambda} \bar{u} \left(\frac{x}{\lambda}, \frac{t}{\lambda^2} \right). \quad (865)$$

In fact, regardless of the above considerations, we can just take a guess that this will be the case, and see if we can justify the guess by finding such solution. The scale invariant \bar{u} would have to be of the form

$$\bar{u}(x, t) = \frac{1}{\sqrt{t}} F\left(\frac{x}{\sqrt{t}}\right). \quad (866)$$

Substituting this into $\bar{u}_t - \bar{u}_{xx} = 0$, which must be satisfied for $t > 0$, we get an ODE for F . It can be solved explicitly, and we again recover the heat kernel (up to a normalizing constant which can be found by integration). The reader can do this calculation as an exercise.

Last time we saw that the heat equation appears in connection with random walks. In the random walks we considered in the last lecture the space-step and the time-step were discrete, so the “events” in space-time which were relevant for the walk formed a certain grid, and the walk was essentially equivalent to a finite-difference scheme. In fact, we could have avoided mentioning random walks altogether and work only with the finite-difference scheme.

Today we will consider the same idea from a slightly different angle, in that our random variables will be “more continuous”. The material covered today can be considered as optional, it will not be needed in what follows.

Let us consider the following modification of the random walk.¹⁹⁶ We will consider a particle moving along a one dimensional line with coordinate x .¹⁹⁷ The particle moves in random steps. The size of individual steps will not be fixed, but some “global” restrictions of course have to be imposed so that we get something non-trivial. We assume that at time $t = 0$ we the particle is located at $x = 0$. Then, between times $t = 0$ and $t = t$ the particle makes n random steps y_1, y_2, \dots, y_n subject to the constraint

$$y_1^2 + \dots + y_n^2 = t. \quad (867)$$

Note that if all steps had the same size $|y_1| = |y_2| = \dots = |y_n| = h$, then (867) would give $h = \sqrt{\frac{t}{n}}$, with $h^2 = \frac{t}{n}$ being the time-step, as in the situation considered in the last lecture. However, this time we will impose only the “global constraint” (867), the steps do not have to a specified size (as long as the global constraint is satisfied). After these steps are made, the new position of the particle will be

$$x = y_1 + \dots + y_n. \quad (868)$$

We now must specify the probability law for the “events” (y_1, \dots, y_n) satisfying (867). A natural assumption is that the vectors (y_1, \dots, y_n) are distributed over the $n - 1$ dimensional sphere of radius \sqrt{t} uniformly with respect to the canonical “surface measure” on the sphere. Let us denote the surface measure by μ_n^t . The measure is normalized so that the total mass is 1. We can write

$$\mu_n^t = \frac{1}{|S^{n-1}| t^{\frac{n-1}{2}}} \sigma_{n-1}^t = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}} t^{\frac{n-1}{2}}} \sigma_{n-1}^t, \quad (869)$$

where σ_{n-1}^t is the canonical surface measure,¹⁹⁸ and we have used (755).

¹⁹⁶It is probably safe to assume that it was already known to 19th century classics. It can be found for example in the well-known 1923 paper “Differential Space” by N. Wiener.

¹⁹⁷The generalization to the higher dimensional case when $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ is straightforward.

¹⁹⁸It can be considered as the restriction of the $n - 1$ -dimensional Hausdorff measure \mathcal{H}^{n-1} to the sphere (868).

We wish to find the probability distribution of x given by (868) in the limit $n \rightarrow \infty$.

We now pause for a moment to explain some definitions used in this context. In general, if X, Y are metric spaces, $f: X \rightarrow Y$ is a continuous mapping, and μ is a Borel measure on X , we define the *push-forward* of μ by f to be the measure ν defined on Y by

$$\nu(\mathcal{O}) = \mu(f^{-1}(\mathcal{O})). \quad (870)$$

We will use the usual notation $f_{\#}\mu$ for ν . Note that ν can also be defined by

$$\int_Y \varphi(y) d\nu(y) = \int_X \varphi(f(x)) d\mu(x), \quad \varphi \in \mathcal{C}_0(Y), \quad (871)$$

where $\mathcal{C}_0(Y)$ denotes continuous compactly supported functions on Y .¹⁹⁹

In the language just introduced, the probability distribution of x referred to above is

$$\nu_n^t = f_{n\#}(\mu_n^t), \quad (872)$$

where

$$f_n(y) = y_1 + \cdots + y_n, \quad (873)$$

and μ_n^t is given by (869). The measure ν_n^t can be calculated explicitly. First, we note that instead of working with f_n given by (873), we can work with

$$f_n(y) = \sqrt{n} y_1. \quad (874)$$

This is because (873) and (874) are related by an orthogonal rotation which leaves the measures μ_n^t unchanged. Therefore in what follows we will write

$$x = \sqrt{n} y_1. \quad (875)$$

To calculate $f_{n\#}(\mu_n^t)$, we first push-forward μ_n^t by the projection $(y_1, y_2, \dots, y_n) \rightarrow y_1$ on the y_1 -axis, and then we stretch it by the map $y_1 \rightarrow \sqrt{n} y_1$. The push-forward under the projection is easy to calculate:²⁰⁰ it is a measure supported in $[-\sqrt{t}, \sqrt{t}]$ given by

$$\frac{|S^{n-2}|(t - y_1^2)^{\frac{n-2}{2}}}{|S^{n-1}|t^{\frac{n-1}{2}}} \left(1 - \frac{y_1^2}{t}\right)^{-\frac{1}{2}} dy_1. \quad (876)$$

Now we need to stretch this measure by \sqrt{n} , which can be done by simply putting $y_1 = \frac{x}{\sqrt{n}}$. After this substitution, using (893), we obtain

$$\nu_n^t = \frac{1}{\sqrt{2\pi t}} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\sqrt{\frac{n}{2}}} \left(1 - \frac{x^2}{\frac{n}{2} 2t}\right)^{\frac{n}{2} - \frac{3}{2}} dx. \quad (877)$$

¹⁹⁹If $\mu(X) = +\infty$, some assumptions are needed so that the integral on the right-hand side of (871) is well-defined - we leave the exact formulation of these to the reader as an exercise.

²⁰⁰Hint: if you do the calculation for $n = 2$ and $n = 3$, the case of general n becomes clear.

We note (e. g. from Stirling's formula) that

$$\lim_{n \rightarrow \infty} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\sqrt{\frac{n}{2}}} = 1, \quad (878)$$

and recalling that

$$\lim_{n \rightarrow \infty} (1 + \frac{a}{n})^n = e^a, \quad (879)$$

we infer

$$\nu_n^t \rightarrow \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx, \quad n \rightarrow \infty, \quad (880)$$

which is the heat kernel for

$$u_t - \frac{1}{2}u_{xx} = 0. \quad (881)$$

We see that our new version of the random walk also leads to the heat kernel.

But where does the heat equation come from in this picture? In the discrete random walk we had a simple "local law" which was quite clearly a discrete version of the heat equation. Can we see the heat equation in the above version of the random walk directly, not just by an ex post facto verification that the limit (880) satisfies (881)?

To consider this issue, it is advantageous to slightly change our point of view. In the above derivation, we considered the measure ν_n^t in some sense separately for each t , as projections of the measures μ_n^t from the spheres of radii \sqrt{t} by the map $y \rightarrow \sqrt{n}y_1$. The time t played a role of a relatively passive parameter. To see the heat equation more directly, it is better to view the measures ν_n^t as time slices of a space-time object, which comes from a projection of some "global" measure in the space $y \in \mathbb{R}^n$ (and note just on the spheres) onto the space-time (and not just the time-slices $t = \text{const.}$). More precisely, we will be projecting some measure μ_n on \mathbb{R}^n to the space-time $\mathbb{R} \times \mathbb{R}$ by

$$F_n(y) = \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \sqrt{n}y_1 \\ |y|^2 \end{pmatrix}. \quad (882)$$

The measure μ_n should have the property that its "slices" by the spheres $\{y, |y|^2 = t\}$ should project by F_n onto the measures ν_n^t above. This means

$$F_{n\#}(\mu_n) = \int_0^\infty f_{n\#}(\mu_n^t) dt. \quad (883)$$

It is not hard to guess what μ_n should be, without much calculation. One might be tempted to take $\mu_n = \frac{1}{|S^{n-1}||y|^{n-1}} dy$, but this is not quite correct, since, using the notation $r = |y|$, we have $t = r^2$ and hence $dt = 2r dr$. The $2r$ in front of dr shows that we should really take

$$\mu_n = \frac{2}{|S^{n-1}||y|^{n-2}} dy. \quad (884)$$

To check that this is consistent with (883), we note that for each fixed t we have $f_{n\#}(\mu_n^t)(\mathbb{R}) = 1$. Letting χ_R to be the characteristic function of a ball of radius R , we see that (883) implies

$$\int_{\mathbb{R}^n} \chi_R d\mu_n = R^2, \quad R > 0, \quad (885)$$

and this specifies (884) uniquely among all measures invariant under the rotations of \mathbb{R}^n .

We note that the measure μ_n is of the form

$$\mu_n = v_n(y) dy \quad (886)$$

and the function v_n satisfies

$$\Delta v_n = -2(n-2)\delta, \quad (887)$$

where δ is the Dirac mass at $y = 0$ in \mathbb{R}^n . Writing

$$F_{n\#}(v_n dy) = u_n(x, t) dx dt, \quad (888)$$

we expect that equation (887) should result in some constraint on the density $u_n(x, t)$. Rather than determining the constraint directly from (888), it is easier to use the equation (871), which in our case is

$$\int_{\mathbb{R} \times \mathbb{R}} u_n(x, t) \psi(x, t) dx dt = \int_{\mathbb{R}^n} v_n(y) \psi(F_n(y)) dy, \quad (889)$$

for each smooth, compactly supported $\psi: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. In view of (887), we take ψ so that

$$\psi(F_n(y)) = \Delta_y \varphi(F_n(y)). \quad (890)$$

We have

$$\int_{\mathbb{R}^n} u_n(y) \Delta_y \varphi(F_n(y)) dy = -2(n-2) \varphi(0, 0), \quad (891)$$

by (887). We calculate

$$\Delta_y \varphi(F_n(y)) = n\varphi_{xx}(x, t) + 2n\varphi_t(x, t) + 4x\varphi_{xt}(x, t) + 4t\varphi_{tt}(x, t), \quad (892)$$

where $x = \sqrt{n}y_1$ and $t = |y|^2$. Combining (889), (891), and (892), we see that

$$\int_{\mathbb{R} \times \mathbb{R}} u_n \left(-\varphi_t - \frac{1}{2}\varphi_{xx} - \frac{2}{n}x\varphi_{xt} - \frac{2}{n}t\varphi_{tt} \right) dx dt = \left(1 - \frac{2}{n}\right) \varphi(0, 0). \quad (893)$$

In the limit $n \rightarrow \infty$ we obtain (the weak form of) the heat equation. We see that the heat equation for the limit $u(x, t)$ is a consequence of equation (887).

We can also proceed similarly as in the last lecture with the discrete random walk approximations u_h . The functions u_n satisfy a uniform bound

$$\int_{\mathbb{R}^n} u_n(x, t) dx = 1, \quad (894)$$

and even if we had not calculated their limit before we see that we can choose a subsequence converging to some tempered distribution \bar{u} which, in view of (893), will satisfy

$$\bar{u}_t - \frac{1}{2}\bar{u}_{xx} = \delta. \quad (895)$$

By uniqueness of the solutions of (895) in tempered distributions, we can conclude, without the explicit calculation of the limit, that the limit \bar{u} has to be the fundamental solution of the heat operator $\partial_t - \frac{1}{2}\partial_x^2$, confirming the result of the explicit calculation (880).

The above calculations suggest that caloric functions can be thought of, in some sense, as harmonic functions in very high dimensional spaces, as we mentioned in lecture 52.²⁰¹

²⁰¹For additional references where related issues are discussed, the reader can consult for example the paper “Geometry of Differential Space” by H. P. McKean, *Annals of Probability*, 1973, Vol. 1, No. 2, 197-206, or notes from a course by T. Tao available online at <http://terrytao.wordpress.com/2008/04/27/285g-lecture-9-comparison-geometry-the-high-dimensional-limit-and-perelman-reduced-volume/>

Today we will discuss another feature of the heat equation: it is a *gradient flow*. Let us first look at some examples of gradient flows in finite dimension. The simplest situation is the following: assume $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth function satisfying $f(x) \rightarrow \infty$ as $x \rightarrow \infty$, and consider the ODE

$$\dot{x} = -\nabla f(x), \quad (896)$$

where we use the usual notation $\dot{x} = \frac{dx}{dt}$. In comparison with general autonomous n -dimensional ODE

$$\dot{x} = b(x), \quad (897)$$

where $b(x) = (b_1(x), \dots, b_n(x))$ is a smooth vector field in \mathbb{R}^n , the solutions of (896) have a much simpler behavior. The trajectories $x(t)$ follow the lines of the “steepest descent” of the function f . They are perpendicular to the level sets $\{f = \text{const.}\}$ and in the case when all the critical points of f are non-degenerate²⁰² the trajectories typically converge to the local minima of f , except for the boundaries between the “basins of attraction” of the various local minima. The trajectories originating at such boundaries typically converge to saddle points²⁰³. This is much simpler behavior than what is possible for solutions of (897), where one can meet all the intricate phenomena of chaos, strange attractors, etc. Non-trivial questions arise even for the gradient flows²⁰⁴, but one can nevertheless say that if we know that a flow is a gradient flow, it usually simplifies its study to a large degree. For example, by taking the scalar product of (896) with \dot{x} we obtain

$$|\dot{x}|^2 = -(\nabla f(x), \dot{x}) = -\frac{d}{dt}f(x). \quad (898)$$

Integration in t now gives

$$f(x(t_1)) - f(x(t_2)) = \int_{t_1}^{t_2} |\dot{x}(t)|^2 dt, \quad (899)$$

which often represents a useful a-priori bound on the integral on the right, when f is bounded from below. (Note that we can let $t_2 \rightarrow \infty$ in that case.)

Using (896), we can also re-write (899) as

$$f(x(t_1)) - f(x(t_2)) = \int_{t_1}^{t_2} |\nabla f(x(t))|^2 dt. \quad (900)$$

²⁰²Recall that x is a critical point of f if $\nabla f(x) = 0$. A non-degenerate critical point is a critical point x where an additional condition that the Hessian matrix $\nabla^2 f(x)$ is non-singular is satisfied.

²⁰³These are the critical points at which the Hessian matrix $\nabla^2 f(x)$ is indefinite.

²⁰⁴For example: assuming only that $f(x) \rightarrow \infty$ as $x \rightarrow \infty$, does any solution of (896) converge to an equilibrium? The answer is non-trivial: no in general, but yes if f is analytic.

If we define that an equation (897) is a gradient flow if and only if $b(x) = -\nabla f(x)$, it is easy to determine which flows are gradient flows: in a simply connected domain, a necessary and sufficient condition for the existence of an f with $b(x) = -\nabla f(x)$ is that the matrix $\nabla b(x)$ be symmetric for each x . Thus for example a linear equation

$$\dot{x} = -Ax \tag{901}$$

is a gradient flow in the above sense if and only if the matrix A is symmetric. If this is the case, the function f is given by $f(x) = \frac{1}{2}(Ax, x)$, where (\cdot, \cdot) denotes the canonical scalar product in \mathbb{R}^n . We see that the scalar product plays an important role in these considerations.

What happens when we take a different scalar product? If an equation is a gradient flow with respect to a different scalar product, the flow should still have the simplifying features of the gradient flows, so we should not really restrict our attention to a particular choice of the scalar product.

Let us consider a symmetric positive definite matrix $G = \{g_{ij}\}_{i,j=1}^n$ and the scalar product

$$(x, y)_G = (Gx, y) = g_{ij}x_i y_j. \tag{902}$$

For which A is equation (901) a gradient flow when we consider the scalar product (902) rather than the canonical scalar product (corresponding to $G = I$)? An easy calculation shows that the condition is

$$GA = A^t G, \tag{903}$$

where A^t is the transpose of A . Different choices of G give us different conditions on A for which we get a gradient flow. A natural question in this context is the following: for which matrices A is (901) a gradient flow with respect to *some*

²⁰⁵Here we write both g_{ij} and x_i, y_j with lower indices, which is often done. However, we should really use the classical Einstein convention and use upper indices for vector coordinates (contra-variant indices) and lower indices for covector coordinates (covariant indices). In this more precise notation one can only sum over repeated indices if one of them is up and one of them is down. For example, the above expression is written as $g_{ij}x^i y^j$. A matrix of a linear map A has coordinates a_j^i and Ax is written as $a_j^i x^j$. A bilinear form $b(x, y)$ should be written as $b_{ij}x^i y^j$. It is symmetric if $b_{ij} = b_{ji}$, and this is the case if and only if it comes from a quadratic form $b_{ij}x^i x^j$. On the other hand the statement that a matrix a_j^i is symmetric does not make sense in this notation, as we cannot exchange upper and lower indices. The notation is ideally suited for clarifying the issues around the gradient flows we discuss, at least in finite dimensions. For example, the derivatives of a function f form a covector and should be written with lower indices f_i . The derivative of f in the direction $\xi = \{\xi^i\}$ (a vector) is then $f_i \xi^i$. If we try to write the gradient flow (896) we get $\dot{x}^i = -f_i$, which does not look good, as we mix upper and lower indices. We first need to raise the indices of f_i (or lower the indices of \dot{x}^i). For that we need a metric g_{ij} and its covariant form g^{ij} . Then we write $\dot{x}^i = g^{ij} f_j$. Similarly, to say that a matrix a_j^i is symmetric, we need to lower the upper index (or raise the lower index) by a metric g_{ij} by $a_{ij} = g_{ik} a_j^k$. The dependence of these notion on a metric is transparent. If one fixes some orthogonal frame and takes all coordinates with respect to that frame, then we do not have to distinguish between the upper and lower indices, as long as we remember that such notation assumes a particular choice of the frame.

scalar product defined as above by some positive definite G ? As an exercise you can prove that a necessary and sufficient condition on A to satisfy (903) for some positive definite G is that A be diagonalizable over the reals.

The more general situation of (897) is similar: if the matrix ∇b is not symmetric, it can still be the case that the equation is a gradient flow with respect to some more general scalar product.²⁰⁶

The above considerations show the importance of the scalar product when dealing with a gradient flow.

The importance of the scalar product for gradient flows also becomes immediately apparent when we try to define a gradient flow of a functional on an abstract linear space, where we are not distracted by existing canonical structures. Let X be a linear space over \mathbb{R} , perhaps with some norm or topology compatible with the linear structure, and let $f: C \rightarrow \mathbb{R}$ be a continuous function. Assume that the derivative $f'(x)$ is well-defined for each $x \in X$ by

$$f'(x)y = \left. \frac{d}{dt} \right|_{t=0} f(x + ty), \quad y \in X \quad (904)$$

and that the map $y \rightarrow f'(x)y$ is linear (and continuous). To define a gradient flow of f , we would need to associate to each $x \in X$ some $b(x) \in X$ related to $f'(x)$. However, the derivative $f'(x)$ naturally belongs to the space X^* of (continuous) linear functionals on X . In general, there is not a canonical way to identify X and X^* , even when the dimension of X is finite.²⁰⁷ In infinite dimensions this is even more transparent. For example, the space $X = L^1(0, 1)$ is separable, whereas its dual $X^* = L^\infty(0, 1)$ is not separable, and hence they cannot be reasonably identified.

To define a gradient flow, we need some mapping

$$T: X^* \rightarrow X. \quad (905)$$

Often the exact domain of T or its exact range do not have to be studied in detail. We can work at a formal level, and establish rigorously only those consequences of the gradient flow structure which we are interested in.

If $T: X^* \rightarrow X$ as above is given, we can define a gradient flow by

$$\dot{x} = Tf'(x). \quad (906)$$

We can use the notation

$$Tf'(x) = \text{grad}_T f(x) \quad (907)$$

and write equation (906) as

$$\dot{x} = -\text{grad}_T f(x). \quad (908)$$

²⁰⁶It can for example be given by a suitable Riemannian metric $g_{ij}(x)$.

²⁰⁷By contrast, there is a canonical way to identify the double dual X^{**} with X without the presence of any additional structures when X is finite-dimensional.

If X is a Hilbert space, then X^* can be identified with X by the scalar product and a canonical $T: X^* \rightarrow X$ is given the Riesz representation theorem for the linear functionals on Hilbert spaces: each $l \in X$ is uniquely represented by a $Tl \in X$ via $l(x) = (x, Tl)$.

Sometimes one can just formally work with a scalar product, without worrying about completeness, and obtain the equations at a formal level, using the gradient flow structure only as an important guidance to a rigorous treatment.

Let now return to the heat equation. For now we will work at a formal level with functions defined on \mathbb{R}^n and various scalar products. The functions are assumed to be “sufficiently regular” and have “sufficiently fast decay” (sometimes together with their derivatives), so that all the operations below are allowed.

Let us first take the functional

$$J(u) = \int_{\mathbb{R}^n} \frac{1}{2} |\nabla u|^2 dx \quad (909)$$

and the standard L^2 - product

$$(u, v) = \int_{\mathbb{R}^n} uv dx. \quad (910)$$

In this case we have, as we have seen during the first semester,

$$J'(u)v = \int_{\mathbb{R}^n} \nabla u \nabla v dx = \int_{\mathbb{R}^n} -\Delta u v dx, \quad (911)$$

and we see that with these choices equation (908) becomes exactly the heat equation

$$u_t = \Delta u. \quad (912)$$

Estimate (899) becomes

$$\int_{\mathbb{R}^n} \frac{1}{2} |\nabla u(x, t_1)|^2 dx - \int_{\mathbb{R}^n} \frac{1}{2} |\nabla u(x, t_2)|^2 dx = \int_{t_1}^{t_2} \int_{\mathbb{R}^n} |u_t(x, t)|^2 dx dt, \quad (913)$$

showing that we should have

$$\int_0^\infty \int_{\mathbb{R}^n} u_t^2 dx dt = \int_{\mathbb{R}^n} \frac{1}{2} |\nabla u(x, 0)|^2 dx. \quad (914)$$

This can be obtained directly by multiplying (912) by u_t and integrating over x and t , using suitable integration by parts. Estimate (900) becomes

$$\int_{\mathbb{R}^n} \frac{1}{2} |\nabla u(x, t_1)|^2 dx - \int_{\mathbb{R}^n} \frac{1}{2} |\nabla u(x, t_2)|^2 dx = \int_{t_1}^{t_2} \int_{\mathbb{R}^n} |\Delta u(x, t)|^2 dx dt. \quad (915)$$

The above is not the only way to represent the heat equation as a gradient flow. There are many other ways. For example, a simple variation of the above is the following. Let us consider

$$J_0(u) = \int_{\mathbb{R}^n} \frac{1}{2} |u|^2 dx, \quad (916)$$

and the scalar product

$$(u, v)_{-1} = \int_{\mathbb{R}^n} [(-\Delta)^{(-1)}u]v dx = \frac{1}{(n-2)|S^{n-1}|} \int_{\mathbb{R}^n} \frac{u(x)v(y)}{|x-y|^{n-2}} dx dy, \quad (917)$$

where we assume $n \geq 3$ for simplicity. Using the Fourier transform, we can also write

$$(u, v)_{-1} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\hat{u}(\xi)\overline{\hat{v}(\xi)}}{|\xi|^2} d\xi. \quad (918)$$

We have

$$J'_0(u)v = \int_{\mathbb{R}^n} uv dx = \int_{\mathbb{R}^n} [(-\Delta)^{(-1)}(-\Delta)u]v dx, \quad (919)$$

showing that the gradient flow of J_0 with respect to the (\cdot, \cdot) scalar product is again the heat equation (912). Equation (899) now becomes

$$\int_{t_1}^{t_2} \|u_t(t)\|_{-1}^2 dt = \int_{\mathbb{R}^n} \frac{1}{2} |u(x, t_1)|^2 dx - \int_{\mathbb{R}^n} \frac{1}{2} |u(x, t_2)|^2 dx, \quad (920)$$

where we use the notation $\|u_t\|_{-1}^2 = (u_t, u_t)_{-1}$. Equation (900) becomes

$$\int_{\mathbb{R}^n} \frac{1}{2} |u(x, t_1)|^2 dx - \int_{\mathbb{R}^n} \frac{1}{2} |u(x, t_2)|^2 dx = \int_{t_1}^{t_2} \int_{\mathbb{R}^n} |\nabla u(x, t)|^2 dx dt. \quad (921)$$

This can be also obtained directly by multiplying (912) by u and integrating over x and t , using suitable integration by parts.

This is the basic “energy estimate” for the heat equation. Important parts of the theory of linear parabolic equations²⁰⁹ can be based on it and its localized versions.²¹⁰ The situation is quite similar to the elliptic case we studied in the first semester.

²⁰⁸The constant $\frac{1}{(2\pi)^n}$ in front of the integral is unimportant for most purposes, as is often omitted from this definition. In this case a constant appears in (917).

²⁰⁹including equations with variable coefficients

²¹⁰This includes existence of weak solutions, their interior and boundary regularity, etc.

The energy inequality (921) we derived last time for the heat equation in \mathbb{R}^n can also be derived, in the same way, for the heat equation in a domain Ω with natural boundary conditions at the boundary $\partial\Omega$. In this case the unknown function is defined on $\Omega \times [t_1, t_2]$ and we can consider various boundary conditions, such as the Dirichlet boundary conditions $u|_{\partial\Omega} = 0$, the Neumann boundary conditions $\frac{\partial u}{\partial \nu} = 0$, or the boundary condition $\frac{\partial u}{\partial \nu} + \gamma u = 0$. The meaning of these condition is the same as discussed in the elliptic case in lecture 30. The conditions can also be taken non-homogeneous, i. e. replacing 0 on the right-hand side by a known function g . In that case we have to add additional terms to the energy inequality, of course, as is also the case when the equations is inhomogeneous, $u_t - \Delta u = f$, with f a known function. We will get to the non-homogeneous cases later.

An important consequence of the energy inequality is the uniqueness of natural initial-boundary-value problems, such as

$$\begin{aligned} u_t - \Delta u &= f(x, t), & (x, t) \in \Omega \times (0, \infty), \\ u|_{\partial\Omega} &= g(x, t), & (x, t) \in \partial\Omega \times (0, \infty), \\ u(x, 0) &= u_0(x), & x \in \Omega, \end{aligned} \quad (922)$$

or various other versions where the Dirichlet condition $u|_{\partial\Omega} = g(x, t)$ is replaced by one of the other conditions mentioned above, or with different conditions of this type imposed on different parts of the boundary. Let us first consider the uniqueness of the problem (922) in the class of “sufficiently regular” solutions, by which we mean the case when the solution is assumed to have differentiability properties with which the proof of the energy inequality is immediate. For example, if the time derivative u_t and the second space derivatives $\nabla^2 u$ are continuous in $\bar{\Omega} \times [t_1, t_2]$, then multiplying the equation by u_1 and integrating by parts presents no difficulty, and (921) is easy to obtain. This immediately gives the uniqueness statement.

The initial-boundary-value problem (922) has at most one solution in the class of “sufficiently regular” functions u .

For the proof we note that the difference of any two solution satisfies the homogeneous problem, with $f = 0, g = 0$ and $u_0 = 0$, and (921) immediately implies that the solution of the homogeneous problem has to vanish identically.

With a small modification, the energy inequality also works when the laplacian in (922) is replaced by operators

$$Lu = -\frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right) + b_j \frac{\partial u}{\partial x_j} - \frac{\partial}{\partial x_j} (\tilde{b}_j u) + cu \quad (923)$$

we studied in the first semester, where the coefficients can now depend on both x and t , so that we have $a_{ij} = a_{ij}(x, t), b_j = b_j(x, t)$, etc. The key assumption again is the ellipticity condition

$$a_{ij} \xi_i \xi_j \geq \nu |\xi|^2, \quad (924)$$

and the boundedness of a_{ij} . We will assume that the other coefficients are also bounded, although this assumption can be relaxed.

The energy inequality is generalized to the equation

$$u_t + Lu = 0 \tag{925}$$

in the following way. Consider a solution of (925) satisfying the homogeneous boundary conditions on $\partial\Omega \times (0, \infty)$. Assume the solution is sufficiently regular. Multiply (925) by u and integrate over Ω . We obtain, in terms of the Lax-Milgram form $A(u, v)$ corresponding to L , as discussed in Lecture 30,

$$\frac{d}{dt} \int_{\Omega} \frac{1}{2} |u(x, t)|^2 dx + A(u, u) = 0. \tag{926}$$

Recalling from lecture 31 that the form $A_{\lambda}(u, u) = A(u, u) + \lambda(u, u)$ is coercive for sufficiently large λ , we write (926) as

$$\frac{d}{dt} \int_{\Omega} \frac{1}{2} |u(x, t)|^2 dx + A_{\lambda}(u, u) = 2\lambda \int_{\Omega} \frac{1}{2} |u(x, t)|^2 dx. \tag{927}$$

The coercivity of A_{λ} now implies

$$\frac{d}{dt} \int_{\Omega} \frac{1}{2} |u(x, t)|^2 dx + \frac{\nu}{2} \int_{\Omega} |\nabla u(x, t)|^2 dx \leq 2\lambda \int_{\Omega} \frac{1}{2} |u(x, t)|^2 dx. \tag{928}$$

This can be rewritten as

$$\frac{d}{dt} \int_{\Omega} \frac{1}{2} e^{-2\lambda t} |u(x, t)|^2 dx + \frac{\nu}{2} \int_{\Omega} e^{-2\lambda t} |\nabla u(x, t)|^2 dx \leq 0. \tag{929}$$

Hence

$$\int_{\Omega} e^{-2\lambda t_2} \frac{1}{2} |u(x, t_2)|^2 dx + \frac{\nu}{2} \int_{t_1}^{t_2} \int_{\Omega} e^{-2\lambda t} |\nabla u(x, t)|^2 dx dt \leq \int_{\Omega} e^{-2\lambda t_1} \frac{1}{2} |u(x, t_1)|^2 dx, \tag{930}$$

which for many purposes is as good as the identity (921) we get for the simple heat equation. For example, it immediately implied uniqueness for many natural initial-boundary-value problems for equation (925), at least for sufficiently regular solutions.

The assumption that the solution be “sufficiently regular” which we work with at the moment is somewhat unpleasant and should be replaced by a condition that u belongs to some natural function space suggested by (930). We will see that that is indeed possible, although the proofs are not completely straightforward. However, before investigating optimal uniqueness statements, it is useful to see what can be proved about the existence of the solutions. Our goal is to find some natural set-up in which both existence and uniqueness can be proved.

There is more than one way to approach the existence results. Our approach here will be through the Galerkin method, which is also suitable for numerical integration of the equations.

To show the main idea of the method, we will consider the initial value problem (922) with $-\Delta$ replaced by L and $f = 0$ and $g = 0$. We first reformulate the problem in the following way: we multiply the equation by $v \in X = H_0^1$ and integrate over x . After an integration by parts, we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} u(x, t) v(x) dx + A(u(\cdot, t), v) &= 0, & v \in X, \quad t \geq 0 \\ \int_{\Omega} u(x, 0) v(x) dx &= \int_{\Omega} u_0(x) v(x) dx, & v \in X. \end{aligned} \quad (931)$$

If we consider other boundary conditions, the space $X = H_0^1$ can be replaced by other spaces $X \subset W^{1,2}$ we considered in connection with elliptic problems in lecture 30.

In (931) we can view the scalar product $\int uv$ as a coordinate of u , and the equations as equations for coordinates of u . In fact, the requirement that the equations are satisfied for each v from some orthogonal basis v_1, v_2, \dots is equivalent to re-writing our equations in the coordinates of this basis.

If we now replace the space X by a finite dimensional space $Y \subset X$ and seek a function $t \rightarrow u(t) \in Y$, the system (931) becomes just a finite-dimensional ODE system. (The requirement that (931) be satisfied for each $v \in Y$ is the same as the requirement that (931) is satisfied for each element v_j of a basis v_1, \dots, v_m of Y .) By general theory of ODE we know that the system has a unique local-in-time solution in some interval $(0, T)$. In fact, since our system is linear and the coefficients do not “blow up”, the solution is global. The finite-dimensional system still satisfies energy estimate (930). To see that, we can write the finite dimensional system as

$$\begin{aligned} \int_{\Omega} \frac{d}{dt} u(x, t) v(x) dx + A(u(\cdot, t), v) &= 0, & v \in Y, \quad t \geq 0 \\ \int_{\Omega} u(x, 0) v(x) dx &= \int_{\Omega} u_0(x) v(x) dx, & v \in Y. \end{aligned} \quad (932)$$

and note that for each t^{211} the equations are true with $v(x) = u(x, t)$, as we now require that $u(\cdot, t) \in Y$ for each t . In other words, we can use (932) with $v(x)$ replaced by $u(x, t)$. Once we have this conclusion, we just follow the proof of (930).

The estimate (930) gives us another way of seeing that the local solution of our ODE system is in fact global. The only way it could not be global is that $u(t)$ would become unbounded in Y on some open interval $(0, T)$, but this is not possible due to the energy estimate.²¹²

²¹¹Strictly speaking, in the coefficients of L are only measurable, we have the pointwise equation only for almost every t . The ODE theory with the Caratheodory assumptions still works. We leave a verification of this to the reader as an exercise.

²¹²Note that this argument does not rely on the linearity of the equations, and can be used also for non-linear equations.

If we check what assumptions on the coefficients of L are needed so that the ODE system is covered by the standard ODE theory using the Caratheodory conditions, we see that it is sufficient to assume that all the coefficients are bounded measurable. If we wish to get the energy estimate (930), we need to assume also the ellipticity condition (924).

We see that it is easy to obtain global solution for the finite dimensional approximations (932). Next we need to establish the convergence of these approximations to some solution of the original problem, if the finite dimensional spaces in some sense approximate the space X . This is not hard and by this limiting procedure we get a weak solution of our original problem. We will discuss the details next time, but even now we see that the overall strategy is somewhat similar to the elliptic case we studied last semester. We first establish the existence of weak solution. Next establish their uniqueness, which will be somewhat more difficult, as we do not really want to assume that the solutions are “sufficiently regular”: we need to establish uniqueness in the same class of functions where we can prove existence. When we have both existence and uniqueness, we know that we are working with the right objects. Finally, we can establish the regularity of the solutions, in a way quite similar to what we did in the first semester for the elliptic case: using a local version of the energy inequality for the derivatives of the solution. We will not enter into the regularity theory in much detail, as it is quite similar to the elliptic case we studied last semester.

Last time we considered the finite-dimensional system (932) and we saw that for each subspace $Y \subset X$ the system has a unique global solution. We now choose a sequence of spaces $Y_m \subset X$, $m = 1, 2, \dots$ which in some sense “converge to X ”. For each m we obtain a solution $u_m :: t \rightarrow u_m(t) \in Y_m$, so we have a sequence of functions $u_m = u_m(x, t)$ which satisfy

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} u_m(x, t) v(x) dx + A(u_m(\cdot, t), v) &= 0, & v \in Y_m, \quad t \geq 0, \\ \int_{\Omega} u_m(x, 0) v(x) dx &= \int_{\Omega} u_0(x) v(x) dx, & v \in Y_m. \end{aligned} \quad (933)$$

The functions $u_m(x, t)$ are of the form

$$u_m(x, t) = \sum_{j=1}^{j=m'} u_j^m(t) v_j^m(x), \quad (934)$$

where $v_1^m, \dots, v_{m'}^m$ is a basis of Y_m , with $m' = m'(m) = \dim Y_m$. Letting

$$b_{ij}^m = \int_{\Omega} v_i^m(x) v_j^m(x) dx, \quad (935)$$

we see that

$$\int_{\Omega} u_m(x, t) v_j(x) dx = \sum_i u_i^m(t) b_{ij}^m. \quad (936)$$

We note that the matrix b_{ij}^m is invertible, as $v_1, \dots, v_{m'}$ form a basis. Therefore the first equation in (933) is of the form

$$\frac{d}{dt} u_i^m = \sum_j \tilde{A}_{ij}^m(t) u_j^m, \quad (937)$$

with the functions \tilde{A}_{ij}^m bounded and measurable. Hence the system (933) is globally solvable, as discussed last time.

The initial condition $u_m(x, 0) \in Y_m$ is given by the second equation of (933). Note that $u_m(x, 0)$ is simply the $L^2(\Omega)$ orthogonal projection of u_0 onto the space Y_m . This means that

$$\int_{\Omega} |u_m(x, 0)|^2 dx \leq \int_{\Omega} |u_0(x)|^2 dx. \quad (938)$$

The functions $u_m(x, t)$ satisfy the energy estimate (930). Therefore, for any given $T > 0$, the functions $u_m(x, t)$ are uniformly bounded in the spaces $L^\infty((0, T), L^2(\Omega))$ and $L^2((0, T), W^{1,2}(\Omega))$. In particular, the functions are uniformly bounded in $L^2(\Omega \times (0, T))$, and we can choose a subsequence, still denoted by u_m , which

converges weakly in $L^2(\Omega \times (0, T))$ to a function $u \in L^2(\Omega \times (0, T))$. Due to the compactness of bounded sets in the various weak topologies, the subsequence u_m will also converge to u weakly* in $L^\infty((0, T), L^2(\Omega))$ and weakly in $L^2((0, T), W^{1,2}(\Omega))$.

It is clear that the function u should in some sense solve the original PDE problem, provided the spaces V_m in some sense “converge” to the space X . A obvious necessary condition is the following: For each $v \in X$ there exists a sequence $v_m \in Y_m$ such that $v_m \rightarrow v$ in $W^{1,2}(\Omega)$. We will see that this condition is also sufficient if we also assume that $Y_1 \subset Y_2 \subset \dots$, which is a quite natural assumption.

It is easy to define a notion of *weak solution* of the original initial-boundary-value problem such that the function u will be a solution. We will discuss it in more detail shortly, for now we can just say that the requirement is, roughly speaking, that u belongs to the spaces suggested by the energy inequality and (931) is satisfied if the t -derivative is taken in the sense of distributions. There are in fact several ways in which the exact definition can be made which are not transparently equivalent.

A key point for a good definition of a weak solution is *uniqueness*. We expect that the solutions of the evolution problems we study here should be unique, and if we cannot prove uniqueness for our weak solutions, our theory cannot be considered as satisfactory. As usual, the definition of the weak solution has to strike the right balance between the existence and uniqueness requirements. For example, the existence of the weak solutions based on the Galerkin approximation as above is quite easy to prove. Once we know that the energy inequality is satisfied for our approximate solutions, we just take the weak limits of the subsequences of our approximate solutions. In one step we establish global-in-time solutions, even for coefficients which are quite irregular, and not so easy to approach by other methods. However, note that in this procedure we lose a lot of information about the solutions. First, even if all our data are smooth, in the limiting procedure we only use the energy estimate (930) and the weak form of the equation. All the other information is lost (unless we make an effort to track what happens to it in the limiting procedure, which is often not easy). Therefore, in some sense, the relative ease with which the global existence of the solutions is established comes at a price: our information about the solution is quite weak, and we have to work harder to establish uniqueness. We do not have enough regularity to prove uniqueness by a simple application of the energy inequality, as we were able to do for “sufficiently regular” solutions. Any information about the smoothness of the solutions beyond the energy estimates is lost (unless we track it during the limiting procedures, which – as already remarked – can be nontrivial), and if we wish to recover it, we have to work on the regularity theory, similarly to the elliptic case in the first semester. ²¹³

²¹³In the case when the coefficients are merely measurable, it is not a-priori obvious how much regularity beyond the energy estimates and – for scalar equations – simple consequences of the maximum principle one should really expect. Questions in this direction are quite difficult and were resolved only in the 1960s.

If we compare the situation with the elliptic theory we studied in the first semester, we see that the situation is quite similar, except that in the elliptic theory the issue of uniqueness was an “automatic” part of the existence theory: the solution given by a coercive Lax-Milgram form or by a minimizer of a convex functional is clearly unique. In the parabolic theory the issue of uniqueness of weak solutions is more subtle and has to be considered with some care. On the other hand, the regularity theory is quite similar in both cases.

Note that the issue of uniqueness is also related to the following questions: when we pass to the limit $m \rightarrow \infty$ for our Galerkin approximations u_m , do we have to take a subsequence? If we can show the uniqueness for the weak solutions, and we know that the limit of any subsequence is a weak solution, then by uniqueness we see that the limit has always be the same, and the whole sequence u_m actually converges to the weak solution, without the need to take subsequences. Establishing this directly is non-trivial.

Let us now formulate the notion of a weak solution for the initial-boundary-value problem

$$\begin{aligned} u_t + Lu &= 0, & \text{in } \Omega \times (0, T), \\ u|_{\partial\Omega} &= 0, & \text{in } \partial\Omega \times (0, T), \\ u(x, 0) &= u_0(x), & \text{in } \Omega, \end{aligned} \quad (939)$$

where L is the operator (924). We take the homogeneous right-hand side and the homogeneous Dirichlet boundary conditions for simplicity, but it is easy to check that the same method works in more general situations with only minor modifications, including non-zero right-hand sides, and all the types of non-homogeneous boundary conditions we have considered for the elliptic problems in the first semester.²¹⁴ To emphasize that the method is not tied to the Dirichlet boundary condition, we will keep using the notation with the space $X \subset W^{1,2}(\Omega)$ from the last lecture. For the homogeneous Dirichlet boundary condition we of course take $X = H_0^1(\Omega)$.

The definition of a weak solution essentially consists of three different types of requirements. The first requirement is that u belongs to a suitable function space. The second requirement is that some weak form of the equations is satisfied. And the third requirement is that the initial condition and the boundary conditions are satisfied in a suitable weak sense. These conditions can be intertwined. For example, the condition

$$u \in L^2(0, T; X) \quad (940)$$

for $X = H_0^1(\Omega)$ implies that the homogeneous Dirichlet boundary condition is satisfied at the lateral boundary. (On the other hand, when we deal with the Neumann boundary condition, we take $X = W^{1,2}(\Omega)$, which by itself does not impose any boundary condition. In this case the boundary condition will be imposed through the weak form of the equation and the freedom to choose the test functions from $W^{1,2}(\Omega)$, as it was the case in the elliptic case.)

In view of the energy inequality (930), all the function spaces considered in this context are usually contained in the space $L^2(0, T; X)$

To obtain a weak form of the equation we take a smooth test function $\varphi(x, t)$ defined in $\bar{\Omega} \times [0, T)$ such that $\varphi(\cdot, t)$ belongs to X for each t , and that φ is supported in $\bar{\Omega} \times [0, T - \delta]$ for some $\delta > 0$. In particular, $\varphi(x, T) = 0$. Such

²¹⁴In fact, in order to understand how to transfer the various boundary conditions we considered for the elliptic problem to the correct boundary conditions on the lateral boundary $\partial\Omega \times (0, T)$ in (939), it is often useful to consider the time derivative u_t just as a right-hand side to in the elliptic problem $Lu = -u_t$ at the corresponding time level. This interpretation, in spite of appearing quite naive and perhaps even problematic, can be surprisingly useful, although one has to be somewhat careful with it.

functions will be called *smooth admissible test functions*. Multiplying the first equation in (939) by φ and (formally) integrating by parts, we obtain

$$-\int_{\Omega} u_0(x)\varphi(x,0) dx - \int_0^T \int_{\Omega} u\varphi_t + \int_0^T A(u(\cdot,t), \varphi(\cdot,t)) dt = 0, \quad (941)$$

where A is the Lax-Milgram form (403) corresponding to L . Note that (941) takes into account the initial condition $u(x,0) = u_0(x)$, and the choice of the test functions reflects the boundary condition at the lateral boundary $\partial\Omega \times (0,T)$.

Definition

A weak solution of the initial-boundary-value problem (939) is a function $u \in L^2(0,T; X)$ which satisfies (941) for each smooth admissible test function φ .²¹⁵

This is, in some sense, a “bare minimum” definition of the weak solutions, if we wish to phrase the definition in terms of the Lax-Milgram form. The solutions which we construct through the Galerkin method will have additional properties to those stated in the definition. For example, if $u_0 \in L^2(\Omega)$, we see from the energy inequality for the Galerkin approximations u_m implies that the functions u_m are uniformly bounded in $L^\infty(0,T; L^2(\Omega))$, and we expect the same from the limit. Also, we see that by a simple density argument we can extend the class of φ for which (941) holds from smooth functions (with some additional properties) to functions with distributional derivatives in $L^2_{x,t}$ (and some additional properties). We could try to collect all such possible improvements, and put them in the definition of the weak solutions as a requirement.²¹⁶ The rationale would be that if we know that the expected solutions have a certain property, why not put the property in the definition? It then obviously restricts the class of the solutions, and reduces the chance that our definition will allow some unwelcome parasitic solutions. Such an approach is reasonable as long as our definition will cover some natural classes of equations, such as the equations with bounded measurable coefficients (+ the ellipticity assumption (924)). This is why one can find in the literature various versions of the definitions, differing in how many “known properties” of general classes of solutions are incorporated into the definition. The most important test of the definition is whether we can prove existence and uniqueness of the solutions. As we already discussed, the more requirements we put in the definition, the more difficult it is to prove the existence, but the easier it is to prove uniqueness. Vice versa, the fewer

²¹⁵In case the domain Ω is unbounded (such as $\Omega = \mathbb{R}^n$) and the space X is not contained in $L^2(\Omega)$, we can make our life easier by adding the condition $u \in L^2(\Omega \times (0,T))$, or simply replace X by a space containing $L^2(\Omega)$. For example, we can replace $X = H_0^1(\Omega)$ by $X = W_0^{1,2}(\Omega)$. If we have terms such as cuv or $b_j \frac{\partial u}{\partial x_j} v$ in the Lax-Milgram form the requirement $L^2(\Omega) \subset X$ is in fact a necessity. If the requirement $u \in L^2(\Omega \times (0,T))$ is not immediately satisfied from the definitions, we have to do some more work in the uniqueness proof. Here we will focus on bounded domains and will not analyze these issues, which may arise for unbounded domains.

²¹⁶Taking this trend to the extreme, in case of smooth data, we could require that the solution be smooth. After all, in the end it is possible to prove the existence of a smooth solution in this case.

assumptions we impose, the easier is to prove the existence, but the harder it is to prove uniqueness.

With the definition above, it is quite easy to prove the existence when $u_0 \in L^2(\Omega)$ ²¹⁷, but we will have to work a little harder to prove uniqueness. The important thing is that uniqueness can be proved. That is why the definition is reasonable.

Let us now turn to the existence proof. In the last lecture we have constructed a sequence of approximate solutions u_m , and it is easy to see that u_m satisfy

$$-\int_{\Omega} u_0(x) \varphi_m(x, 0) dx - \int_0^T \int_{\Omega} u_m \varphi_{mt} + \int_0^T \int_{\Omega} A(u_m(\cdot, t), \varphi_m(\cdot, t)) dt = 0, \quad (942)$$

for each test admissible function $\varphi_m(x, t)$ which is smooth in t and has the additional property that the function $x \rightarrow \varphi_m(x, t)$ belongs to the space Y_m for each $t \in [0, T)$. By the energy inequality for the approximate solutions, the sequence u_m is bounded in $L^2(0, T; X)$. We can choose a subsequence of the sequence u_m , still denoted by u_m , such that u_m converges weakly in $L^2(\Omega \times (0, T))$ to a function $u \in L^2(\Omega \times (0, T))$. By elementary functional analysis we also have that u_m converges weakly to u in $L^2(0, T; X)$. Let φ be an arbitrary smooth admissible test function. We claim that we can find a sequence of admissible test functions φ_m which are smooth in t , the functions $x \rightarrow \varphi_m(x, t)$ belong to V_m for each t , $\varphi_m(x, 0)$ converge weakly to $\varphi(x, 0)$ in $L^2(\Omega)$, and φ_m converge to φ strongly in $L^2(0, T; X)$. Let us for the moment take the existence of such an approximation as proven. Then it is easy to finish the proof: we just pass to the limit in (942), and obtain (941). The only fact we need to check for this limiting procedure that the following statement: if w_m converge to w weakly in $L^2(\Omega \times (0, T))$ and θ_m converge strongly to θ in $L^2(\Omega \times (0, T))$, then

$$\int_0^T \int_{\Omega} w_m \theta_m dx dt \rightarrow \int_0^T \int_{\Omega} w \theta dx dt, \quad m \rightarrow \infty. \quad (943)$$

We leave this to the reader as an exercise.²¹⁸

To finish the proof, we need to justify the strong approximation of φ by the sequence φ_m with the required properties. This is not hard. First, we note that we can approximate φ by finite sums of the form $\sum_j \psi_j(x) \eta_j(t)$, with smooth $\psi_j \in X$ and smooth $\eta_j: [0, T) \rightarrow \mathbb{R}$ vanishing close to T .²¹⁹ Given such an

²¹⁷This assumption is reasonable for bounded domains, but in case of unbounded domains, such as $\Omega = \mathbb{R}^n$ it may be too restrictive in some situations. One can work with local versions of the energy inequality to remedy that. We will not study this aspect in this course.

²¹⁸As an additional exercise, the reader can construct examples showing that for the passage to the limit in (943) it is not enough to assume that both convergencies $w_m \rightarrow w$ and $\theta_m \rightarrow \theta$ are only weak.

²¹⁹Strictly speaking, the existence of such an approximation may not be immediately obvious, but it follows from standard results, such as the Stone-Weierstrass approximation theorem.

approximation, we can replace each ψ_j by its approximation $\psi_{mj} \in Y_m$. and take

$$\varphi_m(x, t) = \sum_j \psi_{mj} \eta_m(t). \quad (944)$$

This finishes our existence proof. We can summarize what we proved as follows:

Theorem

If $u_0 \in L^2(\Omega)$, then the initial-boundary-value problem (939) has at least one weak solution.

Remarks:

1. As we have already indicated, the theorem remains true for non-zero right-hand side²²⁰ and for all the types of boundary conditions at $\partial\Omega \times (0, T)$ which we considered for the elliptic problems in the first semester.
2. Note that for a general function $v \in L^2(0, T; X)$ the “initial condition” $v(x, 0)$ is not well-defined. To see this, consider simply functions constant in x , i. e. $v(x, t) = f(t)$. Such a function belongs to $L^2(0, T; X)$ if and only if $f \in L^2(0, T)$. However, for such f the value $f(0)$ is in general not well-defined. We can make sense of the initial condition $u(x, 0) = u_0(x)$ only because u is not a general function in $L^2(0, T; X)$, but it satisfies (941), the weak form of the equation. The equation implies that for any $\psi \in X$ ²²¹ the function

$$f_\psi(t) = \int_\Omega u(x, t) \psi(x) dx \quad (945)$$

has a distributional derivative in $L^2(0, T)$. (To see that, it is enough to take $\varphi(x, t) = \psi(x) \eta(t)$ in (941).) In particular, the function is uniformly continuous, and therefore the values $f_\psi(t)$ are well-defined for each $t \in [0, T]$. We see that $x \rightarrow u(x, t)$ is defined, at least as a distribution, for *every* t , not just almost every t . This is certainly not true for general elements $v \in L^2(0, T; X)$. The equation provides an additional regularity. This additional regularity will also be used in our proof of uniqueness (together with the energy inequality).

²²⁰The natural form of the right-hand side in the context of the energy inequality (930) is $f + \frac{\partial f_j}{\partial x_j}$ for with $f, f_j \in L^2(\Omega \times (0, T))$.

²²¹Strictly speaking, we should first say “for any smooth ψ ”, and then add that the case of general $\psi \in X$ is justified by suitable approximations.

Lecture 60, 3/2/2001

We will be looking in more detail at the function $u \in L^2(0, T; X)$ satisfying

$$\int_0^T \int_{\Omega} u(-\varphi_t) + a_i(x, t)\varphi_{x_i} dx dt = 0 \quad (946)$$

for certain $a_i \in L^2(\Omega \times (0, T))$ and each smooth *admissible* $\varphi(x, t)$. In a slight deviation from the last lecture, this time we will say that φ is admissible if $\varphi(\cdot, t)$ belongs to X for each t and vanishes for t outside a compact subset of $(0, T)$. We assume that Ω is bounded, and therefore we have $L^2(0, T; X) \subset L^2(\Omega \times (0, T))$.

We note that by a simple approximation argument we can infer that (967) is true not only for the smooth admissible test functions, but also for all admissible test functions with the distributional derivatives φ_t, φ_{x_i} in L^2 .

Equation (967) implies that

$$\frac{\partial u}{\partial t} = \frac{\partial a_i}{\partial x_i} \quad (947)$$

in $\Omega \times (0, T)$ in the sense of distributions. As we assume that $a_i \in L^2(\Omega \times (0, T))$, the information from (968) is sometimes expressed as

$$u_t \in L^2(0, T; H^{-1}(\Omega)). \quad (948)$$

Such notation is commonly used in the literature, and therefore we explain it in some detail. First we define the space $H^{-1}(\Omega)$ as the space of all distributions $v \in \mathcal{D}'(\Omega)$ such that

$$\|v\|_{H^{-1}(\Omega)} = \sup_{\substack{\varphi \in \mathcal{D}(\Omega) \\ \|\varphi\|_{H_0^1(\Omega)} \leq 1}} \langle v, \varphi \rangle < +\infty. \quad (949)$$

Clearly $H^{-1}(\Omega)$ is one possible representation of the dual space of the space $H_0^1(\Omega)$. Any $l \in H^{-1}(\Omega)$ can be identified with an element $u \in H_0^1(\Omega)$ via the scalar product in $H_0^1(\Omega)$:

$$\int_{\Omega} \nabla u \nabla v = l(v), \quad v \in H_0^1(\Omega). \quad (950)$$

As we saw in the first semester, this correspond to $(-\Delta)^{-1}l = u$, where the laplacian is inverted with the boundary condition $u|_{\partial\Omega} = 0$. For example, if $b(x) = (b_1(x), \dots, b_n(x))$ is an L^2 -vector field in Ω , then $\operatorname{div} b \in H^{-1}(\Omega)$, with

$$\|\operatorname{div} b\|_{H^{-1}(\Omega)} \leq \|b\|_{L^2(\Omega)}. \quad (951)$$

If we have a vector field $(a_1(x, t), \dots, a_n(x, t))$ in $L^2(0, T; X)$, it makes sense to write

$$\frac{\partial a_i}{\partial x_i} \in L^2(0, T; H^{-1}(\Omega)). \quad (952)$$

In view of (968), we can therefore write (970).²²²

Note that the spaces $L^2(0, T; H_0^1(\Omega))$ and $L^2(0, T; H^{-1}(\Omega))$ are “in duality”, and if $u \in L^2(0, T; H_0^1(\Omega))$ with $u_t \in L^2(0, T; H^{-1}(\Omega))$, then the expression $\langle u_t(t), u(t) \rangle$ is an integrable function of t . This makes it plausible that the formula

$$\int_{\Omega} \frac{1}{2} |u(x, t_2)|^2 dx - \int_{\Omega} \frac{1}{2} |u(x, t_1)|^2 dx = \int_{t_1}^{t_2} \int_{\Omega} u_t u dx dt \quad (953)$$

is valid (in the right interpretation), and the function $t \rightarrow u(\cdot, t)$ is uniformly continuous in as a map from $[0, T]$ into $L^2(\Omega)$. We will see that this is indeed the case. We will prove (953) in the form

$$\int_{\Omega} \frac{1}{2} |u(x, t_2)|^2 dx - \int_{\Omega} \frac{1}{2} |u(x, t_1)|^2 dx = \int_{t_1}^{t_2} \int_{\Omega} -u_{x_i} a_i dx dt \quad (954)$$

directly from (967). This will include showing that the quantities on the left-hand side are well-defined for each $t_1, t_2 \in [0, T]$ (and, in fact, in $[0, T]$).

We will prove (953) by using suitable approximations for which it is transparently true.²²³

For $\varepsilon > 0$ we consider a mollifiers $\phi_{\varepsilon} = \frac{1}{\varepsilon} \phi(\frac{x}{\varepsilon})$ on the real line \mathbb{R} , where $\phi: \mathbb{R} \rightarrow \mathbb{R}$ is smooth, compactly supported and $\int_{\mathbb{R}} \phi = 1$. The functions ϕ is usually taken to be symmetric about the origin, but for our purposes here it is more convenient to take a ϕ which is supported in $(-1, 0)$. For a locally function $v: \Omega \times (0, T) \rightarrow \mathbb{R}$ we define its smoothing in time $v_{\varepsilon}: \Omega \times (0, T - \varepsilon) \rightarrow \mathbb{R}$ by

$$v_{\varepsilon}(x, t) = \int_{\mathbb{R}} v(x, t - s) \phi_{\varepsilon}(s) ds. \quad (955)$$

²²²We have not discussed the exact definition of spaces such as $L^2(0, T; X)$ for some Hilbert space X . Usually one defines $L^2(0, T; X)$ as the closure of the set of X -valued “simple functions” on $(0, T)$ equipped with the obvious norm. (It is a good exercise to check that with this definition we for example have $L^2(0, T; L^2(\Omega)) \sim L^2(\Omega \times (0, T))$). For our purposes here we can also define spaces such as $L^2(0, T; X)$ as completions of suitable spaces of smooth functions in the corresponding norm. For example, for a bounded domain Ω we can define $L^2(0, T; H_0^1(\Omega))$ as the closure of the space of smooth, compactly supported function in $\Omega \times (0, T)$ in the norm $(\int_0^T \int_{\Omega} |\nabla v|^2 dx dt)^{\frac{1}{2}}$, where ∇ denotes the spatial gradient, as usual (as opposed to the full space-time gradient, where we would also include the time derivative). In the case of bounded domains Ω the space $L^2(0, T; H_0^1(\Omega))$ is sometimes also denoted by $W_0^{1,0}(\Omega \times (0, T))$. For unbounded domains the last space usually means $L^2(0, T; W_0^{1,2}(\Omega))$, in which the norm is $(\int_0^T \int_{\Omega} (|\nabla v|^2 + |v|^2) dx dt)^{\frac{1}{2}}$. For bounded domains the two norms are equivalent, due to the weak Poincare inequality (lecture 21). Also, we have not discussed the precise definitions of differentiability of Hilbert-space valued functions. These subtleties will not play an important role for us, as we will work with equation (967), which has a precise meaning without any additional definitions.

²²³We essentially follow the book *Linear and Quasilinear Equations of Parabolic Type* by O. Ladyzhenskaya, V. Solonnikov, and N. Ural'tseva, Section 3.4.

²²⁴As an exercise in the Lebesgue you can verify that this function is well-defined almost everywhere.

Let $u \in L^2(0, T; X)$. Then u_ε belongs to the space $\mathcal{C}([0, T - \varepsilon], X)$ of continuous functions in $[0, T - \varepsilon]$ with values in X . Moreover, if u satisfies (967), then u_ε satisfies (967) with a_i replaced by $a_{i\varepsilon}$ (and φ compactly supported in $(0, T - \varepsilon)$ as a function of t). Let us take $0 < \varepsilon, \varepsilon' < \varepsilon_1$ and set

$$v = u_\varepsilon - u_{\varepsilon'}, \quad b_i = a_{i\varepsilon} - a_{i\varepsilon'}. \quad (956)$$

Then

$$\int_0^T \int_\Omega v(-\varphi_t) + b_i(x, t)\varphi_{x_i} dx dt = 0 \quad (957)$$

for every admissible test function φ supported vanishing for $t > T - \varepsilon_1$. The function v in some sense represents an error term in the approximations u_ε , and (957) can be thought of as an equation describing the propagation of the error term. Working with the error function v , rather than with u_ε , makes things simpler.

Let $\eta: (0, T) \rightarrow \mathbb{R}$ be smooth and compactly supported in $(0, T - \varepsilon_1)$. It is easy to see that

$$\varphi(x, t) = v(x, t)\eta(t) \quad (958)$$

is an admissible test function: all its derivatives $\partial_t, \partial_{x_i}$ are in $L^2(\Omega \times (0, T))$ and the requirements on the support are trivially satisfied. Hence we have

$$\int_0^T \int_\Omega (-vv_t\eta - v^2\eta_t + b_i(x, t)v_{x_i}\eta(t)) dx dt = 0. \quad (959)$$

Using $vv_t = (\frac{1}{2}v^2)_t$ and a simple integration by parts (note that we have enough regularity in t to be able to do this), we have

$$\int_0^T \int_\Omega -\frac{1}{2}v^2\eta'(t) + b_iv_{x_i}\eta(t) dx dt = 0. \quad (960)$$

Recalling that v is continuous in t (with values in $X \subset L^2(\Omega)$), we now choose $0 < t_1 < t_2 < T - \varepsilon_1$ and use (960) with a test function η approximating the following function

$$\bar{\eta}(t) = 0, t \notin [t_1, t_2], \quad \bar{\eta}(t) = \frac{t_2 - t}{t_2 - t_1}, t \in [t_1, t_2]. \quad (961)$$

For example, we can take $\eta = \eta_\delta = \phi_\delta * \bar{\eta}$ and let $\delta \rightarrow 0$. Due to the continuity of v in t we obtain

$$\int_\Omega \frac{1}{2}|v(x, t_1)|^2 dx = \int_{t_1}^T \int_\Omega -b_iv_{x_i}\eta + \frac{1}{2}v^2\bar{\eta}'(t) dx dt. \quad (962)$$

We note that

$$\int_0^T \int_\Omega (|b_i|^2 + |v|^2 + |\nabla v|^2) dx dt \leq \theta(\varepsilon_1), \quad (963)$$

with $\theta(\varepsilon_1) \rightarrow 0$ as $\varepsilon_1 \rightarrow 0$, and therefore (962) and a simple application of Cauchy-Schwartz inequality shows that for each $\delta > 0$ the integrals $\int_{\Omega} \frac{1}{2} |v(x, t_1)|^2 dx$ approach zero uniformly in $t_1 \in [0, T - \delta]$ as $\varepsilon_1 \rightarrow 0$. This means that the functions u_{ε} converge to u uniformly in $\mathcal{C}([0, T - \delta]; L^2(\Omega))$ for each $\delta > 0$. This means

$$u \in \mathcal{C}([0, T]; L^2(\Omega)), \quad (964)$$

which is one of the important consequences of (967). If we now repeat the procedure above with $v = u_{\varepsilon}$, $b = a_{i\varepsilon}$ and $\bar{\eta} = \chi_{(t_1, t_2)}$ and use (964), we obtain (955). Also, note that if we work from T towards $t < T$ rather than from 0 towards $t > 0$, we obtain that in fact

$$u \in \mathcal{C}([0, T]; L^2(\Omega)) \quad (965)$$

and (955) is satisfied for each $0 \leq t_1 \leq t_2 \leq T$.

We can summarize what we proved as follows

Theorem

Assume u is as described in the first paragraph of this lecture. Then $u \in \mathcal{C}([0, T], L^2(\Omega))$ and

$$\int_{\Omega} \frac{1}{2} |u(x, t_2)|^2 dx - \int_{\Omega} \frac{1}{2} |u(x, t_1)|^2 dx = \int_{t_1}^{t_2} \int_{\Omega} -u_{x_i} a_i dx dt \quad (966)$$

for each $0 \leq t_1 \leq t_2 \leq T$.

Lecture 61, 3/4/2011

The proof of identity (966) from the last lecture can be easily generalized to the situation when we add some lower-order terms to equation (967), and using this approach we can prove energy identity (930) from lecture 57 for weak solutions of problem (939), with the definition of the weak solution given in lecture 59. Therefore we can complement the existence theorem from lecture 59 by an equally important uniqueness result:

Theorem

The initial-boundary-value problem (939) has at most one weak solution.

Combining the two theorems, we obtain:

Corollary

The initial-boundary-value problem (939) has precisely one weak solution.

As we already discussed previously (lectures 58 and 59), it is important that we can prove both existence and uniqueness. It shows that our definition of the weak solution is good. The same theorem is true for non-homogeneous boundary conditions, under appropriate assumptions, of course. The proofs are natural modifications of the proof above. We refer the reader to the book “Linear and Quasilinear Equations of Parabolic Type by O. Ladyzhenskaya, V. Solonnikov, and N. Ural’tseva”, Section 3, for a detailed exposition.

If the coefficients of the equation $u_t + Lu = 0$ are smooth, the solution will be smooth. This is true locally: if the coefficients are smooth in a open space-time set, the solutions will be smooth there, regardless of the smoothness of the coefficients in other regions of space-time.²²⁵ This regularity result can be proved by localizing the energy inequality, similarly to what we have seen in lecture 33 for the elliptic case.

We will illustrate the main idea on the simple case of the heat equation. The equations with variable coefficients and lower-order terms can be treated in a similar way, except that we have to write more terms.

In analogy with the local elliptic estimates in lecture 33, let us consider a solution of the heat equation

$$u_t - \Delta u = 0 \tag{967}$$

in parabolic balls

$$Q_R = B_R \times (-R^2, 0). \tag{968}$$

²²⁵By “smooth” we mean C^∞ . For elliptic equations we studied in the first semester the statement remained true even when “smooth” meant “analytic”. This is not the case for the parabolic equations. Even for the constant coefficients heat equation, local smooth solutions may not be analytic in time (although they are analytic in x), as one can see from inspecting the fundamental solution $\Gamma(x, t)$ at $t = 0$ away from $x = 0$.

We introduce the “local parabolic energy norm”

$$\|u\|_R = \left(\sup_{-R^2 < t < 0} \int_{B_R} |u(x, t)|^2 dx + \int_{-R^2}^0 \int_{B_R} |\nabla u(x, t)|^2 dx dt \right)^{\frac{1}{2}}, \quad (969)$$

where ∇ denotes the spatial gradient, as usual. (The time derivative is not included in ∇ .)

The parabolic version of the basic elliptic estimate (461) is the following:

$$\|u\|_{R_1}^2 \leq \frac{C}{(R_0 - R_1)^2} \int_{Q_{R_0}} |u|^2 dx dt \quad (970)$$

where $R_1 < R_0$. Once this estimate is proved, it can be applied to the spatial derivatives of the equation, similarly to (462) in the elliptic case. Once we control the spatial derivatives of u , we also control the time derivative from the equation.

The proof of (970) is quite similar to the elliptic case. We take a cut-off function η with $\eta = 1$ in Q_{R_1} which is compactly supported in Q_{R_0} , with the spatial gradients of the order $1/(R_0 - R_1)$ and the time derivative of the order $1/(R_0 - R_1)^2$. We multiply (967) by $u\eta^2$ and integrate over B_{R_0} (for a given time-level) We obtain

$$\frac{d}{dt} \int_{B_{R_0}} \frac{1}{2} |u(x, t)|^2 \eta^2 dx + \int_{B_{R_0}} |\nabla u(x, t)|^2 \eta^2 dx = \quad (971)$$

$$\int_{B_{R_0}} (2u^2 \eta \eta_t - 2\eta \nabla u u \nabla \eta) dx. \quad (972)$$

We now estimate the second term in the integral on the right-hand side in the same way as we estimated the right-hand side of (466) and end up with

$$\frac{d}{dt} \int_{B_{R_0}} \frac{1}{2} |u(x, t)|^2 \eta^2 dx + \frac{1}{2} \int_{B_{R_0}} |\nabla u(x, t)|^2 \eta^2 dx \leq \quad (973)$$

$$\int_{B_{R_0}} (2u^2 \eta \eta_t + 2u^2 |\nabla \eta|^2) dx. \quad (974)$$

Estimate (970) can now be obtained by integration over t . The main idea of this calculation works for quite general equations of the form $u_t + Lu = f$, except that we have to write more terms. Also, by using the method from the last lecture, the calculation is in fact valid also for the weak solutions.

We remark that if we only work with the heat equation, we do not have to use the trick of multiplying with $u\eta^2$. We can just multiply by $u\eta$ and do one more integration by parts in the term $\int_{B_{R_0}} -u \nabla u \nabla \eta$, to obtain a term $\int_{B_{R_0}} \frac{1}{2} u^2 \Delta \eta$, which gives the required estimate. However, for more general operators L we may not have enough regularity to proceed this way, and the first approach is probably the most natural way to proceed.

Boundary regularity can also be approached via local energy estimates analogous to (970), essentially following the ideas in the elliptic case (lecture 36). We will not go into the details, but we should mention one additional issue which comes up in the boundary regularity for the initial-boundary-value problems such as (939). If the solution is smooth up to the boundary of $\Omega \times (0, T)$, then obviously $u_t = 0$ on the lateral boundary $\partial\Omega \times (0, T)$. By the equation this means that also $Lu = 0$ on the lateral boundary. On the other hand, at $t = 0$ we have to have $u_t(x, 0) = -Lu(x, 0) = -Lu_0(x)$. This means that for a solution to be smooth at $t = 0$, we must have $Lu_0(x) = 0$ at $\partial\Omega$. This is an additional *compatibility condition* which is necessary for smoothness. Without this condition the solution will not be smooth at $t = 0$ at $\partial\Omega$, even if all the data are smooth.

Homework assignment 1

due Wednesday, March 23

Let $L = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$ be a partial differential operator in \mathbb{R}^n with constant coefficients.

Decide which of the following statements are true:

1. If u is a smooth compactly supported function on \mathbb{R}^n satisfying $Lu = 0$ in \mathbb{R}^n , then $u \equiv 0$.
2. If u is a compactly supported distribution in \mathbb{R}^n satisfying $Lu = 0$ in \mathbb{R}^n , then $u \equiv 0$.
3. If $u \in L^1(\mathbb{R}^n)$ such that $Lu = 0$ (in distributions) in \mathbb{R}^n , then $u \equiv 0$.
4. If $u \in L^2(\mathbb{R}^n)$ such that $Lu = 0$ (in distributions) in \mathbb{R}^n , then $u \equiv 0$.
- 5*. (Optional) If $u \in L^p(\mathbb{R}^n)$ for some $p \in [1, 2]$, and $Lu = 0$ (in distributions) in \mathbb{R}^n then $u \equiv 0$. (Hint: when $p \in [1, 2]$ we can write $f \in L^p(\mathbb{R}^n)$ as a sum of an L^1 -function and an L^2 -function.)
- 6*. (Optional) If $u \in L^p(\mathbb{R}^n)$ for some $p \in [1, \infty)$ and $Lu = 0$ (in distributions) in \mathbb{R}^n , then $u \equiv 0$. (Hint: consider the equation $\Delta u + a^2 u = 0$ in \mathbb{R}^3 , and calculate a radially symmetric solution, either by solving an ODE, or by Fourier transformation.)

Lecture 62, 3/7/2011

In lecture 6 we talked about the maximum principle, strong maximum principle, and Harnack inequality for harmonic functions. Today we will talk about these in the context of “caloric functions” - the solutions of the heat equation. We will see that many of the arguments used for harmonic functions also work (with some modifications) for caloric functions. As we have already seen, when dealing with the heat equation, it is useful to introduce the parabolic balls

$$Q_R = B_R \times (-R^2, 0]. \quad (975)$$

For the balls centered at $z = (x, t) \in \mathbb{R}^n \times (t_1, t_2)$ we use the notation

$$Q_{z,R} = B_{z,R} \times (t - R^2, t]. \quad (976)$$

We will also use the notation

$$\partial_{\text{par}}Q_{z,R} = B_{z,R} \times \{t - R^2\} \cup \partial B_{z,R} \times [t - R^2, t]. \quad (977)$$

For an open set $\Omega \subset \mathbb{R}^n$ we will consider the parabolic cylinders $\Omega \times (t_1, t_2]$. We define

$$\partial_{\text{par}}(\Omega \times (t_1, t_2]) = \Omega \times \{t_1\} \cup \partial\Omega \times [t_1, t_2]. \quad (978)$$

The maximum principle is quite easy to generalize to caloric functions, and, in fact, to the solutions of equation

$$u_t - a_{ij}(x, t)u_{x_i x_j} + b_i(x, t)u_{x_i} = u_t + Lu = 0 \quad (979)$$

with “sufficiently regular” coefficients $a_{ij}(x, t), b_i(x, t)$ satisfying the ellipticity condition (924).

Lemma (Maximum Principle)

Let u be a $C_t^1 C_x^2$ solution of (979) in a cylinder $Q = \Omega \times (t_1, t_2]$ and assume u is continuous up to the parabolic boundary $\partial_{\text{par}}Q$. Then

$$\sup_Q u = \sup_{\partial_{\text{par}}Q} u. \quad (980)$$

Proof

The proof is similar to proof 1 of the elliptic maximum principle in lecture 6. If (980) fails, then it also fails for $u_\varepsilon(x, t) = u(x, t) - \varepsilon t$ for some small $\varepsilon > 0$. Let $\bar{z} = (\bar{x}, \bar{t})$ be a point of \bar{Q} where u_ε attains its maximum. We note that \bar{z} cannot be at $\partial_{\text{par}}Q$, due to our assumptions. This means that $Lu_\varepsilon(\bar{x}, \bar{t}) \geq 0$ ²²⁶ $u_{\varepsilon t}(\bar{x}, \bar{t}) \geq 0$, and therefore $u_{\varepsilon t}(\bar{x}, \bar{t}) + Lu_\varepsilon(\bar{x}, \bar{t}) \geq 0$. At the same time, we have $u_{\varepsilon t} + Lu = -\varepsilon < 0$ everywhere in Q . This is a contradiction and the proof is finished.

²²⁶See (572)

Remark

The reader can easily check that we can replace cylinders Q by more general space-time domains, if we are careful with the definition of the parabolic boundary for such domains.

We have seen that the maximum principle for caloric function is the same as the harmonic principle for the harmonic functions: roughly speaking, the maximum is attained at the corresponding boundary. In this case the analogy between caloric functions and harmonic functions is complete. However, this is not the case when we consider the *strong maximum principle*. We recall from lecture 6 that the strong maximum principle for harmonic function implies that a harmonic function h in a ball B_R such that $h \leq h(x_0)$ in B_R for some $x_0 \in B_R$ must be constant. An analogous statement for caloric functions and parabolic balls Q_R fails. To see that, let us consider a caloric function u in Q_1 with the boundary condition

$$u(x, t) = g(t), \quad (x, t) \in \partial_{\text{par}}Q_1, \quad (981)$$

where $g(t) = 1$ for $t \leq -\frac{1}{2}$ and $g(t) < 1$ for $t > -\frac{1}{2}$. (We note that when g is smooth the corresponding initial-boundary-value problem is uniquely solvable in the class of smooth functions by the existence and regularity theory discussed in the last few lectures.) This corresponds to the situation when a body is at a constant temperature $u = 1$ up to time $t = -\frac{1}{2}$ and then for times $t > -1/2$ we cool the boundary to a lower temperature $g(t) < 1$. The caloric function u defined in this way in Q_1 clearly satisfies $u \leq 1$ in Q_1 and $u(z_0) = 1$ for many interior points $z_0 \in Q_1$, but is not constant. The difference between the heat equation and the Laplace equation in this respect is that for the Laplace equation (and any other elliptic equation) information about disturbances propagate in all directions. On the other hand, for the heat equation this is not the case in variable t . With respect to this variable all disturbances propagate only in one direction: from an earlier time to a later time. What happens after $t = t_0$ will have no effect on what had happened before $t = t_0$. The correct modification of the strong maximum principle is as follows:

Theorem (Strong Maximum Principle)

Let u be a continuous solution of (979) in a cylinder $Q = \Omega \times (t_1, t_2]$, where $\Omega \subset \mathbb{R}^n$ is open and connected. If $u \leq u(\bar{x}, t_2)$ in Q for some $\bar{x} \in \Omega$, then $u = u(\bar{x}, t_2)$ in Q .

This can be proved by using a parabolic version of the Hopf lemma from lecture 41. We will not go into the details at this point. For the heat equation $u_t - \Delta u = 0$ the statement will follow from the representation formulae below.

We now turn to the parabolic Harnack inequality. The literal translation of the Harnack inequality (46) for the harmonic functions to the case of the caloric

function fails, by the same examples we used above to illustrate the failure of the naive version of the strong maximum principle for caloric functions.²²⁷

We have seen in lecture 6 that the Harnack inequality for the harmonic functions is a relatively easy consequence of the mean-value formula (40), together with estimate (37). We have shown an analogue of (??) for caloric functions in lecture 52, see (799). We will now discuss analogues of the mean-value property for caloric functions.

Let u be a caloric function in the parabolic ball Q_1 . In analogy with the mean-value formula for harmonic functions, we consider a representation formula

$$u(0,0) = \int_{Q_1} u(z)A(z) dz \quad \text{for each bounded caloric } u \text{ in } Q_1, \quad (982)$$

where A is a suitable function on Q_1 , the properties of which will be specified more precisely as we proceed. The main requirements will be that A satisfy

$$\int_{Q_1} A(z) dz = 1, \quad (983)$$

$$A \geq \varepsilon(K) > 0 \quad \text{on any compact subset } K \text{ of the interior of } Q_1. \quad (984)$$

The existence of functions satisfying (982)–(984) may not be completely obvious, although we will see that it is not hard. For now we will take it for granted, and will address the existence issue later.

Let us first look at some simple consequences of representation (982). We note that we can scale to formula²²⁸ to apply to $Q_{z,R}$:

$$u(z) = \int_{Q_{z,R}} u(z')A_R(z' - z) dz', \quad (985)$$

with

$$A_R(z) = \frac{1}{R^{n+2}} A\left(\frac{z}{R}\right). \quad (986)$$

The formula immediately implies the strong maximum principle, by an argument similar to the one used for harmonic functions in lecture 6. Let u be caloric in $Q = \Omega \times (t_1, t_2)$ and let

$$M = \sup_Q u. \quad (987)$$

Assume that

$$u(\bar{x}, t_2) = M \quad (988)$$

²²⁷Recall that the Harnack inequality can be considered as a quantitative version of the strong maximum principle, see lecture 6. Therefore the failure of the strong maximum principle implies the failure of Harnack inequality.

²²⁸by parabolic scaling $(x, t) \rightarrow (\lambda x, \lambda^2 t)$

for some $\bar{x} \in \Omega$. Let $\bar{z} = (\bar{x}, t_2)$ and let $Q_{\bar{z}, R} \subset Q$. We have

$$M = u(\bar{z}) = \int_{Q_{\bar{z}, R}} A_R(z - \bar{z})u(z) dz \leq \int_{Q_{\bar{z}, R}} A_R(z - \bar{z})M dz = M. \quad (989)$$

We see that we have to have $u = M$ everywhere in $Q_{\bar{z}, R}$ in order for the inequality not to be strict. Now we can repeat this argument with \bar{z} replaced by any point z_1 of $Q_{\bar{z}, R}$. and $Q_{\bar{z}, R}$ replaces by $Q_{z_1, R_1} \subset Q$. Assuming that Ω is connected and iterating the procedure, it is not hard to see that $u = M$ in Q .²²⁹

Let us not turn to the Harnack inequality. Let $Q = \Omega \times (t_1, t_2)$ be as above, and let K_1, K_2 be two compact subsets of Q such that

$$K_1 \subset \Omega \times (t_3 + \tau, t_2], K_2 \subset \Omega \times (t_1, t_3), \quad (990)$$

where

$$t_1 < t_3 < t_3 + \tau < t_2. \quad (991)$$

Theorem (Harnack Inequality)

For each compact sets K_1, K_2 as above there exist $c = c(Q, K_1, K_2) > 0$ such that for any caloric function $u \geq 0$ in Q we have

$$\inf_{K_1} u \geq c \sup_{K_2} u. \quad (992)$$

Proof

The proof is quite similar to the proof of the Harnack inequality for harmonic functions in lecture 6. Assuming the statement fails, we can find a sequence of caloric functions u_n and points $z_{1,n} \in K_1, z_{2,n} \in K_2$ such that

- $u_n(z_{2,n}) = \sup_{K_2} u_n = 1$
- $u_n(z_{1,n}) \rightarrow 0$
- $z_{1,n} \rightarrow z_1 \in K_1$

By replacing functions u_n by their small shift, enlarging K_1, K_2 slightly, while shrinking Ω slightly, we can assume without loss of generality that in fact $z_{1,n} = z_1$. Let $\bar{Q}_{z_1, R} \subset Q$. We have

$$u_n(z_1) = \int A_R(z - z_1)u_n(z) dz \quad (993)$$

and since $u_n \geq 0$, this means, due to (984), that $u_n \rightarrow 0$ in $L^1(K)$ for any compact subset K contained in the interior of $Q_{z_1, R}$. By estimate (798) this

²²⁹We leave the details of this argument to the reader as an exercise.

means that $u_n \rightarrow 0$ uniformly in K . We can now use this argument again with z_1 replaced by any z' from the interior of a ball $Q_{z',R'} \subset Q$ to infer that $u_n \rightarrow 0$ uniformly in any compact subset of the union of such balls $Q_{z',R'}$. By repeatedly using this same argument we see that u_n will converge to 0 uniformly in K_2 . This contradicts $u_n(z_{2,n}) = 1$, and the proof is finished.

It remains to establish the existence of the function A in (982) with the required properties. Let us first consider a heuristic argument for its existence. (We will comment later on what is necessary to make the argument rigorous.) We consider a representation formula for caloric functions in Q_1 , somewhat similar to the spherical mean value formula (39) for harmonic functions. Our considerations are similar to those in lecture 53, see formulae (825), (826), and (828). Let v be a solution of the problem

$$v_t + \Delta v = 0 \quad \text{in } Q_1, \quad (994)$$

$$v(\cdot, 0) = -\delta, \quad (995)$$

$$v|_{\partial B_1 \times [-1,0]} = 0. \quad (996)$$

Note that (994) is a *backward* heat equation, which can be thought of as the usual heat equation with time going in the opposite direction. That is why the initial condition is at $t = 0$. The existence of the solution v can be established as follows. Denoting the standard heat kernel by Γ as usual, we note that the backward heat kernel

$$\Gamma_b(x, t) = \Gamma(x, -t) \quad (997)$$

satisfies (994) and (995). We can now seek v in the form

$$v = \Gamma_b + w \quad (998)$$

and we get a boundary value problem for w of the form

$$w_t + \Delta w = 0 \quad \text{in } Q_1, \quad (999)$$

$$w(\cdot, 0) = 0, \quad (1000)$$

$$v|_{\partial B_1 \times [-1,0]} = -\Gamma_b|_{\partial B_1 \times [-1,0]}. \quad (1001)$$

By our existence and regularity theory for the heat equation (which applies also to the backward heat equation, modulo the obvious changes), the last problem has a unique smooth solution, and therefore the existence of v is established. By the maximum principle we have $v \geq 0$ in Q_1 .²³⁰ Assume that u is a caloric function in Q_1 , smooth up to the parabolic boundary. We have

$$0 = \int_{B_1 \times (-1,0)} u(v_t + \Delta v) dx dt = \int_{B_1} uv dx|_{t=-1}^{t=0} + \int_{\partial B_1 \times (-1,0)} u \frac{\partial v}{\partial \nu} dx dt, \quad (1002)$$

²³⁰We have $v(0,0) = +\infty$, which can still be considered as non-negative.

and hence

$$u(0,0) = \int_{B_1} v(x,-1)u(x,-1) dx + \int_{\partial B_1 \times (-1,0)} -\frac{\partial v}{\partial \nu} u dx dt. \quad (1003)$$

This can be thought of as

$$u(0,0) = \int_{\partial_{\text{par}} Q_1} u(z)a(z) dz, \quad (1004)$$

where a is function on $\partial_{\text{par}} Q_1$ which is smooth away from the set $\partial B_1 \times \{-1\}$, where it may have a discontinuity. The maximum principle implies that $v \geq 0$ in Q_1 and therefore $a \geq 0$. Formula (1004) can be thought of as a parabolic analogy of the spherical mean value formula (39) for harmonic functions. Heuristically one expects that $a > 0$, except possibly at $B_1 \times \{-1\}$, where we expect a to be discontinuous, with limit 0 if we approach it from $B_1 \times \{-1\}$, and > 0 if we approach it from the lateral boundary $\partial B_1 \times (-1,0)$. All this can be established rigorously if we use the parabolic version of Hopf's lemma. Hopf's lemma can be considered as a slightly stronger version of the strong maximum principle, but weaker than the Harnack inequality. The important point is that it can be proved along similar lines as the elliptic version (lecture 41, (594)), and the proof is independent of any representation formulae, using only the (weak) maximum principle.

Once we have a formula of the form (1004), we can scale it to balls Q_R , by

$$u(0,0) = \int_{\partial_{\text{par}} Q_R} u(x,t) \frac{1}{R^{n+1}} a\left(\frac{x}{R}, \frac{t}{R^2}\right) dz = \int_{\partial_{\text{par}} Q_R} u(z)a_R(z) dz. \quad (1005)$$

Let us write this as

$$u(0,0) = \int u(z) d\mu_R(z), \quad (1006)$$

where μ_R is the measure supported at $\partial_{\text{par}} Q_R$ given by $a_R(z) dz$. We can now take a smooth function φ on $(0,1)$ which is positive everywhere, with $\int_0^1 \varphi = 1$, average (1007) over R with respect to $\varphi(R) dR$. This gives

$$u(0,0) = \int u(z) d\mu(z), \quad \mu = \int_0^1 \mu_R \varphi(R) dR. \quad (1007)$$

The measure μ is easily seen to be of the form $A(z) dz$ with $A \geq 0$ satisfying $\int A(z) dz = 1$ and "almost" satisfying (984), but not quite – it may not be bounded away from zero at the paraboloid $\{t = -|x|^2\}$, due to the fact that $a(z)$ approaches 0 on the rim of the bottom part of the boundary of Q_1 . This defect can be fixed easily, for example by additional averaging. We note that the function $\tilde{A} = A * A$ (the space-time convolution) is supported in $\tilde{Q} = B_2 \times (-2,0)$ and is continuous and strictly positive there. Moreover,

$$u(0,0) = \int_{\tilde{Q}} u(z)\tilde{A}(z) dz \quad (1008)$$

²³¹We will slightly abuse notation by writing dz for the n -dimensional surface measure on $\partial_{\text{par}} Q_1$.

for each function u which is integrable and caloric in \tilde{Q} . It is easy to change \tilde{Q} to Q_1 in (1008): instead of starting the whole construction with Q_1 , we start it with $B_{\frac{1}{2}} \times (-\frac{1}{2}, 0)$, and after going through the same procedure as above, we obtain (1008) with \tilde{Q} replaced by Q_1 .

The above procedure for constructing a representation formula of the form (982) is quite natural, but it is not the simplest way to arrive at such a representation. Also, if we wish to prove the strong maximum principle via representation formulae, we should not rely on it in the proof of the formulae. There is another way to do the construction, in some sense a more elementary one, which we now explain.²³²

We will first look at the formula (982) from a slightly different point of view. Let V be given by

$$V_t + \Delta V = -\delta(x, t) + A, \quad V = 0 \text{ for } t > 0, \quad V(x, t) \rightarrow 0 \text{ as } x \rightarrow \infty \text{ for each } t, \quad (1009)$$

where the function A is extended by 0 outside Q_1 , and $\delta(x, t)$ is the space-time Dirac mass at $(x, t) = 0$. We have

$$V = \Gamma_b - \Gamma_b * A. \quad (1010)$$

We claim that the validity of (982) for any (integrable) u caloric in Q_1 is equivalent to

$$\text{support } V \subset \overline{Q_1}. \quad (1011)$$

In other words, the function A has to have a property analogous to the radial distribution of mass in a ball in the case of the gravitational potential, which we noticed in lecture 1: away from the support of the ball the potential is exactly the same as if we were looking at all the mass concentrated at the center of the ball. The meaning of (1011) is the same: outside of Q_1 the (backward) “heat potential” of A is exactly the same as if all the sources were moved to the origin $(x, t) = (0, 0)$.

To see that (982) and (1011) are equivalent, we first notice that (1011) easily implies (982) by simple integration by parts. On the other hand, if the support of V is not contained in Q_1 we consider a smooth f supported in a compact disjoint with Q_1 , such that $\int V f \, dx \, dt \neq 0$. We set $u = \Gamma * f$, where Γ is the (forward) heat kernel. Then u is caloric in Q_1 and we have

$$0 \neq \int f V = \int (u_t - \Delta u) V = \int u (-V_t - \Delta V) = \int u (\delta - A), \quad (1012)$$

showing that (982) fails for this particular u , which is caloric in a neighborhood of Q_1 .

²³²See also L.C.Evans’s textbook Partial Differential Equations

We see that one way to come up with formulae of the type (982) is to consider functions V of the form

$$V = \Gamma_b - W, \quad (1013)$$

where $W = \Gamma_b$ outside Q_1 . We have

$$V_t + \Delta V = \delta - (W_t + \Delta W) \quad (1014)$$

and setting

$$A = W_t + \Delta W \quad (1015)$$

we obtain (1010) and (1011), and hence also (982). Of course, if we wish to have $A \geq 0$, the function W has to satisfy additional conditions.

It is natural to try to seek W in the form

$$W = F(\Gamma_b), \quad (1016)$$

where $F(s) = s$ for $s \leq s_0$ and F is smooth, bounded, with $F^{(k)}(s) \rightarrow 0$ as $s \rightarrow \infty$ for $k = 1, 2, \dots$. Assuming W is as in (1016) with F satisfying these restrictions, we calculate

$$W_t + \Delta W = F''(\Gamma_b)|\nabla\Gamma_b|^2. \quad (1017)$$

The right-hand side of (1017) is easily evaluated using

$$F''(\Gamma_b)|\nabla\Gamma_b|^2 = F''(\Gamma_b)\Gamma_b^2|\nabla\log\Gamma_b|^2 = F''(\Gamma_b)\Gamma_b^2\frac{|x|^2}{4t^2}. \quad (1018)$$

It is easy to see that we can take F defined by

$$F(s) = s, \quad s \leq s_0, \quad F''(s) = -\frac{s_0}{s^2}, \quad s > s_0, \quad F' \text{ continuous}. \quad (1019)$$

Let us define

$$\Omega_R = \{\Gamma_b \geq \frac{1}{R^n}\}. \quad (1020)$$

Defining F_R by the construction of F above with $s_0 = \frac{1}{R^n}$, and $W_R = F_R(\Gamma_b)$, we see that the function $V_R = \Gamma_b - W_R$ is supported in Q_{Q_R} and satisfies

$$V_{Rt} + \Delta V_R = -\delta + \frac{|x|^2}{4t^2 R^n} \chi_{\Omega_R}, \quad (1021)$$

where χ_{Ω_R} denotes the characteristic function of the ball Ω_R . This gives the representation formula

$$u(0, 0) = \frac{1}{R^n} \int_{\Omega_R} u(x, t) \frac{|x|^2}{4t^2} dx dt, \quad (1022)$$

²³³Formally we have $W_t + \Delta W = F'(\Gamma_b)(\Gamma_{bt} + \Delta\Gamma_b) + F''(\Gamma_b)|\nabla\Gamma_b|^2$. We have $\Gamma_{bt} + \Delta\Gamma_b = -\delta(x, t)$, and since $F'(s) \rightarrow 0$ as $s \rightarrow \infty$, we see that we should take $-F'(\Gamma_b)\delta = 0$.

for caloric functions u integrable in Ω_R . which can be found in the textbook “Partial Differential Equations” by L. C. Evans. Different choices of F will lead to different formulae, but (1022) seems to be the simplest choice.

Note that the derivation of (1022) is explicit and does not rely on any properties of general caloric functions, such as the strong maximum principle (not even the maximum principle).

The representation (1022) is, for a suitable R , of the form (982), except that assumption (984) is not quite satisfied. There will be areas of Q_1 where A will vanish. However, this can be fixed easily by additional averaging. For example, one can proceed as follows. In the formula

$$u(0, 0) = \int u(z)A(z) dz \quad (1023)$$

we can express each $u(z)$ as

$$u(z) = \int u(z_1)A_{R(z)}(z_1 - z) dz_1, \quad (1024)$$

where

$$A_R(z') = \frac{1}{R^{n+2}} A\left(\frac{x'}{R}, \frac{t'}{R^2}\right). \quad (1025)$$

The function $R = R(z)$ in (1024) is, say, continuous and non-negative, and such that the balls $z_1 + \Omega_{R(z_1)}$ stay inside Q_1 . Letting $A^{(1)}(z_1) = \int A(z)A_{R(z)}(z_1 - z) dz$, we see that we can replace A by $A^{(1)}$ in (1023). This way the support of A can be enlarged in a more flexible way than just by using the scaling $A \rightarrow A_R$. We can clearly achieve that $A^{(1)} \geq \varepsilon = \varepsilon(z_1)$ in some parabolic neighborhood of any point in $z_1 + \Omega_{R_1}$ with $z_1 \in \Omega_R$ and R_1 such that $z_1 + \Omega_{R_1} \subset Q_1$. This procedure can now be repeated with A replaced by $A^{(1)}$ to obtain $A^{(2)}$, then $A^{(3)}$, etc. An important point is that the tangent plane to the boundary of Ω_R at $z = 0$ is the plane $t = 0$ in $\mathbb{R}^n \times \mathbb{R}$. This is why we can “travel” almost parallel to the planes $t = \text{const}$ when extending the support of A using the above procedure. From the above argument we get that for each compact $K \subset Q_1$ there exists representation A_K of the form (1023) with A replaced by A_K , with support A_K being contained in the interior of Q_1 and $A_K \geq \varepsilon = \varepsilon(K) > 0$ on K . We can then take a sequence

$$K_1 \subset K_2 \subset \dots \subset \text{interior } Q_1, \quad \cup_k K_k = \text{interior } Q_1, \quad (1026)$$

and define

$$A = \sum_{j=1}^{\infty} 2^{-j} A_{K_j}, \quad (1027)$$

which will have the required properties. This finished the proof of the representation (982) with A satisfying (983) and (984).

It is also possible to prove both the strong maximum principle and the Harnack inequality by working with the balls $\Omega_{z,R}$ and formula (1022) in the proofs, and essentially using the same approach as we did when working with $Q_{z,R}$.

It is perhaps worth mentioning an analogy of (1004) in the context of the balls Ω_R . We expect a representation of the form

$$u(0,0) = \int_{\partial\Omega_R} u(z)\mathbf{a}(z) dz, \quad (1028)$$

where dz denotes the n -dimensional surface measure on $\partial\Omega_R$. The explicit form of \mathbf{a} can be determined by using (1018) with

$$F(s) = \max\{s, s_0\}, \quad s_0 = \frac{1}{R^n}. \quad (1029)$$

Then

$$F''(s) = -\delta(s - s_0) \quad (1030)$$

and

$$F''(\Gamma_b) = -\delta(\Gamma_b - s_0). \quad (1031)$$

In general, if $f: \mathbb{R}^m \rightarrow \mathbb{R}$ is a function which is smooth in a neighborhood of $\{f = 0\}$, and $\nabla f \neq 0$ at $\{f = 0\}$ we have

$$\delta(f) = \frac{1}{|\nabla f|} \mathcal{H}^{m-1}|_{\{f=0\}}, \quad (1032)$$

where $\mathcal{H}^{m-1}|_{\{f=0\}}$ denotes the restriction of the $(m-1)$ -dimensional Hausdorff measure to the surface $\{f = 0\}$.²³⁴ From (1031) and (1018) we obtain that in (1028) we should take

$$\mathbf{a}(z) = \frac{|\nabla\Gamma_b|^2}{(\Gamma_{bt}^2 + |\nabla\Gamma_b|^2)^{\frac{1}{2}}}. \quad (1033)$$

The calculation of this expression can be simplified by using

$$\frac{|\nabla\Gamma_b|^2}{(\Gamma_{bt}^2 + |\nabla\Gamma_b|^2)^{\frac{1}{2}}} = \Gamma_b \frac{|\nabla \log \Gamma_b|^2}{((\log \Gamma_b)_t)^2 + |\nabla \log \Gamma_b|^2)^{\frac{1}{2}}}. \quad (1034)$$

Note that expressions for the derivatives of $\log \Gamma_b$ are quite simple, and it is not hard to carry out the differentiations in (1034) to obtain an explicit formula. We leave the calculation to the interested reader as an exercise. (Note that $\Gamma_b = \frac{1}{R^n}$ on the set relevant for the calculation, and $|x|$ and t are related through $\Gamma_b(x, t) = \frac{1}{R^n}$.)

²³⁴Justifying this formula is a good exercise.

We have covered in some detail the scalar elliptic and parabolic equations of second order. We have already mentioned earlier that the variational methods we have used (i. e. the methods based on the Lax-Milgram lemma/energy estimates for elliptic equations and Galerkin approximations/parabolic energy estimates for parabolic equations) work also for more general classes of elliptic/parabolic equation. There are some new issues which appear in this context, which we will discuss today.

One special feature of the scalar elliptic/parabolic equations of the second order is the maximum principle and related deeper properties of the solutions. When we go to more general classes of elliptic/parabolic equations, the maximum principle is often no longer valid for the solutions (although in some special cases one may perhaps find certain quantities constructed from the solution which do satisfy the maximum principle).

There is another issue which has to be addressed: what is the correct “ellipticity condition”? So far our basic assumption in the context of elliptic/parabolic equations was

$$a_{ij}\xi_i\xi_j \geq \nu|\xi|^2, \quad (1035)$$

see e. g. lecture 30 (455) and lecture 57 (924). This condition was crucial for practically all results. So far we have not really discussed its necessity, but it is probably fair to say that the condition is “almost necessary”.²³⁵ In the context of more general classes of equations, we need look at the ellipticity condition in more detail.

Let consider the following situation. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and consider functions

$$u: \Omega \rightarrow \mathbb{R}^m \quad (1036)$$

defined on Ω . We will write

$$u = (u^1, \dots, u^m), \quad (1037)$$

using latin upper indices for the components of u . The points of Ω will be denoted by

$$x = (x^1, \dots, x^n) = \{x^\alpha\}_{\alpha=1}^n, \quad (1038)$$

with greek upper indices, which will be more convenient in the current context. The gradient ∇u of such a function can be identified with a $m \times n$ matrix

$$\nabla u = \{u_{,\alpha}^k\} = \left\{ \frac{\partial u^k}{\partial x^\alpha} \right\}, \quad k = 1, \dots, m, \quad \alpha = 1, \dots, n. \quad (1039)$$

²³⁵It can be weakened in some situations (if we adjust the corresponding results) to allow for some degeneracies, in the sense that we can allow $\nu = \nu(x)$ with $\nu(x)$ approaching zero along some exceptional set. A precise study of this can get quite difficult and technical, and we will not consider these questions here.

The set of all $m \times n$ matrices will be denoted by $M^{m \times n}$. The matrices will be denoted by X , or X_α^k (which can mean both the matrix and its coordinates).

The space of functions $u: \Omega \rightarrow \mathbb{R}^m$ such that each component of u is in $H_0^1(\Omega)$ will still be denoted by $H_0^1(\Omega)$, or – if we wish to emphasize we are dealing with vector-valued function – by $H_0^1(\Omega, \mathbb{R}^m)$.

Let us now consider a quadratic form on $M^{m \times n}$ given by

$$X \rightarrow Q(X) = a_{kl}^{\alpha\beta} X_\beta^l X_\alpha^k. \quad (1040)$$

We wish to determine under which conditions the form

$$u \rightarrow \int_\Omega Q(\nabla u) = \int_\Omega a_{kl}^{\alpha\beta} u_{,\beta}^l u_{,\alpha}^k dx \quad (1041)$$

is coercive on $H_0^1(\Omega)$ in the sense that

$$\int_\Omega Q(\nabla u) dx \geq \nu \int_\Omega |\nabla u|^2 dx, \quad u \in H_0^1(\Omega), \quad (1042)$$

for some $\nu > 0$. For now we assume that the coefficients $a_{kl}^{\alpha\beta}$ are constant, independent of x .

An obvious sufficient condition is that

$$Q(X) \geq \nu |X|^2, \quad X \in M^{m \times n}, \quad (1043)$$

which is what we have assumed in the scalar case, corresponding to $m = 1$. In this case one can in fact assume that $a_{kl}^{\alpha\beta}$ depend on X , and (1042) is trivially true. In the scalar case ($m = 1$) condition (1043) is also necessary for (1042), and this is also true when $n = 1$ (for any m).²³⁶ However, for $n \geq 2, m \geq 2$ condition (1043) is no longer necessary, as the following example shows.

Example

Consider $m = n = 2$ and

$$Q(X) = \nu |X|^2 + \kappa \det X. \quad (1044)$$

Exercise:

Show that when $n = m = 2$, we have

$$\int_\Omega \det(\nabla u(x)) dx = 0, \quad u \in H_0^1(\Omega). \quad (1045)$$

Hint: use $\det \nabla u = (u^1 u_2^2)_{,1} - (u^1 u_1^2)_{,2}$. Note that (1045) also has a clear geometric meaning.

²³⁶You can try to find a simple proof of these facts. It is not hard, but it requires an idea which might not be obvious if you see such things for the first time. We prove below a more general statement.

We see from (1045) that Q given by (1044) will satisfy (1042) for any κ . For sufficiently large κ the pointwise condition (1043) is obviously not satisfied.

Theorem

In the notation above, the coercivity condition (1042) is satisfied if and only if

$$Q(X) \geq \nu|X|^2, \quad X_\alpha^k = \xi_\alpha \lambda^k, \quad \xi \in R^n, \lambda = (\lambda^1, \dots, \lambda^m) \in \mathbb{R}^m. \quad 237 \quad (1046)$$

In other words, $Q(X) \geq \nu|X|^2$ is only required to hold for very special matrices: the matrices X with $\text{rank } X = 1$. In the previous example, we have $\det X = 0$ for any rank-one matrix X , and therefore form (1044) satisfies (1046).

Proof of the theorem

We can extend any $u \in H_0^1(\Omega)$ by $u = 0$ outside Ω and consider it as an element of $H_0^1(\mathbb{R}^n)$. Clearly $\int_\Omega Q(\nabla u) = \int_{\mathbb{R}^n} Q(\nabla u)$. Using Plancherel's formula (493), we see that

$$\int_{\mathbb{R}^n} Q(\nabla u) dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} a_{kl}^{\alpha\beta} (i\xi_\alpha \hat{u}^k(\xi)) \overline{(i\xi_\beta \hat{u}^l(\xi))} d\xi \geq \quad (1047)$$

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \nu |\xi|^2 |\hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} \nu |\nabla u(x)|^2 dx. \quad (1048)$$

We have used that we can assume $a_{kl}^{\alpha\beta} = a_{lk}^{\beta\alpha}$. Then for each ξ the matrix $\tilde{a}_{kl} = a_{kl}^{\alpha\beta} \xi_\alpha \xi_\beta$ is symmetric, and for any complex vector $z^k = \lambda^k + i\mu^k$, where λ and μ are real vectors, we have $\tilde{a}_{kl} z^k \bar{z}^l = \tilde{a}_{kl} (\lambda^k \lambda^l + \mu^k \mu^l)$, giving the pointwise inequality between the integrands of the last integral of (1047) and the first integral of (1048). This shows that (1046) is sufficient.

To see that it is also necessary, let us assume that for some $\xi \in \mathbb{R}^n$, $|\xi| = 1$ and $\lambda \in \mathbb{R}^m$, $|\lambda| = 1$ we have $a_{kl}^{\alpha\beta} \xi_\alpha \xi_\beta \lambda^k \lambda^l < \nu$. Let $f(t) = \sin(t)$, let ψ be a non-negative compactly supported function compactly in Ω with $\int \psi^2 = 1$ and consider

$$u_\kappa^k(x) = \frac{1}{\kappa} f(\kappa \xi \cdot x) \psi(x) \lambda^k. \quad (1049)$$

Then

$$u_{\kappa,\alpha}^k(x) = f'(\kappa \xi \cdot x) \psi \xi_\alpha \lambda^k + O\left(\frac{1}{\kappa}\right), \quad \kappa \rightarrow \infty. \quad (1050)$$

We have

$$\int_\Omega |\nabla u_\kappa(x)|^2 dx = \frac{1}{2} + O\left(\frac{1}{\kappa}\right), \quad \kappa \rightarrow \infty. \quad (1051)$$

On the other hand, we also have

$$\int_\Omega a_{kl}^{\alpha\beta} u_{\kappa,\beta}^l u_{\kappa,\alpha}^k = \frac{1}{2} a_{kl}^{\alpha\beta} \xi_\alpha \xi_\beta \lambda^k \lambda^l + O\left(\frac{1}{\kappa}\right), \quad \kappa \rightarrow \infty. \quad (1052)$$

²³⁷Note that here λ^k means the k -th component of the vector λ , and not the k -th power of a scalar λ .

Taking κ sufficiently large, we see that

$$\int_{\Omega} \nu |\nabla u_{\kappa}(x)|^2 dx > \int_{\Omega} a_{kl}^{\alpha\beta} u_{\kappa,\beta}^l u_{\kappa,\alpha}^k dx, \quad \kappa \text{ sufficiently large.} \quad (1053)$$

This finishes the proof of the theorem.

Remarks

(i) Note that the statement of the theorem gives some information even if $m = 1$ or $n = 1$. In particular, it shows that the assumption $a_{ij}\xi_i\xi_j \geq \nu|\xi|^2$ we have used in the scalar case is necessary for coercivity.

(ii) In the proof it was important that the coefficients $a_{kl}^{\alpha\beta}$ are constant (=independent of x). In general, when $a_{kl}^{\alpha\beta}$ depend on x the form $\int_{\Omega} a_{kl}^{\alpha\beta}(x) u_{\kappa,\beta}^l u_{\kappa,\alpha}^k$ may no longer be coercive, even though the condition (1046) is satisfied at each point x . One can nevertheless prove the following statement

Theorem (a special case of Gårding's inequality)

Assume the coefficients $a_{kl}^{\alpha\beta}(x)$ are uniformly continuous in Ω and that (1046) is satisfied. Then there exists $\varkappa > 0$ such that for each $u \in H_0^1(\Omega)$ we have

$$\int_{\Omega} \left(a_{kl}^{\alpha\beta}(x) u_{,\beta}^l u_{,\alpha}^k + \varkappa |u|^2 \right) dx \geq \int_{\Omega} \frac{\nu}{2} |\nabla u|^2. \quad (1054)$$

We will omit the proof of this theorem, which is based on dividing Ω into small pieces on which $a_{kl}^{\alpha\beta}(x)$ are nearly constant, using a suitable partition of unity. The assumption that $a_{kl}^{\alpha\beta}(x)$ be continuous is important, the result fails if we only assume that $a_{kl}^{\alpha\beta}(x)$ are bounded measurable and satisfy (1046), unless $n = 1$ or $m = 1$.

It should be emphasized that condition (1046) gives coercivity only on the space $H_0^1(\Omega)$. If we allow the functions u to be non-zero even on a small part of the boundary, the coercivity may be lost and a stronger assumption than (1046) (but still weaker than (1043)) has to be imposed. We will not discuss these issues in more detail.²³⁸ We will restrict our attention to an important special case: Lamé's equations of linear elasticity, which we will discuss next time.

²³⁸For a more detailed discussion we refer the reader to two well-known papers of Agmon, Douglis, and Nirenberg on interior and boundary regularity of solutions of general elliptic systems.

An important example of a linear elliptic system where the strict uniform convexity condition (1043) is not satisfied is represented by the equations of linear elasticity. Let us briefly recall the set-up.²³⁹

We consider a body occupying a region $\Omega \subset \mathbb{R}^n$. Assume the body undergoes a deformation $\phi: \Omega \rightarrow \mathbb{R}^n$. The map ϕ is assumed to be a (sufficiently regular) orientation-preserving homeomorphism of Ω onto some other domain $\tilde{\Omega}$. The equations satisfied by the deformation ϕ can in many cases be derived from a variational principle. We assume that we can associate to the deformation ϕ an *elastic energy*, which can be expressed as

$$\mathcal{E}(\phi) = \int_{\Omega} W(\nabla\phi(x)) \, dx \quad (1055)$$

or perhaps

$$\mathcal{E}(\phi) = \int_{\Omega} W(x, \nabla\phi(x)) \, dx. \quad (1056)$$

The notation is the following: at each $x \in \Omega$ the gradient $\nabla\phi(x)$ is considered as an $n \times n$ matrix. The set of all $n \times n$ matrices is denoted by $M^{n \times n}$. The function W is usually called the stored energy function, and in context of (1055) the quantity $W(X)$ represents the energy which is necessary to deform a unit cube of the material by the matrix X . In (1056) the material is not homogeneous and its properties depend on x and $W(x, \cdot)$ describes the material at the point x . In our discussion we will mostly consider (1055), but most of it also applies to (1056).

We assume that $W \geq 0$, with $W(I) = 0$, where I denotes the identity matrix. Moreover, W should obviously satisfy

$$W(RX) = W(X), \quad R \in SO(n), \quad (\text{the special orthogonal group}). \quad (1057)$$

As W is considered only for X with $\det X > 0$, this means that W depends on X only through the symmetric matrix $X^t X$, where X^t denotes the transpose of X . Note that if ϕ is a diffeomorphism of Ω and $\tilde{\Omega}$, then the symmetric matrix $\nabla\phi^t \nabla\phi$ represents the pull-back of the euclidian metric from $\tilde{\Omega}$ into Ω . It is natural that the elastic energy should depend only on this metric.

The above set-up represents the elasticity of “finite deformations”, and the mathematical issues related to it are quite difficult, with many fundamental questions remaining unresolved.

For situations when the deformation is only a small deviation from identity (which should be, for example, most “civil engineering” structures, such as buildings, bridges, etc.), it is reasonable to linearize the theory. We write $\phi(x) = x + u(x)$ where u is the *displacement vector*. We have $\nabla\phi(x) = I + \nabla u$,

²³⁹A good text on linear elasticity is for example the book *The Theory of Elasticity* by Landau and Lifschitz.

it is often reasonable to assume that ∇u is small.²⁴⁰ We then consider the function $F(Y) = W(I + Y)$. We note that $F \geq 0$ and $DF(0) = 0$. Therefore the first non-zero term in the Taylor expansion of F is the quadratic term. Since W depends on X only through $X^t X$, the function F must depend on Y only through $Y + Y^t + Y^t Y$. Since $F(0) = 0$ and $DF(0) = 0$, the quadratic part of F can depend of Y only through $Y + Y^t$. The symmetric matrix $\frac{1}{2}(Y + Y^t)$ is called the (linearized) strain tensor corresponding to Y . The quadratic part of F must be a quadratic form of the strain tensor. Hence for the quadratic part Q of F at 0 we have

$$Q(Y) = \tilde{Q}\left(\frac{1}{2}(Y + Y^t)\right). \quad (1058)$$

Given a displacement vector $u: \Omega \rightarrow \mathbb{R}^n$, the (linear) strain tensor corresponding to u is defined by

$$e_{ij} = e_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}). \quad (1059)$$

In the linear theory the elastic energy will be give by a positive-definite quadratic form of e_{ij}

$$\mathcal{E}_{\text{lin}}(u) = \int_{\Omega} \frac{1}{2} c_{ijkl} e_{ij} e_{kl} dx, \quad e_{ij} \text{ given by (1059)}, \quad (1060)$$

where $c_{ijkl} = c_{jikl} = c_{klij}$, and the coercivity condition

$$\frac{1}{2} c_{ijkl} e_{ij} e_{kl} \geq \nu |e|^2 \quad (1061)$$

is satisfied for some $\nu > 0$, where $|e|^2 = e_{ij} e_{ij}$ (with summation over repeated indices). As an exercise you can check that in dimension $n = 3$ the quadratic form (1059) is determined by 21 coefficients. Symmetries can reduce the number of parameters which is needed to characterize a material, but, in general, one indeed needs 21 different parameters to describe real materials in this context. When the material is isotropic, e. g. its properties are independent of direction, the number of parameters reduces to only two. As an exercise, show that the isotropy can be characterized by the condition

$$\tilde{Q}(RSR^t) = \tilde{Q}(S), \quad S \in M^{n \times n} \text{ is symmetric, } R \in SO(n), \quad (1062)$$

where \tilde{Q} is as in (1058). Moreover all such \tilde{Q} are given by

$$\tilde{Q}(S) = \mu |S|^2 + \frac{\lambda}{2} (\text{Tr } S)^2. \quad (1063)$$

Hence a linear, homogeneous, isotropic material is described by

$$\mathcal{E}_{\text{lin}}(u) = \int_{\Omega} \mu |e|^2 + \frac{\lambda}{2} (e_{kk})^2 \quad (\text{summation over repeated indices}). \quad (1064)$$

²⁴⁰There is also a slightly more general theory in which we assume that $\nabla \phi$ is close to $SO(n)$, although not necessarily close to I . For example a long thin rod is a good example where this modification would be relevant.

Note that the integrand is a convex quadratic function of ∇u , but it is degenerate, in the sense that it vanishes on some non-trivial subspace (antisymmetric matrices). Hence (1043) cannot be satisfied. For materials which are isotropic but not homogeneous, the coefficients μ and λ can depend on x . The quadratic functional satisfies the ellipticity condition (1046), as one can easily verify directly. In fact, for (1064) the coercivity in $H_0^1(\Omega)$ (in the constant coefficient case) can be proved directly, by simple integration by parts, without the need to use Fourier transformation. For $u \in H_0^1(\Omega)$ we have

$$\int_{\Omega} e_{ij}e_{ij} = \int_{\Omega} \frac{1}{2} (u_{i,j}u_{i,j} + u_{i,j}u_{j,i}) = \int_{\Omega} \frac{1}{2} (u_{i,j}u_{i,j} + u_{i,i}u_{j,j}). \quad (1065)$$

Therefore, for constant μ, λ (independent of x), and $u \in H_0^1(\Omega)$ we have

$$\mathcal{E}_{\text{lin}}(u) = \int_{\Omega} \frac{1}{2} \mu |\nabla u|^2 + \frac{1}{2} (\mu + \lambda) (\text{div } u)^2, \quad (1066)$$

which is uniformly convex in ∇u . It should be emphasized that this form of the functional is only good for Dirichlet boundary conditions. If parts of the boundary are “free” minimizing (1066) can give the wrong boundary condition. The situation is similar to example (1044): we can add to the integrand in (1064) a suitable combination $M(\nabla u)$ of 2×2 sub-determinants of the matrix ∇u which will make it uniformly convex. We also emphasize that this is possible only when the coefficients μ and λ are constant.

The Euler-Lagrange equations corresponding functional

$$\mathcal{E}_{\text{lin}}(u) = \int_{\Omega} \mu |e|^2 + \frac{\lambda}{2} (e_{kk})^2 - uf(x) \, dx \quad (1067)$$

and (with constant μ, λ) also to

$$\int_{\Omega} \frac{1}{2} \mu |\nabla u|^2 + \frac{1}{2} (\mu + \lambda) (\text{div } u)^2 - uf(x) \, dx \quad (1068)$$

(again with constant μ, λ are

$$-\mu \Delta u - (\mu + \lambda) \nabla \text{div } u = f(x). \quad (1069)$$

When we minimize of $H_0^1(\Omega)$ the boundary condition is of course $u|_{\partial\Omega} = 0$. The system (1069) is known as *Lamé's system*, or *Lamé's equations*. It describes small deformation of a homogeneous isotropic materials. For non-homogenous, non-isotropic materials one works with (1060) with c_{ijkl} depending on x . The corresponding Euler-Lagrange equations are

$$-\tau_{ij,j} = f_i(x), \quad \tau_{ij} = c_{ijkl}e_{kl}, \quad (\text{summation over repeated indices}) \quad (1070)$$

The (symmetric) matrix τ_{ij} is called the *stress tensor*, and it represents tensions in the material. It is an important quantity, as in general design should be such

that the stresses do not exceed some limits which are considered safe for a given material.

When the coefficients are not constant, or the boundary conditions are not Dirichlet, the coercivity of the elastic energy \mathcal{E}_{lin} is harder to prove. The main result in this area is *Korn's inequality*. We will not go into details, but we at least formulate the following result.

Theorem (a special case of Korn's inequality)

Let Ω be a bounded domain with a Lipschitz continuous boundary. Let $\Gamma_1 \subset \Gamma = \partial\Omega$ be a set of positive measure, and let

$$\mathcal{X} = W_{\Gamma_1}^{1,2} = \{u \in W^{1,2}(\Omega, \mathbb{R}^n), u|_{\Gamma_1} = 0\}. \quad (1071)$$

There exists $\nu > 0$ such that

$$\int_{\Omega} |e(u)|^2 dx \geq \nu \|u\|_{W^{1,2}}^2, \quad u \in \mathcal{X}. \quad (1072)$$

We will not go into the proof.²⁴¹ However, as an exercise which illustrates some of the issues involved, you can consider a special case. Note that the inequality implies that if $u \in \mathcal{X}$ and $e(u) = 0$, then $u = 0$. Try to prove this directly!

It is clear from this example that in the vector-valued case the question of coercivity of general quadratic forms (1041) on subspaces of $W^{1,2}(\Omega)$ which are larger than $H_0^1(\Omega)$, such as for example the space $W_{\Gamma_1}^{1,2}(\Omega, \mathbb{R}^n)$ above, can be quite non-trivial, which is what we aimed to illustrate.

Finally, we should at least mention the general definition of ellipticity for (systems of) PDEs with constant coefficients. Let $\Omega \subset \mathbb{R}^n$ and consider functions $u: \Omega \rightarrow \mathbb{R}^k$. For such functions we consider constant coefficient operators L of the form

$$Lu = \sum_{|\alpha| \leq m} A_{\alpha} (-i\partial)^{\alpha} u, \quad (1073)$$

where A_{α} are $n \times k$ matrices. The *principal part* of L is

$$L_m u = \sum_{|\alpha|=m} A_{\alpha} (-i\partial)^{\alpha} u. \quad (1074)$$

The *symbol* of the operator L is

$$L(\xi) = \sum_{|\alpha| \leq m} A_{\alpha} \xi^{\alpha}, \quad (1075)$$

²⁴¹For modern development concerning Korn's inequalities we refer the reader to the paper Friesecke, Gero; James, Richard D.; Müller, Stefan A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity. *Comm. Pure Appl. Math.* 55 (2002), no. 11, 1461-1506.

where, as usual, $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$.

By definition, the operator L is *elliptic* if the symbol $L_m(\xi)$ of its principal part satisfies the condition that the matrix $L_m(\xi)$ (the *principal symbol*) is non-singular (in the sense that its kernel is trivial) whenever $\xi \neq 0$.

This class of operators is quite general and cannot be covered by the methods we used, via the quadratic forms and the Lax-Milgram lemma. Nevertheless, solutions of $Lu = 0$ still have nice properties. For example, they are analytic. The discussion of the correct boundary condition and boundary regularity is quite non-trivial. These topics are covered in the well-known paper by Agmon, Douglis, and Nirenberg mentioned earlier.²⁴²

A good reference for the class of operators which can be treated via the variational methods (including the equations of linear elasticity and the Korn inequality) is the book *Les méthodes directes en théorie des équations elliptiques* by J. Nečas.²⁴³

²⁴²Agmon, S.; Douglis, A.; Nirenberg, L. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. *Comm. Pure Appl. Math.* 12 1959 623727.

Agmon, S.; Douglis, A.; Nirenberg, L. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II. *Comm. Pure Appl. Math.* 17 1964 3592.

²⁴³An English edition of the book by Springer is expected to appear soon.

Today we start discussing dispersive equations. The behavior of the solutions of these equations can be quite different from what we have seen so far for elliptic and parabolic equations. Roughly speaking, one can say that the solutions of the elliptic and parabolic equations try to become as constant as possible, at least as far as small scales (or high Fourier frequencies) are concerned. Fast oscillations in the solutions are suppressed, and there is a smoothing effect. Energy is “damped”, at least for high frequencies. Regularity is typically a local phenomenon, at least in the linear case and smooth coefficients. Also, the parabolic equation are usually ill-posed for evolution in the opposite time-direction.

On the other hand, dispersive equations typically conserve energy, tend to display periodic behavior unless energy can escape, and possible smoothing effects cannot be attributed to damping, but to “dispersion”, which means, roughly speaking, that waves of different frequencies travel at different speeds. Moreover, the “arrow of time” is not important: well-posedness in one direction of time typically gives also well-posedness in the opposite direction.

We will start by studying a particular examples of a dispersive equations, the Schrödinger equation. As far as classical PDEs are concerned, it is a relatively new equation, written down by E. Schrödinger around 1926.²⁴⁴

The unknown function in the Schrödinger equation is a *complex valued* function $u = u(x, t)$ defined for $x \in \mathbb{R}^n$ and t in some time interval (t_1, t_2) (which can be taken to be \mathbb{R}). The variable x can also be restricted to some domain $\Omega \subset \mathbb{R}^n$, in which case a boundary condition should be imposed on $\partial\Omega$. For now we will concentrate on the case $\Omega = \mathbb{R}^n$.

The equation is

$$iu_t + \Delta u = 0. \quad ^{245} \tag{1076}$$

The meaning of the function u is not as easy to interpret in comparison what we have seen for other equations we have studied so far, where the solution

²⁴⁴For comparison, let us recall the dates for other equations:

- wave equation: 1747 (d’Alembert),
- Euler’s equations for ideal incompressible fluids: 1757,
- Laplace’s equation: 1784;
- heat equation: 1807 (J. Fourier),
- Navier-Stokes equations: 1820s-1840s,
- Maxwell’s equations: 1860s.

²⁴⁵This is the simplest “non-dimensionalized” form of the equation, in which units are chosen so that the coefficients come out in a simplest possible way, and the quantum particle is “free” (no forces). The classical form of the equations, incorporating potential forces, is

$$i\hbar \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2m} \Delta \psi + V(x)\psi,$$

where \hbar is the Planck constant/ 2π , m is the mass of the particle, and V is the potential describing forces acting on the particle.

has a clear intuitive meaning. In fact, after the discovery of the equation in 1926 it took some time before most physicists agreed on what $u(x, t)$ really represents.²⁴⁶ According to the now standard view proposed by Max Born, the function u should be viewed as “probability amplitude”²⁴⁷ (which can be complex valued) and the function $u\bar{u} = |u|^2$ should be viewed as probability density.²⁴⁸

The situation we have here is somewhat different to what we have seen with the Laplace equation or the heat equation. For these equations the heuristic meaning of the solution was more or less clear, and we could use it to make some good guesses about what solutions do. With the Schrödinger equation we will proceed differently. We will try to understand the equation directly from various formulae, relying only partially on the notion that it should describe some kind of waves, which however can exhibit quite different behavior from, say, electromagnetic waves or acoustic waves. After we have seen enough examples of how solutions behave, we will be able to start to understand how solutions behave even without doing calculations.²⁴⁹

We should note that Schrödinger equation appears not only in the context of Quantum Mechanics, but also in more classical situations. For example, some limiting regimes of 3d incompressible Euler’s equations lead to (non-linear) Schrödinger equation (as well as other dispersive equations). Also, propagation of light in fiber optics cables is described by (non-linear) Schrödinger equation. The equation has some universal features typical for situations where dispersion of waves plays an important rôle.²⁵⁰

We will start our mathematical investigation by calculating the fundamental solution of equation (1076) in $R^n \times R$. (It should be mentioned already at this point that the fundamental solutions in $\Omega \times R$ for a bounded $\Omega \subset \mathbb{R}^n$ and natural boundary conditions on $\partial\Omega$ behave very differently.)

We will make use of our calculation of the fundamental solution of the heat equation in lecture 51. Let us write the heat equation in a slightly more general form

$$u_t - a\Delta u = 0. \tag{1077}$$

Using the notation $\Gamma(x, t)$ for the fundamental solution (775) of the heat equation, it is easy to check that for any real $a > 0$ the fundamental solution

²⁴⁶See Max Born’s 1954 Nobel lecture for an account of this, it available online at http://nobelprize.org/nobel_prizes/physics/laureates/1954/born-lecture.pdf

²⁴⁷If you have not seen this term before, you can just ignore it for now.

²⁴⁸Interestingly enough, Schrödinger did not seem to be quite happy with this interpretation – see M. Born’s lecture mentioned above.

²⁴⁹The process of learning Quantum Mechanics is in some sense similar. “Everyday experience” does not provide good heuristics about the quantum world, and can in fact be quite unhelpful in this context.

²⁵⁰One might think that, for example, the interpretation in terms of some limiting regimes of incompressible flows might be helpful for developing heuristics about the solution. This only true up to a degree, as behavior of incompressible fluids can be quite counter-intuitive.

of (1077), given by

$$u_t - a\Delta u = \delta(x, t), \quad u \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}) \quad (1078)$$

is given by the function

$$\Gamma_a(x, t) = \Gamma(x, at). \quad (1079)$$

We write out the explicit expression for the reader's convenience

$$\Gamma_a(x, t) = \begin{cases} \frac{1}{(4\pi at)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4at}} & t > 0, \\ 0 & t < 0. \end{cases} \quad (1080)$$

It is not hard to check that this formula works also for complex a as long as $\operatorname{Re} a > 0$. The properties of Γ_a are quite similar to the properties of the heat kernel in this case.²⁵¹ This can also be seen in terms of the Fourier transformation. As in lecture 51, taking the Fourier transformation of (1078) gives

$$\hat{u}(\xi, \tau) = \frac{1}{i\tau + a|\xi|^2}, \quad (1081)$$

which is a well-defined distribution when $\operatorname{Re} a > 0$. (It is also well-defined when $\operatorname{Re} a < 0$, with $a = -1$ corresponding to the fundamental solution of the backward heat equation (780).)

For $a = i$ equation (1078) is the equation for the fundamental solution of the Schrödinger equation (1076).²⁵² In what follows we will focus on this form of the Schrödinger equation. What happens if we take $a = i$ in the above formulae? Formula (1080) for $a = i$ gives a smooth, well defined function for $t > 0$, and there is no problem checking that the function will satisfy

$$u_t - i\Delta u = 0, \quad t > 0. \quad (1082)$$

However, the formula (1081) for $\hat{u}(\xi, \tau)$ becomes ambiguous, as the function

$$\frac{1}{i(\tau + |\xi|^2)} \quad (1083)$$

is not locally integrable and the equation

$$i(\tau + |\xi|^2)\hat{u} = 1 \quad (1084)$$

does not determine the solutions uniquely.

The situation is similar to the simpler case of the equation in \mathbb{R}

$$\frac{d}{dt}v = \delta(t), \quad (1085)$$

²⁵¹There is an additional feature of some oscillation caused by the imaginary part of a , which however does not change the qualitative properties of the function, such as the smoothness away from the origin.

²⁵²modulo an unimportant multiple of i

or

$$i\tau\hat{v} = 1, \quad (1086)$$

which we of course know how to solve. The solutions are of the form

$$v(t) = \theta(t) + \text{const.}, \quad (1087)$$

where θ is the Heaviside function (see lecture 45, (644)).

The constant in (1087) reflect the fact that the solution of (1086) is not unique in the class of tempered distributions. If \hat{v} satisfies (1086), then $\hat{v} + c\delta$ also satisfies (1086). The situation with (1084) is similar, and in some sense we have the ambiguity of (1087) for each ξ . The solution $\hat{u}(\xi, \tau)$ is given as a distribution only modulo the solutions of $i(\tau + |\xi|^2)\hat{u}(\xi) = 0$, such as for example measures supported on the surface $\{\tau + |\xi|^2 = 0\}$. The ambiguity is removed if we demand that $u = 0$ for $t < 0$.

We expect that the limit

$$\Gamma_i = \lim_{\substack{a \rightarrow i \\ \text{Re } a > 0}} \Gamma_a, \quad (1088)$$

(which obviously vanishes for $t < 0$), should give the solution of (1078) for $a = i$. This is indeed the case, but we have to address the following issues

(i) Is Γ_i a well defined distribution in $\mathbb{R}^n \times \mathbb{R}$? Note that this is obvious only for $n = 1$, when Γ_i is a locally integrable function. For general n we have

$$|\Gamma_i(x, t)| = \frac{1}{|4\pi t|^{\frac{n}{2}}}, \quad t > 0, \quad (1089)$$

and this function is not locally integrable for $n \geq 2$.

(ii) Do we have $\partial_t \Gamma_i - i\Delta \Gamma_i = \delta$ in the sense of distributions?

We can also ask the following closely related question:

(iii) If $u_0(x)$ is a sufficiently regular integrable function on \mathbb{R}^n and we define

$$u(x, t) = \int_{\mathbb{R}^n} \Gamma_i(x - y, t) u_0(y) dy, \quad t > 0, \quad (1090)$$

do we have

$$u(x, t) \rightarrow u_0(x), \quad t \rightarrow 0, \quad t > 0 \quad ? \quad (1091)$$

These questions are somewhat more difficult than the corresponding questions for the heat equation (or the kernel Γ_a with $\text{Re } a > 0$), due to the fact that $\Gamma_i(x, t)$ is not integrable in x and in fact does not decay as $x \rightarrow \infty$.

²⁵³Note that for $t > 0$ this is a well-defined function which is smooth if, say, $u_0(x)$ is smooth and compactly supported (or, more generally, belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$.)

The main argument can be seen from the following calculations. For simplicity we first do the calculations for $n = 1$.

Let us consider a function $\varphi: \mathbb{R} \rightarrow \mathbf{C}$ which is “sufficiently regular” and decays “sufficiently fast” as $x \rightarrow \infty$. Let us estimate the integral

$$I(at) = \int_{\mathbb{R}} e^{-\frac{x^2}{4at}} \varphi(x) dx. \quad (1092)$$

as $t \rightarrow 0_+$ and a close to i , with $\operatorname{Re} a \geq 0$. Let us first assume that φ vanishes in a neighborhood of $x = 0$. When this is the case and, moreover, $\operatorname{Re} a > 0$, then as $t \rightarrow 0_+$ the integral $I(at)$ approaches zero exponentially fast, simply due to the fast decay of $e^{-\frac{x^2}{4at}}$. However, the rate of decay depends on $\operatorname{Re} a$ and this argument cannot be used for $a = i$. For $a = i$ there is decay as $t \rightarrow 0_+$, but not due to the point-wise decay of the integrand. The reason for the decay will be in cancelations in the integral due to the oscillatory nature of $e^{-\frac{|x|^2}{4it}}$. A standard way to capture the cancelations is to use integration by parts.²⁵⁴ We have

$$e^{-\frac{|x|^2}{4at}} = -\frac{2at}{x} \frac{\partial}{\partial x} e^{-\frac{|x|^2}{4at}}, \quad (1093)$$

and integration by parts in (1092) gives

$$I(at) = 2at \int_{\mathbb{R}} e^{-\frac{|x|^2}{4at}} \frac{\partial}{\partial x} \frac{\varphi(x)}{x} dx. \quad (1094)$$

Denoting by \tilde{D} the operator $\varphi \rightarrow \frac{\partial}{\partial x} \frac{\varphi(x)}{x}$, we see that

$$I(at) = (2at)^k \int_{\mathbb{R}} e^{-\frac{|x|^2}{4at}} \tilde{D}^k \varphi(x) dx. \quad (1095)$$

We see that if φ is supported away from $x = 0$, then for each integrable derivative of φ we can increase the rate of decay of the integral to zero as $t \rightarrow 0_+$ by one power of t . The integral (1095) is also well-defined and finite when φ has zero of order $2k$ at $x = 0$, and hence in this case $I(at)$ will still decay as t^k (when $\operatorname{Re} a \geq 0$ and $t \rightarrow 0_+$).

What about functions φ which do not vanish at 0? For that we note that every smooth φ can be written as

$$\varphi = \tilde{\varphi} + P(x)e^{-\frac{|x|^2}{2}}, \quad (1096)$$

where $\tilde{\varphi}$ has zero of order l at $x = 0$, and P is a polynomial of order l . The integral

$$\int_{\mathbb{R}} e^{-\frac{|x|^2}{4at}} P(x)e^{-\frac{x^2}{2}} dx \quad (1097)$$

²⁵⁴See for example L. Hörmander’s book *The Analysis of Linear Partial Differential Operators I*, Section 3.3.

can be evaluated explicitly. First, using $\int_{\mathbb{R}} e^{-\frac{x^2}{2z}} dx = \sqrt{2\pi z}$ we see that for $\lambda > 0, t > 0$, and $\operatorname{Re} a \geq 0$ we have

$$\int_{\mathbb{R}} e^{-\frac{x^2}{4at}} e^{-\lambda \frac{x^2}{2}} dx = \int_{\mathbb{R}} e^{-\frac{x^2(1+2\lambda at)}{4at}} = \sqrt{\frac{4\pi at}{1+2\lambda at}}. \quad (1098)$$

Taking derivatives in λ at $\lambda = 1$ we can evaluate (1097) for $P(x) = x^{2k}$, while the integrals with $P(x) = x^{2k+1}$ clearly vanish. We obtain

$$\int_{\mathbb{R}} e^{-\frac{|x|^2}{4at}} P(x) e^{-\frac{x^2}{2}} dx = \sqrt{4\pi at} (P(0) + b_1 t + \cdots + b_l t^l + O(t^{l+1})), \quad t \rightarrow 0_+, \quad (1099)$$

where b_j are suitable constants depending on P and a . The exact form of b_j will not be important.²⁵⁵ Putting (1095), (1096) and (1099) together we see that

$$\int_{\mathbb{R}} e^{-\frac{|x|^2}{4at}} \varphi(x) dx = \sqrt{4\pi at} (\varphi(0) + \tilde{b}_1 t + \cdots + \tilde{b}_l t^l + o(t^l)), \quad t \rightarrow 0_+, \quad (1100)$$

for suitable $\tilde{b}_1, \dots, \tilde{b}_l$, when φ has continuous and integrable derivatives of order $2l$.²⁵⁶ With these calculations, and their variants for $n > 1$ it is not hard to check that the answers to questions (i)–(iii) above is positive and that Γ_i indeed represents a fundamental solution for (1076), and (1090) gives a solution of the initial-value problems $u_t - i\Delta u = 0$, $u(x, 0) = u_0(x)$, for $t > 0$, at least when u_0 is sufficiently regular with sufficiently fast decay to ∞ . We will look at the case $n > 1$ next time.

Remarks

1. The calculation above is a special case of calculations with *oscillatory integrals of the first kind*, see E.M.Stein's book "Harmonic Analysis" (1993), Chapter VIII, for more details.

2. The in the Fourier variables (ξ, τ) , of $\hat{u}(\xi, \tau)$ defined by (1081) for $a = i + \varepsilon$ (with $\varepsilon > 0$) is similar to approximating the solution \hat{v} of (1086) by

$$\hat{v}_\varepsilon = \frac{1}{i\tau + \varepsilon} \quad (1101)$$

and letting $\varepsilon \rightarrow 0_+$. This corresponds to taking the fundamental solution of (1085) as the limit of fundamental solutions of

$$\frac{d}{dt} v + \varepsilon v = \delta(t), \quad \varepsilon > 0, \quad (1102)$$

which, unlike the solution of (1085) is uniquely determined in the class of tempered distributions. It will be supported in $\{t \geq 0\}$. If we take $\varepsilon \rightarrow 0_-$ in (1101),

²⁵⁵Note however that the power series on the right-hand side cannot be convergent, as it clearly cannot converge for $t < 0$.

²⁵⁶These assumptions can be somewhat weakened. When we really need is the decomposition (1096) where $\tilde{D}^k \varphi$ is integrable for $k \leq l$.

we obtain a fundamental solution of (1085) which is supported in $\{t \leq 0\}$. This also corresponds to taking $\varepsilon < 0$ in (1102). Quite similarly, we take $a = i - \varepsilon$, $\varepsilon > 0$ in (1081), we obtain another fundamental solution $\tilde{\Gamma}_i$ of the Schrödinger equation, which will be supported in $\{t \leq 0\}$. That solution is related to the fundamental solution of the backward heat equation in the same way the solution Γ_i above is related to the solution of the forward heat equation, and one has

$$\tilde{\Gamma}_i(x, t) = \begin{cases} -\frac{1}{(4\pi it)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4it}} & t < 0, \\ 0 & t > 0. \end{cases} \quad (1103)$$

The difference

$$K(x, t) = \Gamma_i(x, t) - \tilde{\Gamma}_i(x, t) \quad (1104)$$

is also of interest. It is a distribution satisfying

$$K_t - i\Delta K = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R} \quad (1105)$$

and

$$K(x, t) \rightarrow \delta(x) \quad \text{in } \mathcal{S}'(\mathbb{R}^n) \text{ as } t \rightarrow 0. \quad (1106)$$

The Fourier transform of K is the surface measure on $\{\tau + |\xi|^2 = 0\}$ which projects on the standard Lebesgue measure in \mathbb{R}^n under the projection $(\xi, \tau) \rightarrow \xi$.

The kernel K can also be arrived at in another way. Assume we wish to solve

$$u_t - i\Delta u = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}, \quad (1107)$$

with the initial condition $u(x, 0) = u_0(x)$. Taking Fourier transform in x only, the equations decomposes into a family of one-dimensional ODEs for $\hat{u}(\xi, t)$,

$$\hat{u}_t(\xi, t) + i|\xi|^2 \hat{u}(\xi, t) = 0, \quad (1108)$$

with initial condition $\hat{u}(\xi, 0) = \hat{u}_0(\xi)$. The solution is

$$\hat{u}(\xi, t) = u_0(\xi) e^{-i|\xi|^2 t}, \quad t \in \mathbb{R}. \quad (1109)$$

Taking the inverse Fourier transform, we obtain

$$u(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}_0(\xi) e^{i\xi x - i|\xi|^2 t} d\xi. \quad (1110)$$

For $u_0 = \delta$, we have $\hat{u}_0 = 1$ and – at least formally – the integral (1110) then expresses the inverse Fourier transformation of the above described measure on the surface $\{\tau + |\xi|^2 = 0\}$.

Lecture 66, 3/28/2011

Today we will look at the situations for $n > 1$ with questions (i)–(iii) we dealt with last time for $n = 1$.

Let us first recall the following formula, which follows from (483) and (484).

$$\widehat{e^{-\frac{|x|^2}{2a}}} = (2\pi a)^{\frac{n}{2}} e^{-\frac{a|\xi|^2}{2}}. \quad (1111)$$

Strictly speaking, the calculations in lecture 34 show this only when a is real. In that case we must have $a > 0$, so that the functions represent tempered distributions. For complex a the functions in (1111) represent tempered distribution if and only if $\operatorname{Re} a \geq 0$, and it is not hard to check that the formula remains valid for such a . This includes the case $a = i$ (or more generally, $a = \sigma i$ for real $\sigma \neq 0$). In particular, taking $\xi = 0$ and $a = 2it$ we obtain

$$\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4it}} = (4\pi it)^{\frac{n}{2}}, \quad t \neq 0. \quad (1112)$$

This integral is of course not absolutely convergent, but we can regularize it, for example by adding $e^{-\varepsilon|x|^2}$ to the integrand and taking $\varepsilon \rightarrow 0_+$ or by integrating over the balls B_R and taking $R \rightarrow \infty$. Taking $n = 1$ and $t = \frac{1}{4}$ we obtain the well-known identities for the so-called Fresnel integrals²⁵⁷

$$\int_{\mathbb{R}} \cos x^2 dx = \int_{\mathbb{R}} \sin x^2 dx = \sqrt{\frac{\pi}{2}}. \quad (1113)$$

Returning to the fundamental solution Γ_i from the last lecture, we see from (1112) that, with proper interpretation,

$$\int_{\mathbb{R}^n} \Gamma_i(x, t) dx = 1, \quad (1114)$$

similar to what we have for the heat kernel.²⁵⁸ The kernel Γ_i also has the same scale invariance as the heat kernel:

$$\Gamma_i(x, t) = t^{-\frac{n}{2}} \Gamma_i\left(\frac{x}{\sqrt{t}}, 1\right) \quad (1115)$$

So we see that the functions

$$\Gamma_i(t) = \Gamma_i(\cdot, t) \quad (1116)$$

should for $t \rightarrow 0_+$ in some sense approach the Dirac mass, if we can verify that outside of the origin the oscillations give us enough cancelation so that

²⁵⁷See also http://en.wikipedia.org/wiki/Fresnel_integral

²⁵⁸That this should be true can also be seen by integrating (1078) over x .

$\Gamma_i(t) \rightarrow 0$ in $\mathbb{R}^n \setminus \{0\}$ in distributions when $t \rightarrow 0_+$. To prove this, as well as to show that the answer to questions (i)–(iii) from the last lecture is positive, we will use integration by parts in a way similar to the case $n = 1$ we considered last time.

Let us start by estimating the integral

$$I(at) = \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4at}} \varphi(x) dx, \quad a \neq 0, \operatorname{Re} a \geq 0, t > 0, \quad (1117)$$

where φ is sufficiently regular (with sufficiently fast decay). The exact requirements will be determined in the course of the calculation. As in the case $n = 1$, we use integration by parts to capture the cancelations coming from the oscillations of the integrand. We have

$$e^{-\frac{|x|^2}{4at}} = -2at \frac{x}{|x|^2} \nabla e^{-\frac{|x|^2}{4at}}. \quad (1118)$$

We define an operator \tilde{D} by

$$\tilde{D}\varphi = \operatorname{div} \left(\frac{x}{|x|^2} \varphi \right). \quad (1119)$$

Using (1118), and integration by parts, we can write, at least formally

$$I(at) = 2at \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4at}} \tilde{D}\varphi dx. \quad (1120)$$

Under which assumptions is this formal identity valid?

The reader can verify the following statement as an exercise:

If f is a bounded function in \mathbb{R}^n with bounded derivatives and $v = (v_1, \dots, v_n)$ is an integrable vector field with integrable $\operatorname{div} v$, then $\int_{\mathbb{R}^n} -v \nabla f = \int_{\mathbb{R}^n} f \operatorname{div} v$.

From this we see that a sufficient condition for validity of (1120) is that $\frac{\varphi}{|x|}$ and $\tilde{D}\varphi$ are both integrable. The procedure can be continued: if $\frac{\varphi}{|x|}, \frac{\tilde{D}\varphi}{|x|}, \dots, \frac{\tilde{D}^{k-1}\varphi}{|x|}, \tilde{D}^k\varphi$ are integrable, then

$$I(at) = (2at)^k \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4at}} \tilde{D}^k\varphi dx. \quad (1121)$$

For a general sufficiently smooth φ can now proceed in a way similar to the case $n = 1$ from the last lecture. Assume for simplicity that $\varphi \in \mathcal{S}(\mathbb{R}^n)$. We write

$$\varphi = \tilde{\varphi} + P(x)e^{-\frac{|x|^2}{2}}, \quad (1122)$$

where P is a polynomial of order $2l$ and $\tilde{\varphi}$ has zero of order $2l$ at $x = 0$. Then

$$I(at) = \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4at}} P(x)e^{-\frac{|x|^2}{2}} dx + o(t^l), \quad t \rightarrow 0_+. \quad (1123)$$

The last integral can be written as a product of one-dimensional integrals which were evaluated in the previous lecture, see (1098) and (1099). Putting things together, we obtain

$$I(at) = (4\pi at)(\varphi(0) + b_1 t + \dots b_l t^l + o(t^l)). \quad (1124)$$

Although we will not need it in what follows, it is worth noting that (1121) could be used even for the evaluation of the leading term. For example, when $n = 2$ and $k = 1$, we have

$$\tilde{D}\varphi = \operatorname{div}(\nabla \log |x| \varphi(x)) = 2\pi \delta(x) \varphi(0) + (\nabla \log |x|) \nabla \varphi, \quad (1125)$$

and we see that we obtain the leading term in (1124) directly.

Relation (1124) easily implies that the answer to the equations (i)–(iii) from the previous lecture is positive. For example, let us check that Γ_i is a well-defined distribution. Let $\varphi(x, t)$ be a compactly supported function of $\mathbb{R}^n \times \mathbb{R}$. We wish to show that the integral

$$\int_{\mathbb{R}^n \times \mathbb{R}} \Gamma_i(x, t) \varphi(x, t) dx dt \quad (1126)$$

is well-defined, for example in the sense of (1088). This is an easy consequence of (1124), as (1124) implies that the expression

$$\int_0^\infty \int_{\mathbb{R}^n} \Gamma_a(x, t) \varphi(x, t) dx dt \quad (1127)$$

converges as $a \rightarrow i$, $\operatorname{Re} a > 0$ to

$$\int_0^\infty \left(\int_{\mathbb{R}^n} \Gamma_i(x, t) \varphi(x, t) dx \right) dt \quad (1128)$$

which is well defined, as the inner integral is a bounded function of t by (1124).

Positive answer to question (iii) (previous lecture) is a direct consequence of (1124).

Positive answer to question (ii) (previous lecture) follows from the identities

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} \Gamma_1(-\varphi_t - i\Delta\varphi) = \lim_{t_1 \rightarrow 0^+} \int_{t_1}^\infty \int_{\mathbb{R}^n} \Gamma_i(x, t) (-\varphi_t - i\Delta\varphi) dx dt \quad (1129)$$

and

$$\int_{t_1}^\infty \int_{\mathbb{R}^n} \Gamma_i(x, t) (-\varphi_t - i\Delta\varphi) dx dt = \int_{\mathbb{R}^n} \Gamma_i(x, t_1) \varphi(x, t_1) dx \quad t_1 > 0, \quad (1130)$$

where $\varphi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$. By (1124), the last integral converges to $\varphi(0, 0)$ as $t_1 \rightarrow 0$. (This is another example of Duhamel's principle, discussed in lecture 53 for the heat equation.)

We can also look at these calculations in the Fourier picture. For example let φ be a sufficiently regular initial datum and let

$$u(x, t) = \int_{\mathbb{R}^n} \Gamma_i(x - y, t) \varphi(y) dy, \quad t > 0. \quad (1131)$$

We also have, e. g. by (1111)

$$u(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{\varphi}(\xi) e^{i\xi x - i|\xi|^2 t} d\xi, \quad t > 0. \quad (1132)$$

If the Fourier transform $\hat{\varphi}$ is integrable, then we see from the Lebesgue Dominated Convergence Theorem that (1132) in fact defines a continuous function in $\mathbb{R}^n \times \mathbb{R}$, with $u(x, 0) = \varphi(x)$ (and satisfying (1076) in $\mathbb{R}^n \times \mathbb{R}$ in distribution).²⁵⁹ This is another way of showing that the answer to question (iii) from the previous lecture is positive. The positive answer to (i) and (ii) can also be verified via the Fourier transformation.

In the considerations above we did not try to obtain minimal regularity conditions under which the various expressions converge to the right limits. Questions such as what are the minimal assumptions on φ which still give $u(x, t) \rightarrow \varphi(x)$ for each x (or for almost every x) when $t \rightarrow 0$ can be quite difficult.²⁶⁰

²⁵⁹The extension to $\mathbb{R}^n \times \mathbb{R}$ can also be written as $u(x, t) = \int_{\mathbb{R}^n} K(x - y, t) \varphi(y) dy$, where K is given by (1104).

²⁶⁰See, for example, the paper "Schrödinger equations: pointwise convergence to the initial data" by L. Vega, Proceedings of AMS, Vol. 102, No. 4, 1988.

Lecture 67, 3/30/2011

Today we will discuss some examples of solutions of the Schrödinger equation

$$u_t - i\Delta u = 0 \quad \text{in } \mathbb{R}^n \quad (1133)$$

with the initial condition²⁶¹

$$u(x, 0) = u_0(x). \quad (1134)$$

We will use the the representation through the kernel K defined by (1104)

$$u(x, t) = \int_{\mathbb{R}^n} K(x - y, t) u_0(y) dy, \quad (1135)$$

which is equivalent to the Fourier representation

$$u(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}_0(\xi) e^{i\xi x - i|\xi|^2 t} d\xi. \quad (1136)$$

We assume at this point that u_0 is “sufficiently regular”. This can be interpreted in various ways, depending on which phenomenon we wish to illustrate. Typically we will assume that u_0 has a certain number of derivatives which are integrable.

We have not yet addressed the the question to what degree the solutions are unique. This will be addressed later. For now we will discuss the properties of the solutions given by the representation formulae (1135) or (1136).

Although formally the formulae are very similar to corresponding formulae for the heat equation, there are some important differences in the properties of the solutions. Let us take for example $u_0(x)$ to be compactly supported, with continuous derivatives up to some fixed order l , which is large enough for the formulae to work, but still finite. We can assume that $\nabla^{l+1} u_0$ is discontinuous. Under these assumptions it is easy to see from (1135) that u will be smooth (and, in fact, analytic) in the $\mathbb{R}^n \times (0, \infty)$ and $\mathbb{R}^n \times (-\infty, 0)$. We can now take the same solution but think about $t = -1$ as the initial time. Then u should still be a solution for this shifted initial-value problem. This may be somewhat less obvious than it looks as we define the solution of the initial-value problem by representation formulae, as the integral in the expected formula

$$u(x, -1 + s) = \int K(x - y, s) u(y, -1) dy \quad (1137)$$

is may not be absolutely convergent. Let us nevertheless ignore this issue for now, and assume that $u(x, t)$ gives the right evolution from $u(x, -1)$. The evolution from the locally smooth (and bounded) function $u(x, -1)$ will result

²⁶¹We use this terminology even though the solution is also defined for negative t .

at time $t = 0$ in the non-smooth function $u_0(0)$. This is completely different from what we have for the heat equation. We have to be cautious with some certain formal manipulations with the representation (1137), such as taking derivatives in x .

Let us now look at some more specific examples of the solutions.

Example 1

Let u_0 to be a smooth, radially symmetric, compactly supported in the unit ball, with $u_0 \geq 0$ and $u(0) \sim 1$, $|\nabla u_0| \leq 2$, and the growth of $\sup_x |\nabla^k u_0(x)|$ with k as slow as possible for what is consistent with these assumptions. How does $u(x, t)$ behave? Looking at the representation formula (1134), we expect that

- For small times, $|t| \leq t_1$ with, say, $t_1 \sim 0.1$, the solution will change only slightly. The solution cannot have compact support for any $t \neq 0$, and will have a small (and decaying) oscillatory “tail” for large x , but for the most part it will stay quite close to u_0 in this range of t . In some sense, we can say that not much happens for $|t| \leq t_1$.
- For $t \sim 1$, there will be change of order 1, both for the size of the region where the bulk of the solution is supported, as well as in the profile of u . The change has to be consistent with the conservation laws

$$\int_{\mathbb{R}^n} |\nabla^k u(x, t)|^2 dx = \int_{\mathbb{R}^n} |\nabla^k u_0(x)|^2 dx, \quad (1138)$$

which will be discussed in some detail later. These are easily seen from example from the Fourier representation (1136), but it is not hard to derive them also in the “physical space” (the variable x) by integration by parts. The solution will again have an oscillating, small amplitude “tail” for large x , although the oscillation at the same x will be slower (but with larger amplitude)

- For large t the solution will be pointwise small, with slow oscillations in x for moderate x . The oscillations will be becoming faster and the amplitude smaller as $|x| \rightarrow \infty$. Again, the changes have to be consistent with (1138).

We can in fact illustrate this with an explicit formula. The reader can check easily that the function

$$v(x, t) = \frac{1}{(1 + 2it)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2(1+2it)}} \quad (1139)$$

solves (1133) and

$$v_0(x) = e^{-\frac{|x|^2}{2}}. \quad (1140)$$

Of course, v_0 is not really compactly supported, but the solution still gives a good illustration of the above described behavior.

Example 2

Let us now take a small $\varepsilon > 0$ and consider the initial data

$$u_{\varepsilon 0}(x) = u_0(\varepsilon x). \quad (1141)$$

The corresponding solution will be

$$u_{\varepsilon}(x, t) = u(\varepsilon x, \varepsilon^2 t), \quad (1142)$$

where u is the solution from Example 1. Here we use the fact that (1133) has the same scaling symmetry as the heat equation, see (769). The properties of u_{ε} can be read off from the properties of u . What will be important for us is that for the solution u_{ε} “not much happens” for times t with $|t| \leq \frac{t_1}{\varepsilon^2}$. For very small ε and times $|t| \leq \frac{1}{\varepsilon^2 - \delta}$ nearly nothing happens. (Of course, there is still some activity, but the amplitude of the resulting changes is very small. A first impression of an observer would probably be that the solution is constant in that time interval, and only after having a closer look she would notice that this is not quite true.) This is consistent with the fact that the function

$$u \equiv \text{const.} \quad (1143)$$

is an exact solution of (1133), and the solutions u_{ε} approach (1143) with $\text{const.} = u_0(0)$ as $\varepsilon \rightarrow 0_+$.

We will now use the simple solutions u_{ε} above and the symmetries of the equation to generate more interesting solutions.

Example 3

We will use the following non-trivial symmetry of equation (1133). If $u(x, t)$ is a solution and $b \in \mathbb{R}^n$, then

$$u^b(x, t) = u(x - bt, t) e^{ix \frac{b}{2} - i \frac{|b|^2}{4} t} \quad (1144)$$

is again a solution. This symmetry is related to the invariance of description of a classical particle under the *Galilean transformations* - the changes $(x, t) \rightarrow (x - bt, t)$ of the coordinate system. We see from this symmetry that - modulo uniqueness issues - multiplying initial condition u_0 by $e^{i \frac{b}{2} x}$ results in the change of the solution $u(x, t)$ corresponding to u_0 to $u^b(x, t)$ given by (1144). This transformation is sometimes referred to as a *Galilean boost*.

Let us apply this transformation to the solutions u_{ε} from example 1. The initial data will be $u_{\varepsilon 0}^b(x) = u_{\varepsilon 0}(x) e^{i \frac{b}{2} x}$. This can be interpreted as a “wave packet” - the slowly changing function $u_{\varepsilon 0}$ is modulated by the oscillations of $e^{i \frac{b}{2} x}$. The time evolution of the wave packet is quite simple: for times t with $|t| \leq \frac{t_1}{\varepsilon^2}$ the main feature of the evolution is translation at speed b . During the translation the wave packet is slowly disintegrating - it “radiates away” small disturbances, and gradually loses some “mass” and focus, but it takes time of order $\frac{t_1}{\varepsilon^2}$ before

these effects accumulate to a significant change. After times significantly longer than $\frac{1}{\varepsilon^2}$ the wave packet will completely disintegrate.

If we now take general initial data u_0 , we can write them as a linear combination of suitable wave packets. For example we can take a smooth partition of unity $\phi_k(\xi)$ is the Fourier coordinate ξ such that the size support of ϕ_k is of order ε and write $u_0 = \sum_k u_{0k}$ with u_{0k} being the inverse Fourier transform of $\hat{u}_0\phi_k$. The solution will decompose as $u = \sum u_k$, where the evolution of u_k is the evolution of u_{0k} . Each solution u_k will for while look like a wave packet, and the wave packets with different frequencies will move at different speeds. This produces the dispersion. The high-frequency wave packets will move out very fast, with speed proportional to the frequency. This explains the smoothing effect of the evolution when the initial data are, say, compactly supported. The part of the function responsible for the possible singularities is in the high frequencies, and therefore it will move away. It can however also work in the opposite direction: for initial data which are locally smooth, but with increasing oscillations as $x \rightarrow \infty$, the high-frequency wave packets can arrive quickly from far away and if they are coordinated in exactly the right way, they can produce a non-smooth function.

Last time we looked at the “wave packets” for the Schrödinger equation. The notion of a wave packet can be considered for more general equations. For example, let $P(\xi)$ be a real polynomial in the Fourier coordinate $\xi \in \mathbb{R}^n$. Then the operation

$$u(x) \rightarrow \text{inverse Fourier transform of } P(\xi)\hat{u}(\xi) \quad (1145)$$

corresponds to the differential operator $L = P(-i\partial)$ and the functions

$$u(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}_0(\xi) e^{i(\xi x - P(\xi)t)} dx \quad (1146)$$

formally solve

$$u_t + iLu = 0, \quad u(x, 0) = u_0(x). \quad (1147)$$

Note that if $u_0 \in \mathcal{S}(\mathbb{R}^n)$, then $u(x, t)$ is smooth and well-defined in $\mathbb{R}^n \times \mathbb{R}$, and equations (1147) are satisfied pointwise. If we assume $\hat{u}_0 \in L^1(\mathbb{R}^n)$, the function is still well-defined pointwise, satisfies the second equation of (1147) pointwise, and the first equation in the sense of distributions. In the physics literature the notation often is

$$u(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}_0(\xi) e^{i(\xi x - \omega(\xi)t)} dx. \quad (1148)$$

The relation between ξ and ω is called the *dispersion relation*. It expresses the *angular frequency* ω of the *plane wave* $e^{i(\xi x - \omega t)}$ in terms of the *wave vector* ξ . For equations of the form (1147) the dispersion relation is given by a polynomial, but there are important situations where the relation is given by a more general function.²⁶²

The notion of the wave packet discussed in the last lecture in the context of the Schrödinger equation generalizes to the situation described by (1148). This formula can also produce distinctive wave packets traveling at a quite well-defined speed for a relatively long time, before they disperse. Let us consider a smooth non-zero function $\hat{u}_0(\xi)$ supported near some fixed ξ^0 , say, in the interval $(\xi^0 - \frac{\varepsilon}{2}, \xi^0 + \frac{\varepsilon}{2})$ for some small $\varepsilon > 0$. For $\xi^0 = 0$ the function u_0 will be a slowly varying function in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. For $\xi^0 \neq 0$ we can think of u_0 as a slowly varying function modulated by $e^{i\xi^0 x}$. This is our initial wave packet. (Recall that the shift $\hat{f}(\xi) \rightarrow \hat{f}(\xi - \xi^0)$ on the Fourier side corresponds to $f(x) \rightarrow f(x)e^{i\xi^0 x}$ in the x -variable.) We can say that the frequency of the wave packet is approximately ξ^0 , but at the same time we should keep in mind that there is some ambiguity in ξ^0 as the function \hat{u}_0 is not supported at one point. If we wish to have our wave packet localized in space, it cannot have a sharply defined frequency.²⁶³

²⁶²For example, for the wave equation $u_{tt} - \Delta u = 0$, which we will study in some detail later, we have two “admissible” frequencies ω for each ξ , given by $\omega_+ = |\xi|$ and $\omega_- = -|\xi|$.

²⁶³This is related to the *Uncertainty Principle*.

We will now consider the time evolution of the wave packet u_0 , with \hat{u}_0 as above, over times which are quite smaller than $\frac{1}{\varepsilon^2}$. We assume that $\omega(\xi)$ is a smooth function of ξ . For ξ close to ξ^0 we have

$$\omega(\xi) = \omega(\xi^0) + \nabla\omega(\xi^0)(\xi - \xi^0) + O(|\xi - \xi^0|^2). \quad (1149)$$

On the support of u_0 we have

$$|\xi - \xi^0|^2 \leq \varepsilon^2, \quad (1150)$$

and hence for $t \ll \frac{1}{\varepsilon^2}$ we can write

$$\omega(\xi)t = \omega(\xi^0) + \nabla\omega(\xi^0)(\xi - \xi^0)t + o(1), \quad \xi \in \text{support } \hat{u}_0(\xi). \quad (1151)$$

We see that for $t \ll \frac{1}{\varepsilon^2}$ formula (1148) gives

$$u(x, t) \approx \frac{e^{i\gamma(\xi^0)t}}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}_0(\xi) e^{i\xi(x - \nabla\omega(\xi^0)t)} d\xi = e^{i\gamma(\xi^0)t} u_0(x - \nabla\omega(\xi^0)t), \quad (1152)$$

where $\gamma(\xi^0) = \omega(\xi^0) - \nabla\omega(\xi^0)\xi^0$. We see that the main feature of the evolution of the wave packet is its translation at speed $\nabla\omega(\xi^0)$. There is also the modulation by $e^{i\gamma(\xi^0)t}$ but note that this factor is constant in x and therefore it does not really significantly change the properties of $u(x, t)$ viewed for a fixed t as a function of x .

Our conclusion therefore is that, modulo some error terms and before disintegration, *the wave packets of frequency ξ move at speed $\nabla\omega(\xi)$* . The velocity $\nabla\omega(\xi)$ of the wave packets is called the *group velocity* of the waves.

For example, for the equation

$$u_t = u_{xxx}, \quad x \in \mathbb{R}, t \in \mathbb{R} \quad (1153)$$

(sometimes called the Airy equation) we have $\omega(\xi) = \xi^3$, and therefore the wave packets of frequency ξ move at speed $3\xi^2$ (before disintegrating).

It is important to keep in mind that – except in some special cases – the wave packets are typically losing their focus during the evolution and eventually will disperse. The special cases where this does not happen include the linear transport equation $u_t + a\nabla u = 0$, describing the translation of the function u at speed a (in any dimension), the one-dimensional wave equation $u_{tt} - u_{xx} = 0$ (which we will be studying in some detail later) or more general hyperbolic systems in $1 + 1$ dimension.

Lecture 69, 4/3/2011

Let us consider the initial value problem

$$\begin{aligned} u_t - i\Delta u &= 0, & (x, t) &\in \mathbb{R}^n \times \mathbb{R}, \\ u(x, 0) &= u_0(x), & x &\in \mathbb{R}^n. \end{aligned} \quad (1154)$$

Let us at first assume that $u_0 \in \mathcal{S}(\mathbb{R}^n)$. We have seen in lecture 67 that the solution can be given by Fourier representation formula

$$u(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}_0(\xi) e^{i\xi x - i|\xi|^2 t} d\xi, \quad (1155)$$

which can also be written as

$$\hat{u}(\xi, t) = \hat{u}_0(\xi) e^{-i|\xi|^2 t}. \quad (1156)$$

It is transparent that

$$|\hat{u}(\xi, t)|^2 = |\hat{u}_0(\xi)|^2, \quad t \in \mathbb{R} \quad (1157)$$

and hence for each $s \in \mathbb{R}$

$$\int_{\mathbb{R}^n} |\hat{u}(\xi, t)|^2 |\xi|^{2s} d\xi = \int_{\mathbb{R}^n} |\hat{u}_0(\xi)|^2 |\xi|^{2s} d\xi, \quad t \in \mathbb{R}. \quad (1158)$$

Taking $s = 0$ and using Plancherel's formula (493), we see that

$$\int_{\mathbb{R}^n} |u(x, t)|^2 dx = \int_{\mathbb{R}^n} |u_0(x)|^2 dx, \quad t \in \mathbb{R}. \quad (1159)$$

Taking $s = k$, and using also (480), we obtain

$$\int_{\mathbb{R}^n} |\nabla^k u(x, t)|^2 dx = \int_{\mathbb{R}^n} |\nabla^k u_0(x)|^2 dx, \quad k = 0, 1, 2, \dots, \quad t \in \mathbb{R}, \quad (1160)$$

or, for general $s \in \mathbb{R}$,

$$\|u(t)\|_{H^s(\mathbb{R}^n)} = \|u_0\|_{H^s(\mathbb{R}^n)}, \quad t \in \mathbb{R}. \quad (1161)$$

We see that the evolution given by the Schrödinger equation preserves L^2 -Sobolev regularity of the initial data. This should be contrasted with other notion of smoothness, such as, say, the property that u_0 is smooth and bounded, which is not preserved by the evolution.

So far we have assumed that $u_0 \in \mathcal{S}(\mathbb{R}^n)$. If we have, say, $u_0 \in L^2(\mathbb{R}^n)$, formula (1155) still gives a well-defined function $u \in \mathcal{C}(\mathbb{R}, L^2(\mathbb{R}^n))$, the space of functions u in $\mathbb{R}^n \times \mathbb{R}$ such that $t \rightarrow u(\cdot, t)$ is continuous from \mathbb{R} to $L^2(\mathbb{R}^n)$. The function u will satisfy $u_t - i\Delta u$ in distributions. All this can be also seen by approximations: when $u_0 \in L^2(\mathbb{R}^n)$, we can take a sequence $u_0^k \in \mathcal{S}(\mathbb{R}^n)$

converging to u_0 in $L^2(\mathbb{R}^n)$. The corresponding sequence of solutions u^k will obviously be a Cauchy sequence in $\mathcal{C}(\mathbb{R}, L^2(\mathbb{R}^n))$, and hence will have a limit u in this space, which solves the equation in distributions and satisfies $u(x, 0) = u_0(x)$ (as L^2 -functions). The same considerations work when $L^2(\mathbb{R}^n)$ is replaced by $H^s(\mathbb{R}^n)$.

It is important to note that all the above conclusions were made under the assumption that the solutions or their approximations are given by (1155). Note that we did not prove that, for example, any distributional solution of $u_t - i\Delta u = 0$ in $\mathbb{R}^n \times \mathbb{R}$ which is in $\mathcal{C}(\mathbb{R}, L^2(\mathbb{R}^n))$ satisfies $\|u(t)\|_{L^2(\mathbb{R}^n)} = \|u(0)\|_{L^2(\mathbb{R}^n)}$. Equivalently, we have not really proved that solutions to the initial-value problem (1154) are unique in the class $\mathcal{C}(\mathbb{R}, L^2(\mathbb{R}^n))$. We will prove a uniqueness result below.

Most of the above can also be seen by calculation in the physical space, without using the Fourier transformation. For that we note that the equation $u_t - i\Delta u = 0$ implies that

$$\partial_t(u\bar{u}) = u_t\bar{u} + u\bar{u}_t = i(\bar{u}\Delta u - u\Delta\bar{u}) = i\partial_k(\bar{u}u_{,k} - u\bar{u}_{,k}). \quad (1162)$$

The vector

$$q_k = -i(\bar{u}u_{,k} - u\bar{u}_{,k}) \quad (1163)$$

is real and if we write $u = Ae^{i\phi}$ with A, ϕ real (functions of (x, t)) we have

$$q = 2\nabla\phi \, u\bar{u}, \quad (1164)$$

and we can write

$$(u\bar{u})_t + \operatorname{div} q = (u\bar{u})_t + \operatorname{div}(2\nabla\phi \, u\bar{u}) = 0. \quad (1165)$$

Equation (1165) describes the transport of the “density” $u\bar{u}$ by the “velocity field” $2\nabla\phi$.²⁶⁴ Note that all this is consistent with the picture of a wave packet of frequency ξ moving at speed 2ξ .²⁶⁵

Let φ be a smooth compactly supported function with $\varphi(0) = 0$. Letting $\varphi_\varepsilon(x) = \varphi(\varepsilon x)$ we obtain from (1165)

$$\int_{\mathbb{R}^n} \varphi_\varepsilon |u(x, t_2)|^2 dx = \int_{\mathbb{R}^n} \varphi_\varepsilon |u(x, t_1)|^2 dx + \int_{t_1}^{t_2} \int_{\mathbb{R}^n} q \nabla \varphi_\varepsilon dx dt, \quad (1166)$$

and if the last integral on the right-hand side converges to 0 when $\varepsilon \rightarrow 0_+$, we obtain the conservation law (1159). A straightforward sufficient condition for having

$$\lim_{\varepsilon \rightarrow 0_+} \int_{t_1}^{t_2} \int_{\mathbb{R}^n} q \nabla \varphi_\varepsilon dx dt = 0 \quad (1167)$$

²⁶⁴The classical *equation of continuity* describing the evolution of density $\rho(x, t)$ of some substance moving with the flow given by a velocity field $v(x, t)$ is $\rho_t + \operatorname{div}(v\rho) = 0$. We will discuss transport equations in more detail later in the course.

²⁶⁵If we write the equation as $iu_t + \frac{1}{2}\Delta u = 0$ the term $2\nabla\phi$ in (1164) will be replaced by $\nabla\phi$.

is $u \in L^2(t_1, t_2; H^1(\mathbb{R}^n))$ as the reader can easily verify. As a consequence we have the following statement:

If $u \in L^2(t_1, t_2; H^1(\mathbb{R}^n))$ solves $u_t - i\Delta u = 0$ in distributions in $\mathbb{R}^n \times (t_1, t_2)$, then $u \in \mathcal{C}([t_1, t_2], L^2(\mathbb{R}^n))$ and $\int_{\mathbb{R}^n} |u(x, t)|^2 dx$ is constant in t .

It is a good exercise to prove this results with all details - it is not hard, but one has to be careful.

Note that the result implies uniqueness for (1154) in the class $L^2(t_1, t_2, H^1(\mathbb{R}^n))$, and therefore the solution is given by the representation formula (1155) (assuming that $0 \in [t_1, t_2]$). From this one can infer that in fact $u \in \mathcal{C}([t_1, t_2], L^2(\mathbb{R}^n))$.

We now prove a more general uniqueness result.

Theorem

Assume that $u \in \mathcal{S}'(\mathbb{R}^n \times (t_1, t_2))$ satisfies $u_t - i\Delta u = 0$ in distributions. Then

(i) For each $t \in [t_1, t_2]$ the distribution $u(\cdot, t)$ is well-defined and

(ii) If $u(x, \bar{t}) = 0$ for some $\bar{t} \in [t_1, t_2]$, then $u \equiv 0$.

Proof:

The main idea is to use solutions of the dual equation as test functions. This is an important and quite general method in linear PDEs, going back at least to the proof of Holmgren's Uniqueness Theorem around 1900.²⁶⁶ We have already seen an example of this method in lecture 53, when we proved the uniqueness for the heat equation under quite general assumptions.

Let us consider a standard mollifier ϕ_ε in space-time $R^n \times R$. Assume for simplicity that $\phi_\varepsilon(x, t) = \phi_\varepsilon(-x, -t)$ and let $u_\varepsilon = u * \phi_\varepsilon$. (We have defined this convolution for distributions in lecture 45.) Let $v(x, t)$ be a solution of the dual equation $v_t + i\Delta v = 0$ with $v(x, t_0) = \varphi(x)$ for some $t_0 \in (t_1, t_2)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Let $\eta(t)$ be a smooth compactly supported function in (t_1, t_2) . The function $v\eta$ belongs to $\mathcal{S}'(\mathbb{R}^n \times (t_1, t_2))$. The necessary fast decay in x can be inferred for example from the representation formula (1135), using the calculations in lecture 66.²⁶⁷ By our assumptions, we have

$$\int uv(\eta'(t)) dx dt = \int u[(v\eta)_t + i\Delta(v\eta)] dx dt = 0, \quad (1168)$$

and the same is true for u_ε . For u_ε , which is locally a smooth function, it is admissible to take formally $\eta = \eta_1$ which is smooth except for a jump at $t = t_0$ and satisfies for a suitable small $\tau > 0$

$$\eta_1 = 0, \quad t \in (t_1, t_0), \quad \eta_1 = 1, \quad t \in [t_0, t_0 + \tau), \quad \eta_1 = 0, \quad t \in (t_0 + \tau, t_2). \quad (1169)$$

We obtain

$$\int_{\mathbb{R}^n} u_\varepsilon(x, t_0)\varphi(x) dx = \int_{t_1}^{t_2} \int_{\mathbb{R}^n} u_\varepsilon v \theta dx dt, \quad (1170)$$

²⁶⁶We have not discussed this topic yet in this course.

²⁶⁷Alternatively, one can use the Fourier representation (1155) together with rules Fourier Transform rules from lecture 34 and the Sobolev Imbedding Theorem, also lecture 34.

where θ is a smooth function compactly supported in (t_0, t_2) (and $\theta = \eta'_1$ in (t_0, t_2)). As in (640) we can put the convolution in the last integral on $v\theta$ and obtain

$$\int_{\mathbb{R}^n} u_\varepsilon(x, t_0)\varphi(x) dx = \int_{t_1}^{t_2} \int_{\mathbb{R}^n} u(v\theta)_\varepsilon dx dt. \quad (1171)$$

Now if $\varepsilon \rightarrow 0_+$, then $(v\theta)_\varepsilon \rightarrow v\theta$ in $\mathcal{S}(\mathbb{R}^n \times (t_1, t_2))$ and hence the integral on the right-hand side of (1171) converges to $\langle u, v\theta \rangle$, which we will also write as $\int_{t_1}^{t_2} \int_{\mathbb{R}^n} uv\theta dx dt$. Therefore the left-hand side of (1071) also converges as $\varepsilon \rightarrow 0_+$, and will slightly abuse notation by using the notation $\int_{\mathbb{R}^n} u(x, t_0)\varphi(x) dx$ for the limit. For a general $u \in \mathcal{S}'(\mathbb{R}^n \times (t_1, t_2))$ this expression is not well-defined, but in our case we have shown that it can be naturally defined by

$$\int_{\mathbb{R}^n} u(x, t_0)\varphi(x) dx = \int_{t_1}^{t_2} \int_{\mathbb{R}^n} uv\theta dx dt. \quad (1172)$$

Recalling the definition of v above, we see that v depends not only on φ , but also on t_0 . We will therefore write $v = v^{t_0}$ when we wish to emphasize the dependence on t_0 . Using this notation, we have

$$\int_{\mathbb{R}^n} u(x, t_0)\varphi(x) dx = \int_{t_1}^{t_2} \int_{\mathbb{R}^n} uv^{t_0}\theta dx dt. \quad (1173)$$

We would like to conclude that right-hand side of this identity is uniformly continuous in t_0 even for $t_0 \rightarrow t_1$, $t_0 > t_1$. Strictly speaking, this depends on how exactly we define $\mathcal{S}(\mathbb{R}^n \times (t_1, t_2))$ and its dual. We have not discussed this point yet, as we take the obvious definition: consider the usual definition of $\mathcal{S}(\mathbb{R}^n \times \mathbb{R})$ and define $\mathcal{S}(\mathbb{R}^n \times (t_1, t_2))$ by using the same norms for functions on $\mathbb{R}^n \times (t_1, t_2)$. With this definition the continuity of the right-hand side of (1173) in t_0 is easily checked. Similar arguments can also be used close to t_2 , if we work in the opposite time direction. This proves statement (i) of the theorem. Once we know that the left-hand side of (1173) is well-defined and continuous in t_0 , we can formally use the characteristic function of t_0, \bar{t} (if $t_0 < \bar{t}$) as a test function and obtain

$$\int_{\mathbb{R}^n} u(x, t_0)\varphi(x) dx = \int_{\mathbb{R}^n} u(x, \bar{t})v^{t_0}(x, \bar{t}) dx. \quad (1174)$$

If $u(\cdot, \bar{t}) = 0$, we see that the left-hand side also have to vanish. Since we can choose both t_0 and φ we see that $u \equiv 0$, and the proof is finished.

Lecture 70, 4/5/2011

So far we have studied the Schrödinger equation for “free particles”, when there is no force acting in the particle. Let us now assume the particle moves in a force field given by a potential V (such as the electric potential), so that Newton’s law of motion for a classical particle is

$$\ddot{x} = -\nabla V(x) \quad \text{in suitable units.} \quad (1175)$$

As proposed by Schrödinger, the equation describing a quantum particle in the potential field V is, in suitable units,

$$iu_t + \frac{1}{2}\Delta u - Vu = 0. \quad (1176)$$

Here and in the rest of this lecture we will keep the factor $\frac{1}{2}$ in front of the laplacian, as this form of the equation is more convenient for comparisons with classical formulae for a newtonian particle.

Schrödinger noticed that equation (1175) is contained in (1176) as a certain limiting case. Ever since 1926 this has of course been a standard part of courses on quantum mechanics, and the correspondence between (1175) and (1176) has been also studied by mathematicians in depth. However, if one sees these topics for the first time, the connection of (1175) to (1176) may not be immediately obvious. We will get to this topic soon, but for now let us mention some easy general properties of solutions of (1176), and consider some special solutions.

The reader can prove the following statement as an easy exercise.

For sufficiently regular solutions $u(x, t)$ of (??) with sufficient decay for $x \rightarrow \infty$ we have

$$\frac{d}{dt} \int_{\mathbb{R}^n} |u(x, t)|^2 dx = 0, \quad (1177)$$

and

$$\frac{d}{dt} \int_{\mathbb{R}^n} \left(\frac{1}{2} |\nabla u|^2 + V|u|^2 \right) dx = 0. \quad (1178)$$

The integral in (1178) represents the energy, with $\int_{\mathbb{R}^n} \frac{1}{2} |\nabla u|^2 dx$ representing the kinetic energy and $\int_{\mathbb{R}^n} V|u|^2 dx$ representing the potential energy.

Let us now consider some examples of solutions.

Example 1

Let us consider a special case when the potential V is given by

$$V(x) = ax = a_j x_j, \quad (\text{summation understood}), \quad (1179)$$

where $a = (a_1, \dots, a_n)$ is a fixed vector. It turns out that in this case we can still write the solutions explicitly in terms of solutions of the free equation $iu_t + \frac{1}{2}\Delta u = 0$.²⁶⁸ Although V given by (1179) is very special, we note that a general V looks locally, in a neighborhood of a given point x_0 with $\nabla V(x_0) \neq 0$, approximately as (1179) for a suitable a (which can depend on x_0 , of course).

To calculate solutions of

$$iu_t + \frac{1}{2}\Delta u - axu = 0 \quad (1180)$$

we can use Fourier transformation in x . Using rules for Fourier transformation from lecture 34 (see (480) and (481)), we obtain for $\hat{u} = \hat{u}(\xi, t)$

$$\hat{u}_t - a\nabla_\xi \hat{u} = -\frac{1}{2}i|\xi|^2 \hat{u}. \quad (1181)$$

We note that the terms on the left represent a derivative of \hat{u} in a fixed direction, and setting

$$\eta = \xi + at, \quad \hat{u}(\xi, t) = v(\eta, t), \quad (1182)$$

we obtain

$$v_t = -\frac{1}{2}i|\eta - at|^2 v. \quad (1183)$$

The general solution of this equation is

$$v(\eta, t) = v_0(\eta) e^{-i|\eta|^2 \frac{t}{2} - ia\eta \frac{t^2}{2} - i|a|^2 \frac{t^3}{6}}, \quad (1184)$$

where v_0 is any function (which we will assume to be sufficiently regular and with sufficiently fast decay to at ∞). Going back to the variable ξ and taking the inverse Fourier transform, we obtain

$$u(x, t) = U\left(x + \frac{at^2}{2}, t\right) e^{-iaxt - i|a|^2 \frac{t^3}{6}}, \quad (1185)$$

where U is a solution of the free equation $iu_t + \frac{1}{2}\Delta u = 0$. In fact, one can verify directly that (1185) solves (1180) for any solution U of the free equation. Taking U to be a galilean boost of U_0 (see lecture 67, but keep in mind that our equation now has $\frac{1}{2}$ in front of the laplacian, so we have to adjust some coefficients), we can also write

$$u(x, t) = U_0\left(x - bt + \frac{at^2}{2}, t\right) e^{i(b-at)x - i|b|^2 \frac{t}{2} + iab \frac{t^2}{2} - i|a|^2 \frac{t^3}{6}}. \quad (1186)$$

If we think of U_0 as the solution from Example 2 in lecture 67, for which not much is going on on time scales quite smaller than $\frac{1}{\varepsilon^2}$, we see that in (1186) the wave packet moves according to the formula

$$x(t) = x(0) + bt - \frac{at^2}{2}. \quad (1187)$$

²⁶⁸This calculation should be in the literature, but I did not find an exact reference.

This is exactly what we would get from Newton's law (1175) in the potential field $V = ax$ (which represents constant force $-a$) and initial velocity b . Note the spatial gradient of the phase in (1186) is $b - at$, which is also the velocity of our motion, confirming again the classical newtonian formula for this situation.

We see that for the simple situation $V = ax$ equation (1176) used for certain wave packets reproduces the motion law given by (1175). In the case of a general V , if ∇V does not vanish and does not change much over most of the support of the wave packet, it now looks very plausible that (1176) gives (1175) for the motion of the wave packet, as the situation is locally close to the situation described by the above exact solutions. This argument is of course not a rigorous proof, but it sheds some light on what is going on. (Note that if the changes of ∇V over the support of the wave packet are not small, we can expect that the wave packet might be significantly "deformed" by the differences in the force over the area it occupies, and we do not expect to have simple formulae such as (1185) in that case.)

Example 2

Consider a domain $\Omega \subset \mathbb{R}^n$ and the following potential $V = V_\Omega$:

$$V(x) = 0 \text{ when } x \in \Omega, \quad V(x) = +\infty \text{ when } x \notin \Omega. \quad (1188)$$

(We can consider V as a limiting case of V_m , where $V_m = 0$ in Ω and $V_m = m$ outside Ω .) It is clear from (1178) that any solution u of (1176) with V given by (1188) must vanish outside Ω . Therefore the situation is equivalent to solving the free Schrödinger equation in the domain Ω with the boundary condition $u|_{\partial\Omega} = 0$. In this case there are several types of interesting solutions. We will start by looking at the behavior of the wave packets.

Let us first look at the special case when Ω is a halfspace

$$\Omega = \{x \in \mathbb{R}^n, x_1 > 0\}. \quad (1189)$$

In this case we can solve the equation by extending the initial condition u_0 to all \mathbb{R}^n as an odd function of x_1 and solving the equation in all space \mathbb{R}^n with the extended initial data as the initial condition in \mathbb{R}^n . (It is the same idea we have used in lecture 7 for constructing Green's function for the half-space.) Let us denote the extension of u_0 and u respectively by \tilde{u}_0 and \tilde{u} , where \tilde{u}_0 is defined by $\tilde{u}_0(x_1, \dots, x_n) = -u_0(-x_1, x_2, \dots, x_n)$ for $x_1 < 0$. The condition that \tilde{u} is odd on x_1 is preserved by the time evolution.²⁶⁹ If our initial data is a wave packet moving towards the boundary (possibly at an angle), the extension there \tilde{u}_0 will represent two wave packets: the original one and its image under the extension, which will approach the boundary from the other side. The restriction of \tilde{u} to

²⁶⁹Note that even when the original function u_0 defined in Ω is smooth up to the boundary and satisfies $u|_{\partial\Omega} = 0$, the extended function \tilde{u}_0 can have a jump in the second derivatives across the plane $\{x_1 = 0\}$. As singular behavior is not suppressed by the equation, this may have non-local consequences for the solution, but let us ignore this issue for the moment.

Ω gives the solution u of the original problem. Using this construction, we see that the wave packets will be reflected from the boundary in a way similar to billiard balls. (Of course, we should keep in mind that the wave packets will disintegrate after some time.)

In a half-space the wave packet will hit the boundary only once and after being reflected it will continue moving towards ∞ , before gradually disintegrating. In a bounded domain Ω we expect that as long as a wave packet has not disintegrated and the boundary is close to a half-plane on the scales comparable with the size of the packet, it will be moving in the domain in a way similar to a billiard ball with the speed proportional to its frequency, being reflected from the boundary whenever it hits it. If we consider a non-smooth initial condition u_0 , the high-frequency wave packets, which are responsible for the lack of smoothness, will be moving around the domain very quickly, and we see that the smoothing effects which we have seen for the whole space can no longer be counted on in a bounded domain. ²⁷⁰ Next time we will consider some additional properties of solutions in bounded domains.

²⁷⁰If we average suitable average in time, we can still see some smoothing effects, but the situation is much more subtle than in the whole space.

We will still consider the situation in example 2 from the last lecture: we solve the Schrödinger equation with potential $V = +\infty$ outside Ω and $V = 0$ inside Ω , for a domain $\Omega \subset \mathbb{R}^n$. Today we will assume that Ω is bounded, with smooth boundary, and we will consider solutions which look different from the “wave packets” we have considered in the last few lectures. They will look more like “standing waves” known in theories of classical physics. Their interpretation in terms of “particles” is less straightforward, and one would have to get into the questions of interpretation of quantum mechanics to try to understand these solutions in terms of “particles”.²⁷¹ A newtonian particle corresponds to some special solutions of the Schrödinger equation, but the Schrödinger equation describes a broader range of phenomena than what is described by newtonian mechanics, and some of the phenomena do not have newtonian analogies.

Let us consider the eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3, \dots$ ²⁷² of the laplacian in Ω with the Dirichlet boundary conditions, i. e. the corresponding eigenfunctions satisfy

$$\begin{aligned} -\Delta\phi_k &= \lambda_k\phi_k & \text{in } \Omega, \\ \phi_k|_{\partial\Omega} &= 0. \end{aligned} \tag{1190}$$

We will normalize the ϕ_k so that $\int_{\Omega} |\phi_k|^2 dx = 1$. We know that every complex-valued function f in $L^2(\Omega)$ can be written as

$$f = \sum_{k=1}^{\infty} a_k \phi_k, \tag{1191}$$

where $a_k \in \mathbf{C}$, the series converges in $L^2(\Omega)$, and

$$\int_{\Omega} |f|^2 dx = \sum_k |a_k|^2. \tag{1192}$$

Each ϕ_k can be used to write down an explicit solution of the Schrödinger equation $u_t - i\Delta u = 0$. It is

$$u(x, t) = \phi_k(x)e^{-i\lambda_k t}. \tag{1193}$$

The solution is obviously periodic in t , with period $\frac{2\pi}{\lambda_k}$. These are the “standing wave” solutions. Any linear combination of such solutions is again a solution.

²⁷¹Of course, we expect that every function can be written as a linear combination of wave packets and the equations are linear, so that every solution can be thought of as a linear combination of wave packets. (Vice versa, every wave packet can be thought of as a linear combination of standing waves.) However, the obvious mathematical fact that a linear combination of the solution of the Schrödinger equation is again a solution of the Schrödinger equation does not have a simple analogy in the newtonian mechanics of a particle.

²⁷²The first eigenvalue λ_1 is always simple, so we indeed have $0 < \lambda_1 < \lambda_2 \leq \dots$. We have not proved it in this course so far. It is a classical fact which can be found in many books.

We can use this to write down a formal solution of the initial-boundary value problem

$$\begin{aligned} u_t - i\Delta u &= 0 && \text{in } \Omega \times \mathbb{R}, \\ u|_{\partial\Omega} &= 0 && \text{in } \partial\Omega \times \mathbb{R}, \\ u(x, 0) &= u_0(x) . \end{aligned} \tag{1194}$$

We write

$$u_0 = \sum_k a_k \phi_k \tag{1195}$$

as in (1192), and set

$$u(x, t) = \sum_k a_k \phi_k(x) e^{-i\lambda_k t} . \tag{1196}$$

Let us also denote

$$U_m(x, t) = \sum_{k=1}^m a_k \phi_k(x) e^{-i\lambda_k t} \tag{1197}$$

Assume $u_0 \in L^2(\Omega)$. Then the function U_m is a smooth function in $\Omega \times \mathbb{R}$, which satisfies (1194) with u_0 replaced by

$$U_{0m} = \sum_{k=1}^m a_k \phi_k(x) . \tag{1198}$$

Using the notation $U_m(t)$ for $U_m(\cdot, t)$, we have for $1 \leq m < m'$

$$\|U_{m'}(t) - U_m(t)\|_{L^2}^2 = \sum_{m+1}^{m'} |a_k|^2 , \quad t \in \mathbb{R} . \tag{1199}$$

We see that U_m is a Cauchy sequence in $\mathcal{C}(R, L^2(\Omega))$, and converges in this space to u . In particular $u \in \mathcal{C}(R, L^2(\Omega))$ and $\|u(t)\|_{L^2(\Omega)} = \|u_0\|_{L^2(\Omega)}$ for each t . Since each U_m solves the first equation in (1194), the function u also solves the equation in distributions (in $\Omega \times \mathbb{R}$).

Although the functions U_m transparently vanish at $\partial\Omega \times \mathbb{R}$, the situation with the boundary condition for u given by (1196) with $u_0 \in L^2$ is more tricky. Let us compare the situation with the heat equation, in which case we should use the same formulae with $e^{-i\lambda_k t}$ replaced by $e^{-\lambda_k t}$. In this case, even if the initial condition u_0 is only in L^2 , for any $t > 0$ the series $\sum a_k \phi_k(x) e^{-\lambda_k t}$ will quickly converge to a smooth function vanishing at the boundary.²⁷³ However, for the sums $\sum_k a_k \phi_k(x) e^{-i\lambda_k t}$ it is not obvious in which sense the boundary condition $u_0|_{\partial\Omega}$ should be attained. The key is to take a good definition of weak solutions, which will work with the right class of test functions which will recover the boundary condition in some distributional sense. For example, one can demand that $\int_{\mathbb{R}} \int_{\Omega} u(-\varphi_t - i\Delta\varphi) dx dt = 0$ for each smooth $\varphi = \varphi(x, t)$ vanishing at $\partial\Omega \times \mathbb{R}$, and compactly supported in t . Note the this definition

²⁷³In fact, it still requires some work to show this for a general smooth bounded domain, but at least heuristically it looks very plausible.

allows that the normal derivative of φ at $\partial\Omega \times \mathbb{R}$ does not vanish, and this can be used to show that the weak solution satisfy some weak form of the Dirichlet boundary condition.

With the right definitions, it is possible to prove the existence and uniqueness of weak solutions to problem (1194) for $u_0 \in L^2$, and show that such solutions are given by (1196). When $u_0 \in H_0^1(\Omega)$, the situation is much simpler. This condition can be characterized in terms of the coefficients a_k due to the identity

$$\int_{\Omega} |\nabla u_0|^2 = \sum_k \lambda_k |a_k|^2. \quad (1200)$$

An $L^2(\Omega)$ function is in $H_0^1(\Omega)$ if and only if $\sum_k \lambda_k |a_k|^2$ is finite. If u_0 satisfies this condition, than $u(t)$ given by (1196) will satisfy it for each t , and in this case the boundary condition $u|_{\partial\Omega} = 0$ will be satisfied in terms of traces, see lecture 24. The function u will be in $\mathcal{C}(\mathbb{R}, H_0^1(\Omega))$ and the uniqueness in this class can be proved quite easily by using the energy method from lecture 69. In fact, one can also base the existence theory on the Galerking approximation which we discussed in the context of the heat equations in lecture 58. Let us however set these issues aside for now, and look at some properties of the functions (1197) and (1196), accepting without a detailed proof the heuristics that these should be the right solutions.

Just as the behavior of the solutions $\phi_k e^{-\lambda_k t}$ is characterized by the exponential decay in t , the behavior of $\phi_k e^{-i\lambda_k t}$ is characterized by *periodicity* in t . In dimension $n = 1$ the eigenvalues of the laplacian are of the form $\lambda_k = \kappa k^2$ for a suitable κ (as we assume that Ω is an open interval) and hence all the functions $\phi_k(x) e^{-i\lambda_k t}$ are periodic with period $T = \frac{2\pi}{\kappa}$ (although this may not be the shortest possible period). This means that for $n = 1$ the function u given by (1197) will be periodic in t . This is a significant difference with the case $\Omega = \mathbb{R}^n$. The explanation is that there is no decay mechanism in the equation, there is only the dispersion, the fact that wave packets of different frequencies move at different speeds. In the whole space this does lead to pointwise decay and smoothing, as the fast wave packets escape to ∞ . In a bounded domain they stay around and the situation is more subtle. Also, in $\Omega = \mathbb{R}^n$ each wave packet will eventually disintegrate. It is different in a bounded domain. In fact, in 1d the solution is always periodic in t , so after an initial period of disintegration, the wave packet will restore its “order”, and the process will periodically repeat itself.²⁷⁴

When is the situation in higher dimensions?

In general, the sum of two periodic functions is not periodic, as one can see from the elementary example

$$f(t) = \sin t + \sin \sqrt{2}t. \quad (1201)$$

²⁷⁴For more interesting properties of solutions in 1d see the paper “Dispersive Quantization” by P. Olver, Amer. Math. Monthly 117 (2010) 599-610, or its online version http://www.math.umn.edu/~olver/_w/qd.pdf

Nevertheless, the behavior of these functions “repeats itself” with an error which can be made as small as we wish, if we wait long enough. More precisely, for each $\varepsilon > 0$ there exists $L = L(\varepsilon) > 0$ such that each interval $I \subset \mathbb{R}$ contains a number T which is an ε -almost period, in the sense that

$$|f(t+T) - f(t)| < \varepsilon, \quad t \in \mathbb{R}. \quad (1202)$$

Functions with this property are called *almost periodic*,²⁷⁵ and can be characterized as the uniform limits of trigonometric polynomials.²⁷⁶ The trigonometric polynomials themselves (such as (1202), or the functions $t \rightarrow U_m(x, t)$ in (1197) for a fixed x and m) are examples of *quasi-periodic functions*, which is a slightly more restrictive class.

One can consider the same notions also for vector-valued functions, including Hilbert-space valued functions. For example, the functions $t \rightarrow U_m(t)$ can be considered as quasi-periodic L^2 -valued functions (and also as quasi-periodic H_0^1 -valued functions). The general solutions of the Schrödinger equation in a bounded domain (with homogeneous boundary conditions) is a uniform limit of the quasi-periodic function $U_m(t)$ in the space $\mathcal{C}(\mathbb{R}, L^2(\Omega))$ if $u_0 \in L^2(\Omega)$ or the space $\mathcal{C}(\mathbb{R}, H_0^1(\Omega))$ if $u_0 \in H_0^1(\Omega)$, and therefore are almost periodic. In particular, the solution will repeat its behavior, up to a very small error in the L^2 norm (for $u_0 \in L^2(\Omega)$) or the $H_0^1(\Omega)$ norm (for $u_0 \in H_0^1(\Omega)$). In some exceptional cases the general solutions can be periodic even in higher dimensions, if the eigenvalues are integer multiples of some fixed number.²⁷⁷

It is also worth mentioning the analogy of the kernel (1104) when \mathbb{R}^n is replaced by Ω and the Dirichlet boundary condition is assumed. Using the notation δ for the Dirac mass at 0, we can formally write for $x, y \in \Omega$

$$\delta(x - y) = \sum_k \phi_k(x)\phi_k(y). \quad (1203)$$

Hence the analogy of (1104) for the problem (1190) is the kernel

$$K(x, y) = \sum_k \phi_k(x)\phi_k(y)e^{-i\lambda_k t}. \quad (1204)$$

The series is typically not convergent in a pointwise sense, but it can be defined as a distribution. Questions about precise regularity properties of this distribution can be quite subtle, depending for example on certain number-theoretic properties of λ_k and t and the exact location of x, y in Ω .

²⁷⁵Strictly speaking, we should say “almost periodic in the sense of H. Bohr”. There are other definitions, some of which express the same idea using different norms and are not equivalent. See e. g. the Wikipedia entry for Almost Periodic Functions.

²⁷⁶The proof is non-trivial.

²⁷⁷As an exercise you can determine the condition on $a, b > 0$ under which all solutions of (1190) $\Omega = (0, a) \times (0, b) \subset \mathbb{R}^2$ are periodic.

Homework assignment 2
due on Wednesday, April 27

Let $a \in \mathbb{R}$ and consider the operators in \mathbb{R}^n :

$$L_a u = -\Delta u + 2a \frac{\partial u}{\partial x_1}, \quad (1205)$$

and

$$M_a u = -\Delta u + a^2 u. \quad (1206)$$

1. Show that for $a \neq 0$ the operator M_a has a unique fundamental solution ²⁷⁸ V_a in $\mathcal{S}'(\mathbb{R}^n)$.

2. Show that V_a is radially symmetric, i. e. it is a function of $r = |x|$.

3. Show that $U_a = e^{ax_1} V_a$ is a fundamental solution of L_a .

Optional: Show that U_a is not uniquely determined by the requirements $L_a U_a = \delta$ and $U_a \in \mathcal{S}'(\mathbb{R}^n)$, but that every solution $u \in \mathcal{S}'(\mathbb{R}^n)$ of $L_a u = 0$ is a polynomial. For $a \neq 0$ and $n \geq 2$ the solution U_a is the only fundamental solutions of L_a in $\mathcal{S}'(\mathbb{R}^n)$ with the property that $U_a(x) \rightarrow 0$ as $x \rightarrow \infty$.

4. Calculate U_a and V_a for $n = 1$.

5. Calculate U_a and V_a for $n = 3$. (In this case it can still be calculated in terms of elementary functions, which is not the case for general n .)

Hints for the calculation when $n = 3$:

Method 1: Let $V_a = v(r)$. Derive an ODE for $v(r)$ and solve the ODE by using substitution $v(r) = \frac{w(r)}{r}$.

Method 2: Use Fourier transformation to calculate V_a . The formula for $\hat{V}_a(\xi)$ is easy, but the calculation of the integral in the Fourier inversion formula requires some patience. Calculate the integral formally, disregarding that it is not absolutely convergent. One can do the integral in polar coordinates, integrating first of the sphere²⁷⁹, and then over the radius (using the residue theorem).

6* (Optional). You can also investigate the fundamental solution V_a for complex a . Show that the solution is still unique in $\mathcal{S}'(\mathbb{R}^n)$ when $\operatorname{Re} a \neq 0$. On the other hand, for $a = \kappa i$ with $\kappa \neq 0$ we do not have uniqueness. Find the limits of V_a for $a = i + \varepsilon$ as $\varepsilon \rightarrow 0_+$ and for $a = i - \varepsilon$ as $\varepsilon \rightarrow 0_-$. Calculate the difference of the two limits and its Fourier transformation, and show that the situation has some similarity to the situation with the fundamental solution of the Schrödinger equation we discussed in lecture 65, see (1104).²⁸⁰

²⁷⁸Recall that a fundamental solution of an operator L with constant coefficients is a distribution satisfying $Lu = \delta$, where δ is the Dirac mass at 0.

²⁷⁹and using the special property of S^2 that $\int_{S^2} f(x_1) dx = 2\pi \int_{-1}^1 f(x_1) dx_1$

²⁸⁰There are also differences. For example, the difference of the two solutions in the present example will be smooth.

Lecture 72, 4/11/2011

Today we will continue our discussion of examples with

Example 3

The equation

$$iu_t + \Delta u - V(x)u = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R} \quad (1207)$$

with compactly supported (or at least decaying) potentials V . This situation is one of the main subjects of a large area of research sometimes referred to as “Schrödinger scattering”, with many deep results²⁸¹ Our goal here will be mostly to outline some of the phenomena one can encounter here, without going to rigorous proofs.

For simplicity we will assume that V is compactly supported. Let us first look at a natural modification of the solutions considered in example 2. Let Ω be a bounded domain as before. In example 2 we took $V = +\infty$ outside Ω and $V = 0$ in Ω . We modify this first to $V = c$ outside Ω and $V = 0$ in Ω , and then we subtract c from V (which amounts to replacing u in (1207) by ue^{ict}) so that

$$V = 0 \text{ outside } \Omega \text{ and } V = -c \text{ in } \Omega. \quad (1208)$$

This can be thought of as trying to constrain a quantum particle to the “box” Ω by finite forces. We expect that, at least when c is large, some of the solutions

$$u(x, t) = \phi(x)e^{-i\lambda t} \quad (1209)$$

from the last lecture should persist, modulo some perturbation. It is hence natural to consider the eigenvalue problem

$$-\Delta\phi + V\phi = \lambda\phi \quad \text{in } \mathbb{R}^n, \text{ with “suitable boundary conditions at } \infty\text{”} \quad (1210)$$

The choice of the boundary condition is an important point here. For example, when $\lambda > 0$ and $V \equiv 0$, then (1210) has many solutions. Any ϕ for which the Fourier transform $\hat{\phi}$ is a measure on the sphere $\{|\xi|^2 = \lambda\}$ is a solution, and if the measure will have a smooth density with respect to the surface measure on the sphere, the solutions will decay to 0 as $x \rightarrow \infty$. It is reasonable to expect that even when $V \neq 0$, for large λ similar solutions will persist. These solutions are important in their own right and we return to them later, but solutions analogous to those from example 2 should be more localized, with faster decay at ∞ . A natural “boundary condition” at ∞ in the context of (1210) is $\phi \in L^2$, since a quantum particle described by a wave function $u_0(x)$ with

²⁸¹See, for example the paper “Schrödinger Semigroups” by B. Simon, Bull. AMS, Vol 7, No. 3, 1982; the book “Geometric Scattering Theory” by R. Melrose (Cambridge Univ. Press, 1995); or textbooks of quantum mechanics (where the treatment is often not fully rigorous, but heuristically illuminating).

$\int_{\mathbb{R}^n} |u_0(x)|^2 dx < \infty$ can be thought of as being localized to some area of \mathbb{R}^n ²⁸². Therefore we wish to solve

$$-\Delta\phi + V\phi = \lambda\phi \quad \text{in } \mathbb{R}^n \text{ with } \phi \in L^2(\mathbb{R}^n). \quad (1211)$$

What are the solutions? In one dimension we can take $\Omega = (-\alpha, \alpha)$ and calculate the solutions explicitly. We find that for $c > 0$ we always have at least one solution, and as we increase c we add additional solutions, but we have only finitely many solutions for any finite $c > 0$.²⁸³ The eigenvalues λ for all these solutions are always non-negative. (In fact in this one-dimensional case they are strictly negative.) For $c \leq 0$ there are no solutions $\phi \in L^2(\mathbb{R})$. The situation for $n \geq 2$ is similar, but one cannot rely on explicit calculations to prove it, of course, and the proofs are non-trivial. When our specific V is replaced by a general compactly supported V which is, say, bounded, the conclusions are still similar, but it may happen that there are no eigenvalues. (This would be the case when $c \geq 0$ in our example.)

Let us denote by ϕ_1, \dots, ϕ_m the finitely many solutions of (1211) and with the corresponding eigenvalues $\lambda_1 < \lambda_2 \leq \dots \leq \dots \lambda_m \leq 0$.²⁸⁴ Any linear combination

$$u(x, t) = \sum_k a_k \phi_k(x) e^{-i\lambda_k t} \quad (1212)$$

solves (1207) with the initial condition

$$u_0 = \sum_k a_k \phi_k. \quad (1213)$$

The solutions (1212) are quasi-periodic in t , similar to the solutions U_m from the last lecture, and are localized “near Ω ” in some sense, decaying exponentially fast to 0 as $x \rightarrow \infty$.

Similar conclusions hold for any bounded, compactly supported V in any dimension with the understanding that it can happen that $m = 0$.²⁸⁵ and that

²⁸²One also ask: does (1210) with V given by (1208) have a compactly supported solution? It turns out that the answer is no, but with the methods we have studied so far this result is not accessible. If V were analytic, we could use the fact that solutions of (1210) with analytic V are analytic, which we have not really proved, but we have all the tools to prove it. All one needs for that is to get a suitable estimate for the growth of the derivatives $\nabla^k \phi$ with k , and this can be accomplished by methods we studied in the first semester. However, when V is not analytic, a new idea is needed.

²⁸³As $c \rightarrow \infty$ and we look on the dependence of a given solution $\phi(x, c)$ on c (with $\phi(x, c)$ varying continuously in c) and the normalization $\int_{\mathbb{R}^n} |\phi(x, c)|^2 dx = 1$, the functions $\phi(x, c)$ will converge to eigenfunctions of $-\Delta$ in Ω . The situation in higher dimensions should be similar in principle, but one has to be more careful with the exact formulation in domains where eigenvalues can have higher multiplicity.

²⁸⁴In dimension $n = 1$ we have $\lambda_m < 0$.

²⁸⁵Under the assumption that V has compact support, this is the case when $V \geq 0$. On the other hand, if V is only bounded but not compactly supported, with relatively slow decay to 0 at ∞ , one can have infinitely many eigenfunctions $\phi_k \in L^2(\Omega)$, with the eigenvalues $\lambda_k \leq 0$ accumulating at 0.

the decay corresponding to the eigenfunctions of the eigenvalue $\lambda = 0$ (if it exists) will not be exponential.

The solutions $\phi_k e^{-i\lambda_k t}$ above are called “bound states” in quantum mechanics. Their main feature is that they do not disperse, they “stay around”, unlike the solutions of the free Schrödinger equation, which disperse.

In some sense, the bound states represent particles which are “trapped” by the potential V . One can show that a compactly supported potential V with $V \geq 0$ does not admit such solutions - such potentials cannot trap a quantum particle, although one easily sees that they can trap a newtonian particle with a positive energy. (Think of an area surrounded by “walls”.²⁸⁶)

In addition to the bound states, equation (1207) has solutions which are similar to the wave packets we looked at for the free equation in lectures 67, 68, and 70. For simplicity assume again that $n = 1$ and that V is compactly supported. We can consider a V that is given by (1208), but the same considerations are valid for general bounded compactly supported V and, in fact, for bounded V with sufficiently fast decay at ∞ .²⁸⁷

We imagine that we send a wave packet of frequency ξ from some location quite far out on the left of V towards V . As we have seen in lecture 67, the speed of the wave packet will be 2ξ (as we did not put $\frac{1}{2}$ in front of Δ in (1207)). We assume that the times involved in the following considerations are such that for the free equation the wave packet would mostly keep its identity and its dispersion would not yet be significant. What will happen to the wave packet when it hits the potential V ? It is natural to expect that a part of it will pass through the potential V and continue moving towards $x = +\infty$, and a part of it will be reflected back, and will be moving towards $-\infty$. The frequency of the reflected wave packet can be expected to be $-\xi$. (The “shape” of its amplitude may be different from the original wave packet. For example, it can come in “groups”, each representing a reflection from the potential “walls”.) This can be somewhat visualized by solving the equation

$$-\psi'' + V\psi = \xi^2\psi \quad (1214)$$

with a certain prescribed behavior for both large positive x and large negative x . For large negative x the solution should be a superposition of the stream of original particles, which we can represent as $e^{i\xi x}$ and the reflected particles, which we represent as $R e^{-i\xi x}$, where $R = R(\xi)$ is the “reflection coefficient”. For large positive x the solution should be given by $T e^{i\xi x}$, where $T = T(\xi)$ is the transmission coefficient. Thus our boundary conditions for (1214) are

$$\psi(x) = e^{i\xi x} + R(\xi)e^{-i\xi x}, \quad x \ll 0, \quad (1215)$$

and

$$\psi(x) = T(\xi)e^{i\xi x}, \quad x \gg 0, \quad (1216)$$

²⁸⁶This is related to “quantum tunneling”, the fact that a quantum particle can jump through a barrier even though it does not have enough energy to do it classically. Radioactive decay can be often related to this effect.

²⁸⁷These assumptions can be relaxed still further.

where $R = R(\xi)$ and $T = T(\xi)$ are to be determined. One way to produce such solution is to continue the solution $e^{i\xi x}$ from the right to the left. This continuation produces for large negative x a solution $ae^{-i\xi x} + be^{i\xi x}$. Dividing this solution by b , we obtain ψ . The solution

$$u(x, t) = \psi(x)e^{-i\xi^2 t} \quad (1217)$$

of (1207) can be thought of as describing a stationary situation where particles (or wave packets) at speed ξ are streaming from $x \ll 0$ and are passing through the area where the potential produces forces. Part of the the particles passes through the field, and part is reflected back. The motion of a single wave packet of frequency $\xi > 0$ will of course not be a stationary situation, but the above solution gives some idea about what will happen. The wave packet will move towards V at speed 2ξ and what exactly happens when it hits V can be complicated in the area where V is supported. Nevertheless, the net result of the collision (“after the dust settles down”), is relatively simple: a wave packet moving to the right at speed ξ and a wave packet moving to the left, with speed $-\xi$. The shape of each of these wave packets can be different from the original wave packet.²⁸⁸ The change of amplitude and phase-shift the the two resulting wave packets are related the functions $T(\xi)$ and $R(\xi)$.²⁸⁹ Of course, we still must not forget that all the time the wave packets are disintegrating dues to dispersion and radiating away small amounts of energy.

Let M be the finite dimensional subspace of $L^2(\Omega)$ spanned by the eigenfunctions ϕ_1, \dots, ϕ_m describing the bound states and let Y be its orthogonal complement. Denoting ψ_ξ the functions defined by (1215) and (1216), one can from “linear combinations” $\int \psi_\xi \varphi(\xi) d\xi$ where φ is smooth and compactly supported. The functions from a dense subspace of Y . We emphasize that we now consider ψ_ξ also for $\xi < 0$, in which case the description of ψ_ξ for $\xi > 0$ does not quite fit. However, mathematically there is no problem with the definition for $\xi < 0$. All this is best understood in terms of the spectral theory of the operator $-\Delta u + V$ on L^2 . The functions ϕ_k are eigenfunctions in the usual sense, and the functions ψ_ξ and $\psi_{-\xi}$ resemble some suitably generalized “eigenfunctions” associated with the point ξ^2 of the continuous spectrum of the operator, but we must be cautious with such terminology since clearly $\psi_{\pm\xi}$ do not belong to L^2 . Now one can expect that a general function $u_0 \in L^2$ can be written as

$$u_0 = \sum_{k=1}^m a_k \phi_k + \int \psi_\xi b(\xi) d\xi, \quad (1218)$$

²⁸⁸For example, one can imagine that parts of the original wave packet will reflect several times back and forth, producing “echos” in both direction.

²⁸⁹These functions themselves are very interesting and encode information about V . They are studied in detail in *Inverse Scattering* where we wish to reconstruct V from how it reflects/transmits waves. This is a very important problem which moreover plays a crucial role in the theory of certain “completely integrable” equations, such as the KdV equation $u_t + uu_x = u_{xxx}$. This connection is discussed in any book on Soliton Theory.

for some a_k and $b(\xi)$, with

$$\int_{\mathbb{R}} |u_0|^2 = \sum_{k=1}^m |a_k|^2 + \int |b(\xi)|^2 d\mu(\xi), \quad (1219)$$

where μ is a suitable measure. The solution of the Schrödinger equation (1207) with the initial condition $u(x, 0) = u_0(x)$ will then be given by

$$u(x, t) = \sum_{k=1}^m a_k \phi_k(x) e^{-i\lambda_k t} + \int b(\xi) \psi_\xi(x) e^{-i|\xi|^2 t} d\xi. \quad (1220)$$

The first term on the right-hand side, representing the “bound states”, is similar to what we have seen in bounded domains. It will be quasi-periodic and will not disperse. This is typical for a discrete spectrum. The second term is quite similar to the formula (1155) for the solution of the free equation. It has some similar features, we can think about it in terms of the wave packets which will eventually disperse, except that the motion of the wave packets is more complicated than for the free equation in \mathbb{R}^n : they can partially reflect, stay “almost trapped” by V and be only “leaking to ∞ ” relatively slowly²⁹⁰ etc. The behavior of such solutions does not repeat itself even approximately if we wait sufficiently long, and this can be linked to the continuous spectrum of the operator $-\Delta + V$. General initial data will lead to a combination of these scenarios. Part of the initial data will be trapped in the bound states and the remainder will disperse. Although above we had in mind mostly the 1d situation, the results can be generalized to any dimension, with appropriate modifications.

We see that the behavior of solutions is very rich, and it would take a significant effort to establish the necessary background in the spectral theory and to study these effects rigorously.²⁹¹ Unfortunately we do not have time to go further in this direction. Nevertheless, I thought it was important to mention these results, so that we can appreciate the complexity of behavior of solutions involved in the Schrödinger equation.

²⁹⁰The so called quasi-stationary states, which appear for example when $V \geq 0$ by there is a region with small values of V surrounded by high values of V (“walls”).

²⁹¹And we are still only talking about V rapidly decaying at ∞ . When this assumption is dropped, still new phenomena emerge.

Our goal is to have a closer look at the connection between the Schrödinger equation and classical motion laws for a newtonian particle. For that it is convenient to formulate the laws governing the newtonian particle in term of the *Hamilton-Jacobi equations*. If you have not seen this formulation of classical mechanics before, do not worry, we will derive it as we proceed.

We start by recalling some facts about the linear transport equations of the form

$$u_t + a(x, t)\nabla u + b(x, t)u = 0, \quad (1221)$$

where $u = u(x, t)$ is the unknown real-valued function in the “space-time” $\mathbb{R}^n \times \mathbb{R}$ (or its subset), $a(x, t) = (a_1(x, t), \dots, a_n(x, t))$ is a given time-dependent vector field in \mathbb{R}^n , and $b(x, t)$ is a scalar function. All the quantities are assumed to be as regular as necessary to make our considerations valid.²⁹² The notation $a\nabla u$ means, as usual, $a_j \frac{\partial u}{\partial x_j}$ (summation understood).

In the formulation (1221), the time variable t is singled out to have a somewhat special role, as we think of (1221) as describing time evolution of u , and therefore u_t appears with coefficient 1. This will be natural in our context.²⁹³

Let us first look at the case $b = 0$

$$u_t + a(x, t)\nabla u = 0. \quad (1222)$$

In this case the equation says that u is constant along the trajectories $x = x(t)$ given by the ODE

$$\frac{dx}{dt} = a(x, t). \quad (1223)$$

These solution are called *characteristic curves* of (1222), or *characteristics*. Equation (1222) simply says that the solution is constant along the characteristics:

$$\frac{d}{dt}u(x(t), t) = u_t + a\nabla u = 0 \quad \text{when } x(t) \text{ solves (1223)}. \quad (1224)$$

This immediately suggests a method for finding a solution of (1222) for any initial condition $u(x, 0) = u_0(x)$: on a characteristic curve passing through

²⁹²The question of finding the right setting for (1221) under minimal regularity assumptions which still give existence and uniqueness of solutions is important and has been studied, see for example DiPerna, R. J.; Lions, P. L.; Ordinary differential equations, transport theory and Sobolev spaces, *Invent. Math.* 98 (1989), no. 3, 511–547, and also Ambrosio, L.; Transport equation and Cauchy problem for BV vector fields. *Invent. Math.* 158 (2004), no. 2, 227–260.

²⁹³One can also consider a more symmetric situation in which the equation is $A DU + B = 0$ in \mathbb{R}^{n+1} , with $A = (A_0, A_1, \dots, A_n)$ depending on $x = (x_0, x_1, \dots, x_n)$ and $D = (\partial_0, \partial_1, \dots, \partial_n)$. In a neighborhood of a point where the field A does not vanish we can always write such an equation in the form (1222), as the reader can easily verify.

x define $u = u_0(x)$. Uniqueness is also clear and is left to the reader as an exercise.²⁹⁴

Let ϕ^t be the “flow map” of (1223), defined by the requirement that for any fixed $x \in \mathbb{R}^n$ the curve $t \rightarrow \phi^t(x)$ solves (1223)²⁹⁵ with the initial condition $\phi^t(x)|_{t=0} = x$. The solution of (1224) with the initial condition $u(x, t) = u_0(x)$ can be written in terms of the flow map ϕ^t as

$$u(x, t) = u_0([\phi^t]^{-1}(x)). \quad (1225)$$

The interpretation of (1221) is a simple modification of (1224). We again define the characteristics by (1223) and solve

$$\frac{d}{dt}u(x(t), t) + b(x(t), t)u(x(t), t) = 0, \quad t \in \mathbb{R}^n. \quad (1226)$$

If we are given $u(x, 0) = u_0(x)$ then we choose the solution with $u(x(0), 0) = u_0(x(0))$.

We see that existence and uniqueness theory is simple (if we have a good theory for (1223), which is what we assume here). The study of more detailed properties of solutions may not be easy (depending on the specific question we consider), since the behavior of the solutions of (1223) can be complicated.

An important example of (1221) is the *equation of continuity*

$$u_t + \frac{\partial}{\partial x_j}(a_j u) = 0, \quad (1227)$$

where $a_j = a_j(x, t)$ is considered given.²⁹⁶ The equation is often written as

$$u_t + \operatorname{div}(au) = 0. \quad (1228)$$

We can think of u as a density of some material which moves according to the velocity field $a(x, t)$, in the sense that the “particle” of the material which is located at x at time t moves at speed $a(x, t)$. The flow map $\phi^t(x)$ defined above then describes the trajectories of the particles. If we think about u as density of particles which are moved by the map ϕ^t , then we have for every measurable $\mathcal{O} \subset \mathbb{R}^n$

$$\int_{\phi^t(\mathcal{O})} u(y, t) dy = \int_{\mathcal{O}} u_0(x) dx \quad (1229)$$

²⁹⁴We emphasize again that we assume a to be sufficiently regular. The question what is the minimal regularity of a needed to get good existence and uniqueness of solutions of (1222) with initial data $u(x, 0) = u_0(x)$ is non-trivial, and is discussed for example in the papers of DiPerna-Lions and Ambrosio quoted above.

²⁹⁵We slightly abuse notation in that x in (1223) has a different meaning than x in $t \rightarrow \phi^t(x)$.

²⁹⁶In equations of fluid mechanics this equation appears as a part of a more complicated system of equations with both u and a as unknowns, but here we consider a as given.

for each t , which means that

$$u(\phi^t(x), t) \det \nabla \phi^t(x) = u_0(x). \quad (1230)$$

Equation (1227) can be thought of as an infinitesimal version of (1229). If we take derivative of (1230) in t , we obtain (1227).

The precise statements one can prove is the following:

(i) If a and u_0 are sufficiently regular, than (1227) with initial condition $u(x, 0) = u_0(x)$ has a unique solution. This follows from our comments about the more general equation (1221) above.

(ii) The solution is given by (1230). In view of the uniqueness part of (i), it is enough to verify that $u(x, t)$ given by (1230) satisfies (1227). This is just a matter of a straightforward calculation²⁹⁷.

The property (1229) is transparent if we have (1230), but it can also be derived directly from (1227). For that we use (1227) to show that for open sets with smooth boundary the derivative of the left-hand side with respect to t vanishes.

Another derivation of (1227) is based on the identity

$$\frac{d}{dt} \int_{\mathcal{O}} u(x, t) dx = - \int_{\partial \mathcal{O}} u(a \cdot n) dx, \quad (1231)$$

where \mathcal{O} is any open set with smooth boundary and n is the outward unit normal at the boundary. This identity expresses the idea that the change of “mass” inside \mathcal{O} can only be due to the flux of the material through the boundary.

Still another useful point of view is to view (1227) as a dual equation to (1222). If $v(x, t)$ solves (1222) and $u(x, t)$ solves (1227), then we have, by direct calculation,

$$\frac{d}{dt} \int_{\mathbb{R}^n} u(x, t)v(x, t) dx = 0. \quad (1232)$$

It is not hard to see that the meaning of this identity is essentially the same as that of (1229).

The equation of continuity preserves the “total mass”:

$$\frac{d}{dt} \int_{\mathbb{R}^n} u(x, t) dx = 0, \quad (1233)$$

assuming of course that u is integrable (and that a is smooth and bounded, say). This is a special case of (1229), and also of (1232) when we take $v \equiv 1$.

²⁹⁷The key for the calculation is the formula $d/ds|_{s=0} \det(A+sB) = \text{Trace}(\text{Adj}(A)B)$, where $\text{Adj}(A)$ denotes the adjoint matrix of A . For a non-singular A we have $\text{Adj}(A) = A^{-1} \det A$.

In connection with Schrödinger equation, it is also important to look at equations of the form (1221) which preserve the L^2 norm of u . In that case we require that u^2 satisfies the equation of continuity,

$$(u^2)_t + \operatorname{div}(au^2) = 0, \quad (1234)$$

which suggest that we should take

$$u_t + a\nabla u + \frac{1}{2}(\operatorname{div} a)u = 0. \quad (1235)$$

Any (sufficiently regular) solution of this equation will satisfy (1235), and therefore conserve the L^2 norm. In general, if we wish to conserve the L^p norm, we take

$$u_t + a\nabla u + \frac{1}{p}(\operatorname{div} a)u = 0. \quad (1236)$$

Note that for $p \rightarrow \infty$ we obtain (1222), which does preserve L^∞ norm, so everything is consistent.

As the next tool for our analysis of certain the Schrödinger solutions, we introduce the Burgers equation. In dimension $n = 1$ this is an equation for a scalar function $u = u(x, t)$ which reads

$$u_t + uu_x = 0. \quad (1237)$$

Unlike other equations we have dealt with so far, this is a non-linear equation. However in this case it is easy to understand the *smooth* solutions. (It is different with weak solutions, where the problems are more subtle, but will not consider the topic of weak solutions of (1237) in this course.) If we view (1237) as an equation of the form (1222) with $a = u$, we see that it says that u is constant along the characteristics given by u . This means that the speed (or slope) of each characteristic curve is constant. This shows that a smooth solution of (1237) with the initial value $u(x, 0) = u_0(x)$ satisfies

$$u(x + u_0(x)t, t) = u_0(x). \quad (1238)$$

If u_0 is, say, smooth and compactly supported, then equation (1238) defines u uniquely on some time interval $(0, T)$, but when $u_0(x_1) > u_0(x_2)$ for some $x_1 < x_2$, the characteristic curves originating at $t = 0$ respectively at x_1 and x_2 will eventually intersect, and we see that the equation cannot have a smooth solution defined for all t .

The expression on the left-hand side of (1237) expresses the acceleration of particles moving according to the velocity field $u(x, t)$. The equation for the particle trajectories is

$$\frac{dx}{dt} = u(x, t) \quad (1239)$$

and hence

$$\frac{d^2x}{dt^2} = u_t + uu_x. \quad (1240)$$

Equation (1237) then says that the particles are “free”, in the sense that there are no forces acting on them. Therefore they move along straight lines, which coincide with the characteristics.

All the above easily generalized to higher dimensions. In general dimension n the unknown u is a vector field $u(x, t) = u_1(x, t), \dots, u_n(x, t)$ and the equation is

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = 0, \quad (1241)$$

which is often written as

$$u_t + u \nabla u = 0. \quad (1242)$$

The expression $u_t + u \nabla u$ can again be interpreted as acceleration. The meaning of the equation is the same as for $n = 1$. We can again think of “free particles” moving along straight lines. As before, smooth initial data u_0 can lead to a singularity when two particles collide. Formula (1238) can be used without change.

The Burgers equation and the linear transport equations we introduced last time appear naturally in the context of the relation of the Schrödinger equation and classical mechanics. We will now explain this connection. We start with the following classical calculation.

The classical mechanics can be considered as a limiting case of the quantum mechanics when the Planck constant $\hbar \rightarrow 0$. Therefore when studying the relation between the two theories, it is natural to keep \hbar in the equations. We will therefore write the Schrödinger equation in the form

$$i\hbar u_t + \frac{\hbar^2}{2} \Delta u - Vu = 0. \quad (1243)$$

We will look at particular solutions of (1243), representing certain wave packets. Our goal is to show that in the limit $\hbar \rightarrow 0$ the equations of motion for the wave packet are the equations of classical mechanics. The wave packet we will be looking at are of the form.

$$u(x, t) = A(x, t)e^{i\frac{S(x, t)}{\hbar}} \quad (1244)$$

This is the so-called WKB approximation (after Wentzell-Kramers-Brillouin), developed shortly after the discovery of quantum mechanics.²⁹⁸ The functions A and S are real-valued functions, and they depend of \hbar . In the connection with the limit $\hbar \rightarrow 0$, we can think of A as a smooth suitably localized function which is significantly away from 0 only in the region where we expect the particle to be located. The function S determines the phase of u , and is chosen so that ∇S represents the velocity of the particle.²⁹⁹ The important point is this: we think of \hbar as very small and therefore the function u oscillates rapidly. However, we think of A and S as “nice functions” which are not rapidly oscillating. The rapid oscillations in u are only those which are already transparent in its definition. We do not expect any additional oscillation in A and S . Therefore, although u rapidly oscillates, we assume we know more or less exactly what the oscillations are. Substituting (1244) into (1243), we obtain

$$S_t + \frac{1}{2}|\nabla S|^2 + V - \frac{1}{2}\hbar^2\Delta A = 0 \quad (1245)$$

and

$$A_t + \nabla S \nabla A + \frac{1}{2}\Delta S A = 0. \quad (1246)$$

²⁹⁸For a different approach based on Wigner measures see a paper “Wigner Functions versus WKB-Methods in Multivalued Geometrical Optics” by C. Sparber, P. A. Markowich, and N. J. Mauser. (Thanks to F. Otto for pointing out this reference.)

²⁹⁹More precisely, we should talk about momentum rather than velocity, but we assume that the mass of the particle is 1, so we do not really have to make a distinction in our situation here. However, it is important to keep in mind that, in general, “velocity” is not really the most natural quantity in quantum mechanics.

These equations represent (1243) in the variables A, S . In the limit $\hbar \rightarrow 0$ we formally obtain

$$S_t + \frac{1}{2}|\nabla S|^2 + V = 0 \quad (1247)$$

and

$$A_t + \nabla S \nabla A + \frac{1}{2} \Delta S A = 0. \quad (1248)$$

The important point is that equation (1247) for S is now decoupled from the equation for A . Once S is known, equation (1248) is just a linear transport equation of the type we discussed in the last lecture. It is of the form (1235), and it says that A is transported along the characteristics given by ∇S in such a way that $\int_{\mathbb{R}^n} A^2$ is preserved, or – more precisely – that A^2 satisfies the equation of continuity

$$(A^2)_t + \operatorname{div}(\nabla S \nabla(A^2)) = 0. \quad (1249)$$

Equation (1247) is more complicated. It has been known in Classical Mechanics since 1830s, when it was discovered by W. R. Hamilton. It represents another formulation of Classical Mechanics. It was later used by C. G. Jacobi as a tool for finding solutions of a number of problems in classical mechanics in terms of explicit integrals.³⁰⁰

The connection between (1247) and the classical newtonian

$$\ddot{x} = -\nabla V(x) \quad (1250)$$

can be seen for example in terms of the Burgers equation we discussed last time. We explain this in the next lecture.

In the context of (1250) we usually think in terms of trajectories determined by its initial data: we are given x , and \dot{x} at $t = 0$ and solve (1250) to calculate the trajectory $x(t)$. In the context of the Hamilton-Jacobi equation (1247) one can think of certain of *families of particles* and *families of trajectories*. We imagine that at time $t = 0$ we have at each point x of the space \mathbb{R}^n a particle with speed $\nabla S(x, 0)$, and that each of these particles moves according to (1250). At a later time t we look again at a “general point” point x . There will be some particle at that point, which arrived there from its initial position at time $t = 0$, which was most likely different from x . What will be the speed of this particle? It turns out it will be given by $\nabla S(x, t)$. This may not be immediately obvious, but we will see that it follows easily from our discussion of the Burgers equation last time.

³⁰⁰See for example the book “Mathematical Methods of Classical mechanics” by V. I. Arnold for more details.

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We are interested in the connection between the Hamilton-Jacobi equation (1247) and the Newton law (1250). This can be established in many ways.³⁰¹ We will establish the connection “by hand”, without much discussion the underlying contact geometry, which is a very interesting topic in its own right. the Hamilton-Jacobi equation is

$$S_t + \frac{1}{2}|\nabla S|^2 + V = 0. \quad (1251)$$

Let us denote $v = \nabla S$. We consider v as vector field in $R^n \times (t_1, t_2)$. By taking the gradient of (1251), we obtain

$$v_{it} + v_j v_{j,i} + V_i = 0, \quad (\text{summation understood}). \quad (1252)$$

The key point (which is at the root of all approaches to the problem) is the elementary fact that

$$v_{i,j} = v_{j,i} \quad (1253)$$

since v is a gradient. This implies that

$$v_j v_{j,i} = v_j v_{i,j} \quad (1254)$$

or, in vector notation,

$$\nabla\left(\frac{1}{2}|v|^2\right) = v\nabla v. \quad (1255)$$

This means that (1252) can also be written as

$$v_t + v\nabla v = -\nabla V. \quad (1256)$$

We have already seen in lecture 73 that

$$\dot{x} = v(x, t) \quad (1257)$$

gives

$$\ddot{x} = v_t + v\nabla v \quad (1258)$$

and hence (1289) is just a different form of

$$\ddot{x} = -\nabla V(x). \quad (1259)$$

The difference is that we usually think about (1259) in connection with individual trajectories. On the other hand, equation (1289) can be thought of as an equation for a field of trajectories, which fill up the space, in the sense that (locally) there is a trajectory through each point of the space.

³⁰¹See for examples V.I. Arnold’s “Mathematical Methods of Classical Mechanics” for a geometric approach using based on “contact geometry”, or R. Courant’s “Partial Differential Equations” for a related traditional approach based on determining the characteristics of the equation as intersection of infinitesimally close hyper-planes.

The vector field v which solves (1289) is easily determined from the initial condition $v(x, 0)$ and equation (1259). For each $x \in \mathbb{R}^n$ we solve (1259) with initial position x and initial velocity $v(x, 0)$. This gives us a curve $t \rightarrow \phi^t(x)$ for each $x \in \mathbb{R}^n$. Under reasonable assumptions on the initial velocity field $v(x, 0)$ and the potential $V(x)$ the maps ϕ^t for a smooth family of diffeomorphisms of \mathbb{R}^n for $t \in (t_1, t_2) \ni 0$. Now we define $v(x, t)$ simply by

$$v(\phi^t(x), t) = \frac{d}{dt}\phi^t(x), \quad t \in (t_1, t_2), x \in \mathbb{R}^n. \quad (1260)$$

We see that (1259) can be used to construct a local-in-time solution of (1289). Vice-versa, if we have a solution of (1289), we can get solutions of $\ddot{x} = -\nabla V(x)$ by solving $\dot{x} = v(x, t)$.

Our original PDE is not (1289) but it is (1251). We will construct S solving (1251) with initial data $S(x, 0)$ from the solution v of (1289) satisfying

$$v(x, 0) = \nabla S(x, 0). \quad (1261)$$

The key point is to establish that if v solves (1289) and the vector field was a gradient at $t = 0$, it will also be a gradient at any other time. For that we will write the equation for the anti-symmetric part Ω of the gradient ∇v . The equation for the gradient ∇v is

$$(\nabla v)_t + v\nabla(\nabla v) + (\nabla v)^2 = -\nabla^2 V, \quad (1262)$$

where in $(\nabla v)^2$ we square the matrix ∇v by matrix multiplication, and the first two terms on the left represent the derivative of ∇v along the particle trajectory. Let us write

$$\nabla v = Y + \Omega, \quad (1263)$$

where Y is the symmetric part of ∇v and Ω is the anti-symmetric part of ∇v . Taking the anti-symmetric part of (1262), we obtain

$$\Omega_t + v\nabla\Omega + Y\Omega + \Omega Y = 0. \quad (1264)$$

For each trajectory this can be viewed as a first order equation

$$\frac{d}{dt}\Omega = \mathcal{A}(t)\Omega, \quad (1265)$$

along the trajectory, where $\mathcal{A}(t)\Omega = Y(t)\Omega + \Omega Y(t)$. By standard ODE uniqueness, of $\Omega(0) = 0$, then $\Omega(t) = 0$ for each $t \in (t_1, t_2)$. We see that

If $v(x, 0)$ is a gradient, then $v(x, t)$ will be a gradient for each $t \in (t_1, t_2)$.

It is now easy to check that the equations

$$\nabla S(x, t) = v(x, t), \quad S_t + \frac{1}{2}|v|^2 + V = 0, \quad S(x, t)|_{t=0} = S(x, 0) \quad (1266)$$

uniquely determine a solution of (1251) in $R^n \times (t_1, t_2)$ with initial condition $S(x, 0)$.

We conclude that

Under natural assumptions on V and $S_0 = S_0(x)$, equation (1251) has a unique regular local-in-time solution satisfying $S(x, 0) = S_0(x)$. Moreover, the trajectories of the ODE $\dot{x} = \nabla S(x, t)$ solve Newton's equation (1259).

It is clear from the construction of the solutions of (1251) that – if we wish – we can solve the equation only in a neighborhood of a given trajectory of (1259) with the initial condition $S_0(x)$ given only in a neighborhood of the initial point of that trajectory.

It is also easy to see that the solution may not be well defined for all t . This can happen when two different trajectories intersect and we have conflicting requirements for $v(x, t)$ at the point of intersection.³⁰²

Considerations similar to those above can be used to prove the following classical result, which leads to an important method for integrating the equation of motion (1259).

Assume that $S = S(x, t; \alpha)$ is family of solutions of (1251) depending smoothly on a parameter $\alpha \in (\alpha_1, \alpha_2)$. Then the quantity

$$F = \frac{\partial S}{\partial \alpha} \tag{1267}$$

is a constant of motion along the trajectories of (1259) given by $\dot{x} = \nabla S(x, t; \alpha)$.

The proof is simple: taking a derivative of (1251) with respect to α gives

$$F_t + v \nabla F = 0 \tag{1268}$$

which is exactly what is claimed.

Let us illustrate the last statement by calculating the classical elliptical trajectory of a single planet orbiting in a newtonian potential $V = -\frac{\gamma}{|x|}$. We assume the orbit is in the plane x_1, x_2 , and we will work in the polar coordinates given by $x_1 = r \cos \phi$, $x_2 = r \sin \phi$. In these coordinates, the Hamilton-Jacobi equation is

$$S_t + \frac{1}{2} \left(\frac{\partial S}{\partial r} \right)^2 + \frac{1}{2} \left(\frac{\partial S}{\partial \phi} \right)^2 + V(r) = 0. \tag{1269}$$

We seek solution S in the form

$$S = f(r) + M\phi - Et, \tag{1270}$$

³⁰²It is possible to define various generalized notions of solution which are defined globally and one can still prove uniqueness. (Uniqueness is a key point. The requirement that S is Lipschitz and satisfies (1251) almost everywhere does not lead to uniqueness, and hence this class of solutions is too broad.) The relation of these generalized solutions to equation (1259) is related to variational principles in mechanics. There is a large literature on these topics, the reader can consult for example the paper “Viscosity solutions of Hamilton-Jacobi equations” by M. G. Crandall and P. L. Lions, *Trans. AMS*, 277, No.1, 1983, and the paper “A survey of PDE methods in weak KAM theory” by L. C. Evans, *Comm. Pure Appl. Math.* LVII, 445-480, 2004.

where M, E are parameters (representing the angular momentum and energy respectively), and $f(r) = f(r; M, E)$ is a function of r depending on the parameters M, E . Strictly speaking the function ϕ is not well-defined as a smooth function in $\mathbb{R}^2 \setminus \{0\}$, but let us ignore this issue for now.³⁰³ Equation (1269) gives

$$f(r) = \text{const.} \pm \int \sqrt{2E + \frac{2\gamma}{r} - \frac{M^2}{r^2}} dr . \quad (1271)$$

The equation

$$\frac{\partial S}{\partial M} = \text{const.} \quad (1272)$$

gives

$$\pm \int \frac{\frac{M}{r^2}}{\sqrt{2E + \frac{2\gamma}{r} - \frac{M^2}{r^2}}} dr = -\phi + \phi_0 , \quad (1273)$$

which is the equation of the trajectory. The integral can be evaluated and the reader can verify that the curve $r = r(\phi)$ given by (1273) is, for a suitable range of parameters, an ellipse.

We also have

$$\frac{\partial S}{\partial E} = \text{const.} \quad (1274)$$

and this gives the time dependence $r = r(t)$.

The key for this method is to make a good guess for S , such as (1270). For more examples the reader can check the book of V. I. Arnold quoted above.

³⁰³It can be justified for example by lifting everything to a covering space of $\mathbb{R}^2 \setminus \{0\}$, where ϕ is well-defined, but let us not worry about these details at this stage.

Now that we have some understanding of the Hamilton-Jacobi equation, it is easy to see that in the limit $\hbar \rightarrow 0$ wave packets moving according to (1243) should closely follow trajectories given by Newton's equation (1250). We consider an initial condition u_0 for the Schrödinger equation (1243) of the form

$$u_0(x) = A_0(x)e^{i\frac{S_0(x)}{\hbar}}, \quad (1275)$$

where A_0 is a smooth bump function supported in a neighborhood of a point x_0 . We choose the initial phase function $S_0(x)$ to be $S_0(x) = \xi x$ on the support of A_0 . We can extend it to a smooth compactly function defined on all \mathbb{R}^n , but the values outside of the support of A_0 will not be important to us, as long as the function is smooth. We now solve the equation (1247) by the method of characteristics we discussed in the last lecture. The smooth solution $S(x, t)$ may be defined only for some time interval $(t_1, t_2) \ni 0$. For our purposes here we do not need to solve the solution for x in all space \mathbb{R}^n , we can find only find the solution along the characteristic curves emanating from the support of A_0 . These correspond to Newtonian trajectories starting in the points in the support of A_0 with speed $\nabla S_0 = \xi$. As long as none of these trajectories intersect, the solution $S(x, t)$ is well defined in the region $\mathcal{O} \subset \mathbb{R}^n \times \mathbb{R}$ swept by the trajectories. Note that the behavior of S is quite local: we can define find $S(x, t)$ in \mathcal{O} just from the values of S_0 on the support of A_0 . (It can still happen that some trajectory from outside of the support would intersect some of the trajectories in \mathcal{O} , but this effect is not important for in the first approximation which we are considering here.) Having found S in \mathcal{O} , we now turn to equation (1248) for A . Since S is now given, this is merely a linear transport equation. Its characteristics are exactly the characteristics we calculated when determining S . We saw in lecture 73 that this means that $A(x, t)$ is just transport of A along the characteristics, modified so that $|A|^2$ satisfies the equation of continuity

$$(A^2)_t + \text{div}(\nabla S(A)^2) = 0. \quad (1276)$$

We see that $A(x, t)$ can be thought of as the initial bump functions moved along the newtonian trajectories in such a way that its L^2 norm is preserved. The exact solution of (1243) can be expected to be

$$u(x, t) = A(x, t)e^{i\frac{S(x, t)}{\hbar}} + \text{error terms}, \quad (1277)$$

where the error terms approach 0 in the limit $\hbar \rightarrow 0$. This can be proved, but we will not go into the proof at this point. Our goal was to illustrate the rich structure of the Schrödinger solutions and the connection to newtonian mechanics, and hopefully the above analysis sufficiently makes this point at least at a heuristic level.

We now turn our attention to a more detailed look at the function $S(x, t)$ and its meaning. In the context of classical mechanics, the function S was introduced

by R. W. Hamilton in 1827, with motivation coming originally from geometric optics. We will now briefly explain the connection to optics and how it can be used to understand the meaning of S .

In both optics and mechanics, the function S is related to *variational principles*. These principles describe physical laws in terms of minimization of certain quantities, and S represents the optimal value of the quantity which is minimized. Its derivatives, ∇S and S_t have obvious meaning: ∇S is the velocity³⁰⁴ and $-S_t$ is the energy. The meaning of S itself is more subtle. Let us start by a brief explanation of the role of a function similar to S in geometric optics.³⁰⁵

Let us consider some material in which light propagates. We assume that the speed of propagation can depend on the position x . The dependence of the speed on x is usually expressed in terms of the *index of refraction* $n(x)$ and the speed in the vacuum c , so that

$$\text{speed of light at } x = \frac{c}{n(x)}. \quad (1278)$$

We imagine that light travels along curves usually called rays. The equation for these curves can be derived from the following remarkable principle, often called *Fermat's principle*.

For two points x, y on a light ray which are sufficiently close to each other, the light ray minimizes the time light needs to travel from x to y .

A more precise formulation is the following. Let γ be any curve joining x and y . The time $T(\gamma)$ light needs to travel from x to y along γ is

$$T(\gamma) = \int_{\gamma} \frac{dl}{\frac{c}{n(x')}} = \int_{\gamma} \frac{n(x')}{c} dl, \quad (1279)$$

where dl is the “length element” on the curve (the 1-dimensional Hausdorff measure).³⁰⁶ If $\bar{\gamma}$ is the actual light ray joining x, y , and x, y are sufficiently close, then

$$T(\bar{\gamma}) = \inf\{T(\gamma), \gamma \text{ joins } x \text{ and } y\}. \quad (1281)$$

In terms of elementary Riemannian geometry, this can be formulated as follows. Consider a metric given by

$$g_{ij}(x) = \frac{n(x)}{c} \delta_{ij}. \quad (1282)$$

³⁰⁴We should really say momentum, but recall that we consider particles with unit mass.

³⁰⁵This topic is closely related to the wave equation which we will discuss later, but much of the theory was developed independently of the wave equations, before Maxwell identified light with the solutions of his equations of electromagnetism in 1860s.

³⁰⁶If we think of γ as map from $[0, 1]$ into \mathbb{R}^n , we can also write

$$T(\gamma) = \int_0^1 \frac{n(\gamma(s))}{c} |\gamma'(s)| ds \quad (1280)$$

Then the light rays are exactly the geodesic curves of the metric g_{ij} . The quantity $T(\bar{\gamma})$ in (1280) is the distance of x and y in this metric. Let us write $T(x, y)$ for this distance, i. e.

$$T(x, y) = \inf\{T(\gamma), \gamma \text{ joins } x \text{ and } y\}. \quad (1283)$$

This quantity is defined for any x, y , even if they are not close and the minimizing curve may not be unique.

The minimizing property (1280) has been known to mathematicians since the 17th century. The important discovery of W. R. Hamilton (around 1827) is that the function $T(x, y)$ contains in it all information about the system, and it can be often used to derive effectively what we need.³⁰⁷ It is this function T which is closely related to the function S we studied in the last few lectures. This may not be immediately obvious, since T is a function of two variables the role of which is symmetric, which is different from the function $S(x, t)$ above. However, we will see that $T(x, y)$ can be used to generate functions of one variable similar to $S(x, t)$. When we adapt this procedure from optics to mechanics, we will get exactly the function $S(x, t)$.

³⁰⁷This is a special case of an important principle: if a model is governed by a variational principle, e. i. some quantity is to be minimized, then the function giving the minimal values in terms of the parameters of the problem is quite likely to have some significance.

We will further discuss the function $T(x, y)$ introduced in the last lecture, (1283). We assume the function $n(x)$ involved in the definition of T is smooth and positive, with $0 < \nu_1 \leq n(x) \leq \nu_2$ in \mathbb{R}^n for some ν_1, ν_2 . With these assumptions one can show that when x, y are sufficiently close, there exists a unique curve γ minimizing (1283), and $T(x, y)$ is smooth in both x, y away from the diagonal $x = y$. In general, when x, y are not close, we may lose both uniqueness of the minimizing path and the smoothness of T . Here we will not consider this situation, and we will always assume that T is as regular as needed. The proof of the statements above, such as the smoothness of $T(x, y)$ for x, y close to each other $x \neq y$ requires some work. It is a special case of results about geodesics and the distance function which can be found in textbooks of Riemannian geometry, and we will not go into it in this class. It is a technical point which of course needs to be clarified at a certain stage, but for now I suggest that we accept these properties without proof.³⁰⁸

We will write

$$\frac{\partial T(x, y)}{\partial y} = \eta, \quad \frac{\partial T(x, y)}{\partial x} = -\xi, \quad (1284)$$

where we use the notation $\nabla_x = \frac{\partial}{\partial x}$ and $\nabla_y = \frac{\partial}{\partial y}$. To see the meaning of ξ and η , let us consider the minimizing curve γ joining x to y and let $T(x, y) = d > 0$. Let us consider the surface $\Sigma = \{z, T(x, z) = d\}$ in the neighborhood of the point y . It is not hard to see that $\eta \neq 0$, and hence Σ is smooth near y , and has a tangent plane L_y at the point y . The key point now is that, in our situation, L_y is perpendicular to η with respect to the usual scalar product. This is immediate from the point of view of elementary Riemannian geometry, as the metric defining $T(x, y)$ is at each point a multiple of δ_{ij} .³⁰⁹ One can also verify the statement $\eta \perp L_y$ “by hand”, by “blowing up” the picture near y and showing that if η and L_y were not perpendicular, you could find points z on Σ close to y with $T(x, z) < d$. To see that, one moves along γ almost up to y , and then, very close to y one moves towards Σ in a direction perpendicular to L_y , reaching Σ over distance just slightly smaller than d . This of course requires a calculation which would confirm the heuristics, and we leave this to the reader as an exercise.³¹⁰ Once we know that L_y is perpendicular to η , it is clear that η coincides with the derivative of $T(x, y)$ along the minimizing curve γ at y , and we see that

$$\left| \frac{\partial T(x, y)}{\partial y} \right| = \frac{n(y)}{c}, \quad (1285)$$

³⁰⁸The proof is non-trivial and it leads to the consideration of geodesics which are infinitesimally close to a given geodesic. Such geodesics are best described in terms of *Jacobi fields*. If $\gamma_\varepsilon = \gamma_\varepsilon(s)$ is a family of geodesics, with each of the geodesics parametrized by s , then the Jacobi field of the family is $X(\gamma(s)) = \frac{\partial}{\partial \varepsilon} |_{\varepsilon=0} \gamma_\varepsilon(s)$. The vector field satisfies a linear differential equation along γ (the so called *Jacobi equation*), which can be analyzed in detail. The properties of T can be studied with the help of the Jacobi equation.

³⁰⁹In the geometry terminology, the metric is *conformal* to the Euclidean metric.

³¹⁰If the metric g_{ij} defining the geodesics is not diagonal at y , then η will be perpendicular to L_y with respect to the metric $g_{ij}(y)$, and not necessarily with respect to δ_{ij} .

which is known as the *eikonal equation*. Sometimes it is written as

$$\left| \frac{\partial T(x, y)}{\partial y} \right|^2 = \left(\frac{n(y)}{c} \right)^2. \quad (1286)$$

In a similar way we have

$$\left| \frac{\partial T(x, y)}{\partial x} \right| = \frac{n(x)}{c}. \quad (1287)$$

The meaning of $\xi = -\frac{\partial T(x, y)}{\partial x}$ is similar to that of η . It is a vector of size $\frac{n(x)}{c}$ at the point x , pointing in the direction of the shortest path from x to y . Equation (1286) can be used to find the equation for the geodesics (which can of course be found also in many other ways). For that we take a derivative of (1223) in y and get an equation for the characteristics, similarly to what we did in lecture 75, see equation (1289). The geodesics (= the light rays) coincide with the characteristics. We will look at this in more detail (and for slightly more general equations) later. For now we will focus on different aspects of the situation.

So far we have viewed (1284) as equations determining η and ξ for a given x, y . We can also fix x and ξ , and consider them as equations for y and η , which again brings us to the equation for geodesics. If both x is restricted to some plain L and y is restricted to some plane L' , then equations(1284) can be thought as describing some imaging process in which rays emanating from L at x at a given angle (related to ξ) will reach L' at y at an angle related to η . We see that we can think of $T(x, y)$ as describing some optical device.

As noticed by W. R. Hamilton, the above considerations have an analogy in mechanics. To illustrate the analogy, let us first consider the simple situation which you probably have encountered at some point in a physics class. Consider that upper half-space $\mathcal{O}_+ = \{x_3 > 0\}$ is filled with an optical medium of the index of refraction $n_1(x)$ and the lower half-space \mathcal{O}_- is filled with an optical medium of index of refraction $n_2(x)$. If a light ray passes from the upper \mathcal{O}_+ to \mathcal{O}_- , and its angle with the normal $(0, 0, 1)$ in \mathcal{O}_+ is α_1 and the angle with $(0, 0, -1)$ in \mathcal{O}_- is α_2 , then

$$\frac{\sin \alpha_1}{\sin \alpha_2} = \frac{n_2}{n_1}, \quad (1288)$$

which is *Snell's law*, discovered before Fermat's principle.³¹¹ It is more or less clear that this generalizes to the situation when the boundary between the two media is a more general surface, if the angles are taken with respect to the normals at the point where the ray passed through the boundary.³¹²

Let us now replace the light rays by streams of particles. We assume that particles move in the presence of a potential V which has a *constant* value

³¹¹As an exercise you can derive Snell's law from Fermat's principle.

³¹²It is a good exercise to formulate the precise statement and verify it from Fermat's principle.

V_1 in the \mathcal{O}_+ and a constant value V_2 in \mathcal{O}_- . The energy of the particles is $E > \max\{V_1, V_2\}$. This is an idealized picture in which the force ∇V acts only when the particle passes through the boundary, by giving the particle a “kick” in the direction perpendicular to the boundary. In both \mathcal{O}_+ and \mathcal{O}_- the particles move along straight lines as the force ∇V vanishes in those regions. If the particle passes from \mathcal{O}_+ to \mathcal{O}_- , the component of the velocity parallel to the normal of the dividing surface will change, whereas the other components will remain unchanged. The trajectory will “bend”. One can calculate the angles of the trajectories in \mathcal{O}_+ and \mathcal{O}_- respectively with the corresponding normals, as in the case of Snell’s law. An easy calculation shows

$$\frac{\sin \alpha_1}{\sin \alpha_2} = \frac{\sqrt{2(E - V_2)}}{\sqrt{2(E - V_1)}}. \quad (1289)$$

This can be easily checked also for the case when the boundary is a more general smooth surface than a plane, although at this point we do not wish to use any analogue of the Fermat’s principle to prove this at this point, as our goal is ultimately to derive an analogy of Fermat’s principle in mechanics. We can simply convince ourselves that for the purposes of this mechanical thought experiment we can replace the boundary by the tangent plane at the point of impact. We see that – at least in the above situation – the laws governing the mechanical particles at energy E are the same as the laws of optics, if we choose the index of refraction as $\sqrt{2(E - V)}$. This should generalize to general smooth potentials V by the following argument. We can approximate V by staircase-like piece-wise constant potentials of the form $V_\varepsilon(x) = \chi_\varepsilon(V(x))$, where $\chi_\varepsilon: \mathbb{R} \rightarrow \mathbb{R}$ is a suitable step function with step ε .³¹³ For such potentials the analogy with optics works as above, and we expect that this property should be preserved in the limit $\varepsilon \rightarrow 0$. We therefore reach the remarkable conclusion, known already to Euler³¹⁴ around 1744, that

A particle of energy E in a potential field $V(x)$ moves along the same trajectories as the light rays in a material with index of refraction $n(x) = \sqrt{2(E - V(x))}$.

We have not really fully proved it, although our consideration above should be convincing. We will soon have a more rigorous proof. The optical function $T(x, y)$ will have a mechanical analogy, which will explain the meaning of the function $S(x, t)$ we dealt with in previous lectures.

³¹³For example, if $[x]$ is the integer part of x , we can take $\chi_\varepsilon(x) = \varepsilon[\frac{x}{\varepsilon}]$.

³¹⁴Euler studied the question in connection with the so called Maupertuis’ principle.

We have seen last time that the trajectories of a particle with energy E in the presence of a potential V is the same as the trajectories of light in material with index of refraction $\sqrt{2(E - V)}$. Let $T(x, y) = T(x, y; E)$ be the corresponding “optical function”, which we studied in the last lecture.³¹⁵ Let x, y be a sufficiently close to each other and let $t > 0$. At this stage we consider t as a parameter the value of which will be specified later. The value of $T(x, y)$ is the infimum of

$$\int_0^t \sqrt{2(E - V(z))} |\dot{z}(s)| ds \quad (1290)$$

over the curves $z: [0, t] \rightarrow \mathbb{R}^n$ with $z(0) = x$ and $z(t) = y$. We note that (1290) is a curve integral along the curve $z(s)$ written in a particular parametrization, but its value is independent of the parametrization, the integral is “geometric” in the sense that it only depends on the curve traced by $z(t)$, and not on any particular parametrization. We recall that for $a, b > 0$ we have

$$ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2, \quad \text{with equality} \iff a = b \quad (1291)$$

Hence we have

$$\int_0^t \sqrt{2(E - V)} |\dot{z}| ds \leq \int_0^t (E - V + \frac{1}{2}|\dot{z}|^2) ds, \quad (1292)$$

with equality if and only if $|\dot{z}(s)| = \sqrt{2(E - V(z(s)))}$ for each $s \in [0, t]$. Now instead of minimizing the left-hand side of (1292), we can try to minimize the right-hand side, and see if we can find a minimizer which would satisfy $|\dot{z}(s)| = \sqrt{2(E - V(z(s)))}$ for each $s \in [0, t]$. Since E is fixed, minimizing the right-hand side is the same as minimizing

$$I(z) = \int_0^t \frac{1}{2}|\dot{z}|^2 - V(z) ds. \quad (1293)$$

The equation for the minimizers of $I(z)$ is easily obtained from the condition

$$I'(z)\zeta = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} I(z + \varepsilon\zeta) = 0 \quad (1294)$$

for each smooth $\zeta = \zeta(s)$ with $\zeta(0) = \zeta(t) = 0$. We have

$$\ddot{z} = -\nabla V, \quad (1295)$$

which, quite luckily, is exactly the newtonian equation of motion. This is the celebrated *Hamilton's principle*:

³¹⁵Note that now T does not have dimension of time, since the quantity which is integrated along curves has the dimension of velocity.

The newtonian trajectories passing from x to y in time t are exactly the critical points³¹⁶ of the integral $I(z)$ in (1293).

Equation (1295) implies the energy conservation

$$\frac{1}{2}|\dot{z}|^2 + V = \text{const.} \quad (1296)$$

along the trajectory. For a minimizer of (1293) to be also a minimizer of (1290) for a given E , the condition for equality in (1291) gives

$$\frac{1}{2}|\dot{z}|^2 + V = E \quad \text{along the trajectory.} \quad (1297)$$

This will be satisfied if the constant in (1296) is exactly E . To achieve this, we must choose the right t : it is clear that the energy of the minimizer of (1293) should be high for small t and just above $\max\{V(x), V(y)\}$ for large t , so we should be able to choose t so that the energy is exactly E . Therefore the minimization of (1292) should be the same as the minimization of (1293) for a suitably chose t , so that the energy along the trajectory is E .

For any x, y and t we can define

$$S(x, y, t) = \inf \int_0^t \frac{1}{2}|\dot{z}|^2 - V(z) ds, \quad z(0) = x, \quad z(t) = y, \quad (1298)$$

where the infimum is taken over all smooth curves $z: [0, t] \rightarrow R^n$ satisfying the boundary conditions $z(0) = x, z(t) = y$. The function S is called *action* in Mechanics, and plays a role similar to the function $T(x, y)$ in optics. It contains much of the information about the system. In comparison with optics we have one more variable t , which stems from the fact that the energy is not fixed and its value influences the mechanical “index of refraction” $\sqrt{2(E - V)}$. In some sense, rather than having one optical system, we have a one-parameter family of “optical systems” parametrized by E . So we should also have a one parameter family of analogues of optical functions. The variable t is in some sense dual to the variable E . The relation of $T(x, y) = T(x, y; E)$ defined above and $S(x, y, t)$ can be seen from (1292), assuming we choose t so that in (1292) we have equality:

$$T(x, y; E) = Et + S(x, y, t), \quad \text{where } t = t(x, y, E) \text{ or } E = E(x, y, t) \quad (1299)$$

In the case of the free particle ($V = 0$) we easily calculate

$$T(x, y; E) = \sqrt{2E}|x - y|, \quad S(x, y, t) = \frac{|x - y|^2}{2t} \quad (1300)$$

³¹⁶We will see later that if t is fixed and x, y are sufficiently close (for the given t) the critical points are in fact minimizers

and also

$$t = t(x, y, E) = \frac{|x - y|}{\sqrt{2E}} = \frac{\partial T}{\partial E}, \quad E = E(x, y, t) = \frac{|x - y|^2}{2t^2} = -\frac{\partial S}{\partial t}. \quad (1301)$$

We will see that similar (and additional) relations are true in general, and that $S(x, y, t)$ satisfies the Hamilton-Jacobi equation as a function of either y (with fixed x) or x (with fixed y). Moreover, we will also see that $S(x, y, t)$ in some sense generates the solutions $S(x, t)$ we considered in lecture 75.

Homework Assignment 3

due on Thursday, May 12, 2011

Show that the Schrödinger equation $iu_t + \Delta u = 0$ in $\mathbb{R}^n \times \mathbb{R}$ has a bounded smooth solution $u(x, t)$ with the following properties.

1. $\sup_{x \in \mathbb{R}^n} |\nabla^k u(x, 0)| \leq 10^{-8}$ for $k = 0, 1, \dots, 10^4$.
2. At some time $t_0 > 0$ we have $|u(0, t_0)| \geq 1000$.
3. There exists $T > 0$ such that for each $t \geq T$ we have $\sup_{x \in \mathbb{R}^n} |\nabla^k u(x, t)| \leq 10^{-8}$ for $k = 0, 1, \dots, 10^4$.

Hint: use the fundamental solution and the fact that we can “run the equation” both backwards and forwards.

We continue with investigating properties of the function $S(x, y, t)$ defined last time, see formula (1298). For now we will assume that in the range of x, y, t we will be considering the infimum in (1298) is attained on a unique curve $z(s)$ on $[0, t]$ satisfying the boundary conditions $z(0) = x$ and $z(t) = y$. We first look at some simple sufficient conditions under which this is the case. Let us set

$$I(z) = \int_0^t \frac{1}{2} |\dot{z}|^2 - V(z) \, ds. \quad (1302)$$

The functional I will be considered on the set

$$Z = Z(x, y, t) = \{z \in W^{1,2}(0, t); z(0) = x, z(t) = y\}. \quad (1303)$$

The set Z is clearly affine, and hence the notion of convexity of I on Z makes sense. We also define

$$Z_0 = Z_0(t) = \{\zeta \in W^{1,2}(0, t), \zeta(0) = \zeta(t) = 0\} \quad (1304)$$

We have for any $z \in Z$ and any $\zeta \in W^{1,2}(0, t)$

$$\frac{d^2}{d\varepsilon^2} \Big|_{\varepsilon=0} I(z + \varepsilon\zeta) = \int_0^t |\dot{\zeta}|^2 - D^2V(z)(\zeta, \zeta) \, ds. \quad (1305)$$

If we assume that

$$|D^2V(z, z)(\zeta, \zeta)| \leq c|\zeta|^2 \quad \zeta \in \mathbb{R} \quad (1306)$$

and recall that

$$\int_0^t |\dot{\zeta}|^2 - c|\zeta|^2 \, ds \geq \nu \int_0^t |\dot{\zeta}|^2, \quad \zeta \in Z_0, \quad \text{when } \frac{\pi^2}{t^2} \geq c + \nu, \quad (1307)$$

we see that I is convex on Z when $t \leq \frac{\pi}{\sqrt{c+\nu}}$. In this case there will be a unique minimizer of I in Z , which will also be the unique critical point of I in Z . The condition $I'(z)\zeta = 0$ for each $\zeta \in Z_0$ gives the Euler-Lagrange equation

$$\ddot{z} = -\nabla V(z). \quad (1308)$$

(We first get that the equation is satisfied in a weak sense, but if V is smooth one sees easily that z is also smooth, by a bootstrapping argument, see lecture 40.) Under assumptions (1306), (1307), and assuming V is smooth, the minimizing trajectory $z(s) = z(s, x, y, t)$ will depend smoothly on x, y, t . We will not go into the proof of this statement at this point³¹⁸ but we will assume in what follows that $S(x, y, t)$ can be differentiated as necessary.

³¹⁷This can be seen for example from the Fourier series. It also follows from the variational characterization of the first eigenvalue of the laplacian, but we have not discussed this topic yet.

³¹⁸Although the statement is heuristically very plausible, is not completely trivial to prove it unless one approaches the proof in the right way.

For the minimizer z of I in $Z(x, y, t)$ we will denote

$$p = \dot{z}(0), \quad q = \dot{z}(t), \quad E = \frac{1}{2}|\dot{z}|^2 + V(z). \quad (1309)$$

In other words, the velocity of the minimizing trajectory at x is $p = p(x, y, t)$, the velocity of the minimizing trajectory at y is $q = q(x, y, t)$, and the energy (which is constant along the trajectory) is $E = E(x, y, t)$.

With this notation, we have the following identities, the first two of which are analogous to (1284), with the meaning of ξ and η explained in the discussion following (1284).

$$\frac{\partial S(x, y, t)}{\partial x} = -p, \quad \frac{\partial S(x, y, t)}{\partial y} = q, \quad \frac{\partial S(x, y, t)}{\partial t} = -E. \quad (1310)$$

Heuristically it is not hard to see why these identities should be true. Let $z(s)$ be the minimizing curve between x and y . Consider $S(x, y + \varepsilon h, t)$. We will write the minimizer $z_\varepsilon(s)$ for the trajectory between x and $y + \varepsilon h$ as

$$z_\varepsilon(s) = z(s) + \varepsilon \zeta(s, \varepsilon), \quad (1311)$$

where $\zeta(0, \varepsilon) = 0$ and $\zeta(t, \varepsilon) = h$. Let $\zeta(s) = \zeta(s, 0) = \lim_{\varepsilon \rightarrow 0^+} \zeta(s, \varepsilon)$. In a full proof we would have to prove the existence of this limit, but we will not go into the details of this step. We note that formally we have

$$\frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} (\varepsilon \zeta(s, \varepsilon)) = \zeta(s). \quad (1312)$$

We will again take this formal identity without proof. We have

$$S(x, y + \varepsilon h, t) = \int_0^t \frac{1}{2} |\dot{z}(s) + \varepsilon \dot{\zeta}(s, \varepsilon)|^2 - V(z(s) + \varepsilon \zeta(s, \varepsilon)) \, ds, \quad (1313)$$

and taking $\frac{\partial}{\partial \varepsilon}$ at $\varepsilon = 0$, we obtain, using integration by parts and (1308),

$$\frac{\partial S(x, y, t)}{\partial y} \cdot h = \int_0^t \dot{z} \dot{\zeta} - \nabla V(z) \zeta \, ds = \dot{z}(t) \cdot \zeta(t) = q \cdot h. \quad (1314)$$

This shows that

$$\frac{\partial S(x, y, t)}{\partial y} = q. \quad (1315)$$

The first equation of (1310) can be obtained similarly.

For the last equation of (1310), let us note that

$$S(x, z(t + \tau), t + \tau) = S(x, y, t) + \int_t^{t+\tau} \frac{1}{2} |\dot{z}|^2 - V(z) \, ds. \quad (1316)$$

Taking $\frac{\partial}{\partial \tau}$ of this identity, we obtain, using $\frac{\partial S(x, y, t)}{\partial y} = q$,

$$|q|^2 + \frac{\partial S(x, y, t)}{\partial t} = \frac{1}{2} |q|^2 - V(y), \quad (1317)$$

which implies the third equation of (1310).

Relations (1310) imply that for each fixed y the function $(y, t) \rightarrow S(x, y, t)$ satisfies

$$S_t + \frac{1}{2} |\nabla_y S|^2 + V = 0. \quad (1318)$$

The function $(x, t) \rightarrow S(x, y, t)$ satisfies

$$S_t + \frac{1}{2} |\nabla_x S|^2 + V = 0. \quad (1319)$$

We recall that when $V = 0$, the function $S(x, y, t)$ is given by the explicit formula (1300).

We note that formula (1298) will make sense under quite general assumptions (e. g. when V is bounded from below and continuous) for all (x, y, t) , so $S(x, y, t)$ can be defined globally. However, in general it may not be smooth, even when V is smooth and all its derivatives are bounded. Roughly speaking, this is due to the fact that the minimizing trajectories may not be unique for some (x, y, t) . The function still satisfies (1318) and (1319) in a suitable generalized sense. The investigation of such possibly non-smooth solutions is the subject of the theory of *viscosity solutions*, see a footnote in lecture 75 for references. The important fact is that one can establish both existence and uniqueness in this class of solutions. We will not pursue this topic here.

The functions $S(x, y, t)$ can be viewed from many different angles. For example the relations

$$\frac{\partial S(x, y, t)}{\partial x} = -p, \quad \frac{\partial S(x, y, t)}{\partial y} = q \quad (1320)$$

can be thought of (under some assumptions) as a transformation

$$\phi^t: (x, p) \rightarrow (y, q). \quad (1321)$$

These transformations (which can be defined in more general situations by replacing $S(x, y, t)$ by quite general functions $F(x, y)$) have many remarkable properties³¹⁹ which are studied in *symplectic geometry*. The particle trajectory (together with the velocity) of a particle starting at x with velocity x at time $t = 0$ can be thought of as

$$(x(t), p(t)) = \phi^t(x, p). \quad (1322)$$

The equations of motion

$$\dot{x} = p, \quad \dot{p} = -\nabla V(x) \quad (1323)$$

³¹⁹For example, we have seen that a “gradient graph” $\{p = \nabla f(x)\}$ is mapped again in a gradient graph $\{q = \nabla g(y)\}$. As an exercise, you can also check that the transformation preserves volume. (In fact, it takes the differential form $dp_i \wedge dx_i$ into the differential form $dq_i \wedge dy_i$, with the summation convention understood.)

can be thought of as an infinitesimal version of this transformation.

The function $S(x, y, t)$ can be also used to generate other solutions of the Hamilton-Jacobi equation. For example, let us consider the equation

$$F_t + \frac{1}{2}|\nabla F|^2 + V = 0, \quad (x, t) \in \mathbb{R}^n \times [0, t_1] \quad (1324)$$

with the initial condition $F(x, 0) = F_0(x)$. (In previous lectures we wrote this equation with S instead of F , but in this lecture we use S for the function $S(x, y, t)$ above.)

Let us set

$$F(x, t) = \inf_{y \in \mathbb{R}^n} (S(y, x, t) + F_0(y)) \quad (1325)$$

For each fixed y the function $(x, t) \rightarrow M(y, x, t) = S(y, x, t) + F_0(y)$ satisfies equation (1324), and hence F in (1325) can be viewed as the infimum of a family of functions which already satisfy the equation. We say that F is an *envelope* of the family. It is easy to see that F should satisfy the equation: assuming the infimum is attained at $y(x, t)$ and that $y(x, t)$ can be differentiated in (x, t) , we have

$$\begin{aligned} F(x, t) &= M(y(x, t), x, t), \\ F_t(x, t) &= M_y(y(x, t), x, t)y_t(x, t) + M_t = M_t, \\ \nabla_x F(x, t) &= M_y(y(x, t), x, t)\nabla_x y(x, t) + M_x = M_x, \end{aligned} \quad (1326)$$

where we used $M_y(y(x, t), x, t) = 0$, which is valid at the point of minimum in y .³²⁰

We also note the behavior of $S(x, y, t)$ as $t \rightarrow 0$. It is quite similar to that of the explicit formula (1300) for the case $V = 0$. In particular $S(x, y, t) \rightarrow \infty$ as $t \rightarrow 0_+$ for $x \neq y$, while $S(x, x, t) = 0$ for each $t > 0$. In view of this it is clear from (1325) that at $t \rightarrow 0_+$, the infimum over y is achieved at a point $y(x, t)$ which approaches x as $t \rightarrow 0$, and that $F(x, t) \rightarrow F_0(x)$ as $t \rightarrow 0_+$. Also, the condition $M_y(y(x, t), x, t) = 0$ gives

$$q = -\nabla_y S(y, x, t) = \nabla F_0(y), \quad y = y(x, t), \quad (1327)$$

which means that the velocity of the trajectory starting at y is $\nabla F_0(y)$.

In the special case $V = 0$ we obtain, using (1300), that the solution of

$$F_t + \frac{1}{2}|\nabla_x F|^2 = 0, \quad (x, t) \in \mathbb{R}^n \times (0, t_1), \quad F(x, 0) = F_0(x) \quad (1328)$$

is given for $t > 0$ by

$$F(x, t) = \inf_y \left(\frac{|y - x|^2}{2t} + F_0(y) \right), \quad (1329)$$

³²⁰We note that for this it is enough to have a critical point instead of the minimum and the notion of envelope can be extended to this situation.

which is known as *Hopf's formula*, after Eberhard Hopf. This formula defined F for all $t > 0$, but we have not investigated in what sense the equation $F_t + \frac{1}{2}|\nabla_x F|^2 = 0$ is satisfied after ∇F may become discontinuous (when characteristics corresponding to different values of ∇F meet). This topic has been studied in detail, see for example the textbook "Partial Differential Equations" by L. C. Evans for an exposition.

In the above we viewed the function $S(y, x, t)$ as a function of x with y as a parameter. In the formula (1325) the exact nature of the parameter y was not very important, one can use a similar procedure for any family of solutions $\phi(x, t, \alpha)$ of (1324). Here $\alpha = (\alpha_1, \dots, \alpha_m)$ is a parameter, and for each fixed α the function $x \rightarrow \phi(x, t, \alpha)$ is assumed to satisfy equation (1324). Assume we have such a family of solutions. We wish to investigate under which conditions we can generate all solutions (1324), at least locally, from this family, somewhat similarly to how we generated $F(x, t)$ above from $S(y, x, t)$. In that case the role of α was played by y .

Let $F(x, t)$ be a smooth solution of

$$F_t + \frac{1}{2}|\nabla F|^2 + V = 0. \quad (1330)$$

Let us try to express the derivatives $F_i = F_{x_i}$ in terms of the derivatives $\phi_{x_i}(x, \alpha)$. At a fixed (x, t) we consider the equations for α

$$\begin{aligned} F_1(x, t) &= \phi_{x_1}(x, t, \alpha) \\ F_2(x, t) &= \phi_{x_2}(x, t, \alpha) \\ &\dots \\ F_n(x, t) &= \phi_{x_n}(x, t, \alpha) \end{aligned} \quad (1331)$$

As (x, t) is fixed, this is just a system of n equations for the unknown α . If the equations are to be uniquely solvable for all possible values of $F_i(x, t)$ at least locally, in a neighborhood of some given solution, we clearly should have n unknowns $(\alpha_1, \dots, \alpha_n)$, and Jacobian of the mapping $\alpha \rightarrow \{\phi_{x_i}(x, \alpha)\}_{i=1}^n$ should be non-singular. The Jacobian is

$$\{\phi_{x_i \alpha_j}(x, t, \alpha)\}_{i,j=1}^n. \quad (1332)$$

An n -parameter family $\phi(x, t, \alpha) = \phi(x, t, \alpha_1, \dots, \alpha_n)$ of solutions of (1330) such that the matrix (1332) is non-singular at each point is called a *complete integral* of the equation (1330). We will see that – at least locally – a general solution of (1330) can be expressed in terms of complete integrals.

Let $\phi(x, t, \alpha) = \phi(x, t, \alpha_1, \dots, \alpha_n)$ be a complete integral of (??), as defined at the end of the last lecture. For a fixed (x, t) , we consider the map

$$\alpha \rightarrow \nabla_x \phi(x, t, \alpha). \quad (1333)$$

The matrix $\phi_{x_i \alpha_j}$ is the Jacobi matrix of this mapping, and by our assumptions it is non-singular. By the Implicit Function Theorem this means that for $\bar{\alpha} \in \mathbb{R}^n$ and $\bar{p} = \nabla_x \phi(x, t, \bar{\alpha})$ the mapping (1333) is a diffeomorphism of a neighborhood of $\bar{\alpha}$ onto a neighborhood of \bar{p} . (We recall that (x, t) is fixed at this point.) In what follows we will assume that all the relevant quantities take values in these neighborhoods, so that the map (1333) can be inverted. For typical complete integrals which can be found “in practice” these neighborhoods can be quite large, but they do not always coincide with the whole space \mathbb{R}^n , and therefore in specific examples this issue has to be kept in mind.

Assume that F is a solution of (1330). For each (x, t) we solve (1331) for α , so that we find α as a function of (x, t) . We can write $\alpha = \alpha(x, t)$.

We should comment on notation. When we consider α as a function of (x, t) and we write expressions such as $\nabla_x \phi(x, \alpha)$, there is a danger of confusion. Does this expression mean that we first differentiate for a fixed α and only then consider α as a function of (x, t) , or do we differentiate the function $x \rightarrow \phi(x, \alpha(x, t))$? Will use the notation $\phi_x(x, \alpha)$ for the former situation and the notation $\nabla_x(\phi(x, \alpha))$ for the latter situation in case this ambiguity might lead to a confusion. With this convention we can write $\nabla_x(\phi(x, \alpha)) = \phi_x(x, \alpha) + \phi_\alpha(x, \alpha)\alpha_x$. The notation $\nabla_x \phi(x, \alpha)$ can be used in both ways, and we will only use it when there is no danger of confusion.

We note that we can use (1330) to infer that, in addition to $\nabla_x F = \nabla_x \phi(x, \alpha)$ we also have $F_t(x, t) = \phi_t(x, t, \alpha)$. (Recall that for each fixed α the function $(x, t) \rightarrow \phi(x, t, \alpha)$ solves (1330).) Hence all the derivatives of $F(x, t)$ at (x, t) are “matched”:

$$\begin{aligned} F_t(x, t) &= \phi_t(x, t, \alpha), \\ F_1(x, t) &= \phi_{x_1}(x, t, \alpha), \\ &\dots \\ F_n(x, t) &= \phi_{x_n}(x, t, \alpha). \end{aligned} \quad (1334)$$

As (x, t) varies, α becomes a function of (x, t) , as already discussed. Although we matched the derivatives of ϕ and F the values of the functions themselves, $F(x, t)$ and $\phi(x, t; \alpha(x, t))$, are not necessarily matched. To match those, we need to use a trivial augmentation of the complete integral: from the n -parameter family for solutions $\phi(x, t, \alpha_1, \dots, \alpha_n)$ we obtain $(n + 1)$ -parameter family of solutions

$$\phi(x, t, \alpha_1, \dots, \alpha_n) + \alpha_0 = \phi(x, t, \alpha) + \alpha_0. \quad (1335)$$

As indicated, α still denotes the n -tuple $(\alpha_1, \dots, \alpha_n)$, without α_0 . We can use α_0 to match $F(x, t)$ and $\phi(x, t, \alpha) + \alpha_0$ simply by setting

$$F(x, t) = \phi(x, \alpha(x, t)) + \alpha_0(x, t). \quad (1336)$$

In other words, for each fixed (x, t) the system (1334) determining α is augmented by the additional equation

$$F(x, t) = \phi(x, t, \alpha) + \alpha_0. \quad (1337)$$

We can think about it in the following way. We wish to use the augmented complete integral $\phi(x, t, \alpha) + \alpha_0$ to match the function F and all its derivatives with ϕ and its derivatives in x, t at each point. We have $n + 2$ conditions: the values of the functions, n spatial derivatives and the time derivative. We have $n + 1$ parameters at our disposal, which seems to be one too few. However, we have relation (1330) between the time derivative of F and the spatial derivatives, so the $n + 2$ conditions are not independent - there is one relation between them. Hence $n + 1$ is exactly the right number of parameters. Therefore, at each point we can uniquely solve the system of equations (1334) augmented by (1337). The last equation is of course in some sense trivial, which is why we did not include α_0 in the definition of the complete integral.

Considering now $\alpha = \alpha(x, t)$ and $\alpha_0 = \alpha_0(x, t)$ as functions of (x, t) we can take derivatives of (1337) with respect to x_i and t (this time also differentiating α and α_0) and compare the result with (1334). We obtain

$$\alpha_{0t} + \phi_{\alpha_k} \alpha_{kt} = 0, \quad (1338)$$

and

$$\alpha_{0x_i} + \phi_{\alpha_k} \alpha_{kx_i} = 0, \quad (1339)$$

where in both equation summation over $k = 1, \dots, n$ is understood. The way we arrived at these equations is the following: we assumed we have a solution $F(x, t)$ and the complete integral $\phi(x, t; \alpha)$ and from that we found the functions $\alpha_k(x, t)$, $k = 0, 1, \dots, n$. We can now reverse this procedure: we can start with the complete integral $\phi(x, \alpha)$ and if we solve equations (1338) and (1339), the formula (1336) will give us a solution $F(x, t)$. In this point of view the complete integral can be viewed as a function generating a suitable change of variables. The change of variables is given by

$$p = \phi_x(x, t, \alpha), \quad \beta = -\phi_\alpha(x, t, \alpha), \quad (1340)$$

and we consider these equations as determining a transformation (which depends on t)

$$(x, p) \rightarrow (\alpha, \beta) \quad (1341)$$

In particular, for a given t , if α, β is given, then x, p are given. The equations (1338) and (1339) can be re-written in terms of α and β as

$$\alpha_{0t} - \beta_k \alpha_{kt} = 0, \quad (1342)$$

together with

$$\alpha_{0x_i} - \beta_k \alpha_{kx_i} = 0. \quad (1343)$$

These equations are considered together with (1340). Roughly speaking, our goal is to eliminate β from (1340), (1342), and (1343), and express α as a function of (x, t) . Equations (1342) and (1343) are best understood in terms of differential forms and *contact geometry*.³²¹ We will not go into these topics at this point, but we note the following: equations (1342), and (1343) are purely “geometrical” in the sense that if $\alpha_k(x, t), \beta_k(x, t)$ is a solution and we then $\alpha_k(x'(x, t), t'(x, t)), \beta_k(x'(x, t), t'(x, t))$ is again a solution for any diffeomorphism

$$(x, t) \rightarrow (x'(x, t), t'(x, t)). \quad (1344)$$

Also, note that they are independent of the choice of V in the equation we started with. We can in fact write down many solutions of (1342) and (1343) as follows: let $f = f(\alpha_1, \dots, \alpha_n)$ be any smooth function. Then for any smooth functions $\alpha_1(x, t), \dots, \alpha_n(x, t)$ the function

$$\begin{aligned} \alpha_0(x, t) &= f(\alpha_1(x, t), \dots, \alpha_n(x, t)) \\ \alpha_1(x, t) & \\ \dots & \\ \alpha_n(x, t) & \\ \beta_1(x, t) &= f_{\alpha_1}(\alpha_1(x, t), \dots, \alpha_n(x, t)) \\ \dots & \\ \beta_n(x, t) &= f_{\alpha_n}(\alpha_1(x, t), \dots, \alpha_n(x, t)) \end{aligned} \quad (1345)$$

The verification that these solve (1342) and (1343) is a matter of simple application of the chain rule. (In some sense, all “generic” solutions look like that, at least locally, but we will not discuss the details - they are not necessary for our purposes here.)

The equations for $\beta(x, t)$ in (1345) can be used to eliminate β from (1340). Substituting for beta from (1345), we obtain from the second equation in (1340)

$$f_{\alpha_k}(\alpha) = -\phi_{\alpha_k}(x, t, \alpha), \quad k = 1, 2, \dots, n. \quad (1346)$$

The equation says that α is a critical point (for fixed (x, t) of the function $\alpha \rightarrow f(\alpha) + \phi(x, t, \alpha)$. If this equation can be uniquely solved for α , we obtain the desired expression for α as a function of (x, t) , and (1336) together with the first equation of (1345) then give a solution $F(x, t)$ of (1324). (Note that the situation is somewhat similar to (1325), with α playing the role of y and $f(\alpha)$ playing the role of $S_0(y)$.) The initial condition $F(x, 0)$ is related to the choice of f . It should be emphasized that in many cases we do not expect that (1346) will be solvable of α globally, but only locally, in some open set. In that case $F(x, t)$ will also be defined only locally. The important point about the transformations (1340) given by (1340) is that the variable α is not just a function of x , it can “mix” both x and p .

³²¹In terms of differential forms, equations (1342) and (1343) have a simple meaning: we search for integral sub-manifolds of the field of hyper-planes in the $(2n + 1)$ -dimensional space $\alpha_0, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ given by the one-form $d\alpha_0 - \beta_k d\alpha_k$.

We will not go too much further in this direction - we have only scratched the surface of the large area of contact geometry and the systematical study of transformations of differential equations. Both these topics were initiated by the work of S. Lie around 1872. The interested reader can consult for example the book "Applications of Lie Groups to Differential Equations" by Peter Olver, or the paper "A Brief History of Contact Geometry and Topology" by H. Geiges, *Expositiones Mathematicae* 19 (2001), 25-53.

One can obtain some information about the functions $\alpha(x, t)$ by elementary means. We have seen - see (1252) - that we have

$$(\phi_i)_t + \phi_j(\phi_i)_j + V_i = 0. \quad (1347)$$

and also (since $\phi(x, t, \alpha)$ solve (1324) for any fixed α)

$$\phi_{it} + \phi_j\phi_{ij} + V_i = 0, \quad (1348)$$

where we write f_i, f_{ij} for f_{x_i} and $f_{x_i x_j}$ respectively, and we use the convention discussed before (1334) concerning whether or not the notation implies differentiating $\alpha(x, t)$ in $\phi(x, t, \alpha)$. Carrying out the differentiation in (1347) and subtracting (1348), we obtain

$$\phi_{i\alpha_k}\alpha_{kt} + \phi_{i\alpha_k}\phi_j\alpha_{kx_j} = 0. \quad (1349)$$

Since the matrix $\phi_{i\alpha_k}$ is non-singular by our assumptions, we infer that

$$\alpha_{kt} + \phi_j\alpha_{kx_j} = 0, \quad k = 1, \dots, n, \quad (1350)$$

which is the same as

$$\alpha_{kt} + F_j\alpha_{kx_j} = 0, \quad k = 1, \dots, n. \quad (1351)$$

This means that the parameters $\alpha_k = \alpha_k(x, t)$ are constant along the characteristics of the solution $F(x, t)$. The same is true for $\alpha_0 = \alpha_0(x, t)$, and one can easily establish by calculating $\alpha_{0t} + F_i\alpha_{0x_i} = 0$.

The fact that $\alpha_k(x, t)$ are constant along the characteristics of the solution $F(x, t)$ means that each given characteristic of $F(x, t)$ is also a characteristic of the solution $(x, t) \rightarrow \phi(x, t, \alpha)$ for some fixed α , which depends on the characteristic. Using the statement concerning (1267) in lecture 75, we see that the quantities

$$\beta_k = -\phi_{\alpha_k}(x, t, \alpha(x, t)) \quad (1352)$$

are also constant along the characteristics of $F(x, t)$.

We note that the above consideration also imply that once we know the complete integral $\phi(x, t, \alpha)$, we can determine the characteristics (and hence also the newtonian trajectories) as follows. We again consider the equations

$$p = \phi_x(x, t, \alpha), \quad \beta = -\phi_\alpha(x, t, \alpha), \quad (1353)$$

and consider these equations as determining a t -dependent transformation

$$(x, p) \longleftrightarrow (\alpha, \beta) \quad (1354)$$

If we keep α, β on the right-hand side constant and vary t the variables (x, p) on the left-hand side will move along a curve $(x(t), p(t))$, where $x(t)$ is a newtonian trajectory and $p(t) = \dot{x}(t)$ is its velocity. The initial point $x(0), p(0)$ of the trajectory can be chosen by varying the constants α, β . We see that the knowledge of the complete integral of the Hamilton-Jacobi equation enables us to calculate the trajectories. The calculation of the elliptic orbits of the planetary motion in lecture 75 was a special case of these principles.

The method of finding the complete integral of the Hamilton-Jacobi equation remains since its invention by Jacobi in 1830s one of the most powerful methods for expression solutions of the newtonian (or lagrangian) equations of motion by means of integrals.

We mention some examples.

An example of a complete integral is given by (1270), (1271), with E, M playing the role of α_1, α_2 . Note that the variables E, M “mix” both x and the velocity p . If we only consider changes of variables in the coordinate x , one cannot achieve the simplification.

For the free equation $S_t + \frac{1}{2}|\nabla S|^2 = 0$ we can write down a complete integral in more than one form. For example we have

$$\phi(x, \alpha) = \alpha \cdot x - \frac{1}{2}|\alpha|^2 t. \quad (1355)$$

Another possibility is a version of the function $S(x, y, t)$ we considered before

$$\phi(x, t, y, s) = \frac{|x - y|^2}{2(t - s)}, \quad (1356)$$

Where y plays the role of α . Here we have the extra parameter s which can be used instead of the parameter α_0 above in augmenting the integral.

In dimension $n = 1$ we only need one parameter to obtain a complete integral, and it can be taken to be the energy E . One seeks $S(x, t)$ in the form

$$S(x, t, E) = -Et + f(x, E), \quad (1357)$$

The equation for f is

$$\frac{1}{2}(f')^2 = E - V, \quad (1358)$$

from which we get

$$f(x, E) = \pm \int \sqrt{2(E - V(x))} dx. \quad (1359)$$

The role of α_1 is played by E . The equation

$$\beta_1 = \frac{\partial S}{\partial E} = -t \pm \int \frac{dx}{\sqrt{2(E-V)}} = \text{const.} \quad (1360)$$

gives the trajectory $x(t)$.

In general, it is difficult to find complete integrals for systems in dimension $n \geq 2$ in a closed form. Having such an integral means that we can find the equations for trajectories in a closed form, which is of course difficult in general. One can prove that a complete integral always exists at least locally, but one can rarely find it in a closed form.

We will briefly mention the general first-order equations

$$F(x, u, \nabla u) = 0 \quad (1361)$$

where $u = u(x_1, \dots, x_n)$ and F is smooth function on $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$. The partial derivatives u_{x_i} are traditionally denoted by p_i and we also write $p = \nabla_x u$. With this notation, the equation is often written as

$$F(x, u, p) = 0. \quad (1362)$$

These equations are best understood in terms of the contact geometry we already mention above. Here we will restrict ourself to some simple considerations, which nevertheless should give a good idea about the nature of the equation. Overall, the situation is quite similar to the more special Hamilton-Jacobi equations such as (1324) which we considered in previous lectures. One difference is that in the form (1362) the role of all the variables is the same, there is no distinguishes variable such as the variable t in (1324). The key to understanding the equation is again in the equations for p_i obtained by differentiating (1362). (There is again some geometry behind these equations, which we will not discuss.) Taking $\frac{\partial}{\partial x_i}$ of (1362), we obtain

$$0 = F_{x_i} + F_u u_{x_i} + F_{p_j} p_{j x_i} = F_{x_i} + F_u p_i + F_{p_j} p_{i x_j}. \quad (1363)$$

This equation shows that the information about p_i propagates along integral curves of the vector field $b_i(x) = F_{p_i}(x, u(x), p(x))$. let us assume that a solution u is given. Let $x(s)$ be a curve with

$$\frac{dx_i}{ds} = F_{p_i} = F_{p_i}(x, u(x), p(x)) = b_i(x). \quad (1364)$$

Along the curve we have, from (1363)

$$\frac{dp_i}{ds} = -F_{x_i} - F_u p_i. \quad (1365)$$

To close the equations, we need an equation for u along the curve (1364). The equation is

$$\frac{du}{ds} = u_{x_i} \frac{dx_i}{ds} = p_i F_{p_i}. \quad (1366)$$

We have the system

$$\frac{dx_i}{ds} = F_{p_i}, \quad \frac{dp_i}{ds} = -F_{x_i} - F_u p_i, \quad \frac{du}{ds} = p_i F_{p_i}. \quad (1367)$$

These equations define an ordinary differential equation for the triple (x, u, p) . Assume that Σ is an $n - 1$ -dimensional surface in R^n and that the values of u, p are given for each point $x \in \Sigma$, so that $F(x, u, p) = 0$ on Σ . The Cauchy problem for (1362) consists in finding an extension of u from Σ to a neighborhood of Σ so that $F(x, u, p) = 0$ is satisfied in the neighborhood. If for some $x \in \Sigma$ the vector F_{p_i} is tangent to Σ , then $\frac{dp}{ds}$ and $\frac{du}{ds}$ are given on one hand by the values of u, p on the surface, and on the other hand by (1367). The two requirements may not be consistent, in which case the Cauchy problem is not solvable. Therefore we will consider the Cauchy problem only for the situation when the vectors F_{p_i} are transversal (not tangent) to Σ at any point. In this case the ODE (1367) will give us a system of curves $x(s), u(s)$ in $\mathbb{R}^n \times \mathbb{R}$ parametrized by the points of Σ . In a neighborhood of Σ these curves will fill up an n - dimensional surface, and one can show that in some neighborhood of Σ this surface is a graph of a function solving $F(x, u, p) = 0$. The details of the proof are quite similar to the local existence proof for (1251), which we did in lecture 75, and we omit the details. The procedure also shows that the solution is unique, as at each step is fully determined by the data, there are no “free parameters”.

Above we assumed that the data on Σ were both u and p . If only u is given, we can still calculate the derivatives in the directions tangential to Σ , which gives us $n - 1$ coordinates of p , but it does not give us the derivative in the direction normal to Σ . That has to be determined from the equation. At each point $x \in \Sigma$ we know all the tangential components of p and hence the equation $F(x, u, p) = 0$ should be used to get information about the normal component of p . The normal component may not be determined uniquely by the equation - we may have several solutions. (This happens for example for the simple eikonal equation $|\nabla u|^2 = 1$.) If this happens, the solution of the Cauchy problem is not unique, and if we wish to recover uniqueness, the choice of the normal derivative at u at Σ has to be specified.

The notion of the complete integral is similar to that we have discussed for the Hamilton-Jacobi equation (1324). A function $\phi(x, \alpha)$ with $\alpha = (\alpha_1, \dots, \alpha_n)$ is a complete integral of (1362) if it solves the equation in x for each fixed α , and the matrix

$$\begin{matrix} \phi_{\alpha_1} & \phi_{\alpha_1 x_1} & \dots & \phi_{\alpha_1 x_n} \\ \phi_{\alpha_2} & \phi_{\alpha_2 x_1} & \dots & \phi_{\alpha_2 x_n} \\ \dots & \dots & \dots & \dots \\ \phi_{\alpha_n} & \phi_{\alpha_n x_1} & \dots & \phi_{\alpha_n x_n} \end{matrix} \quad (1368)$$

is non-singular. The last condition comes about as follows. Assume we are given a solution $u(x)$ and we wish to match it with the complete integral. At each

point x we wish to find α so that

$$\begin{aligned} u(x) &= \phi(x, \alpha) \\ u_{x_1}(x) &= \phi_{x_1}(x, \alpha) \\ &\dots \\ u_{x_n}(x) &= \phi_{x_n}(x, \alpha). \end{aligned} \tag{1369}$$

We have $n+1$ quantities on the right-hand side, but they satisfy equation (1362), and therefore we expect that we will need n parameters to match them with the right-hand side, for suitable α . Matrix (1368) is the Jacobian matrix of the map

$$\alpha \rightarrow \begin{pmatrix} \phi(x, \alpha) \\ \phi_{x_1}(x, \alpha) \\ \dots \\ \phi_{x_n}(x, \alpha) \end{pmatrix} \tag{1370}$$

which we wish to be invertible, at least locally. Hence the condition that the matrix be non-singular. Most of the consideration which we did for the complete integral of the Hamilton-Jacobi equation (1324) can be extended the above more general situation, but we will not pursue this topic at this point.

Lecture 81, 5/4/2011

Today we start discussing the wave equation

$$u_{tt} - \Delta u = 0, \tag{1371}$$

and also its non-homogeneous version

$$u_{tt} - \Delta u = f(x, t). \tag{1372}$$

The function u will be assumed to be a scalar function on some subset of the space-time $\mathbb{R}^n \times \mathbb{R}$, with $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

Historically, equation (1371) might have been the first PDE studied in some detail. It appeared in a 1747 paper by d'Alembert, who arrived at it in connection with vibrations of strings. In this context, we imagine that a string vibrates (with small amplitude) around its equilibrium state in the direction perpendicular to the string. The quantity $u(x, t)$ denotes the displacement from the equilibrium at location x and time t . For the usual strings, such as guitar strings, for example, $u(x, t)$ really is a vector in the plane perpendicular to the string, but we can consider the idealized situation in which the string vibrates only in one direction, in which case the function u can be considered as a scalar function. This is the case we will consider.³²² The equation arises in many other situations. For example, it describes the propagation of small-amplitude sound waves in air (and other media). In this case $u(x, t)$ represents the density (or, alternatively, the pressure) at the point x at time t . The wave equation also plays an important role in the analysis of Maxwell's equations of electrodynamics. It is through this connection that the analysis of the wave equation (together with Maxwell's equations) and the group of transformations which leaves it invariant, lead to profound changes in the foundations of Physics, culminating with the special theory of relativity (A. Einstein, 1905) and the general theory of relativity (A. Einstein, 1915).

There are many ways in which the equation can be arrived at. Let us consider for example the derivation based on Hamilton's principle which we discussed in lecture 78. We consider small oscillations of a string, with $u(x, t)$ having the same meaning as described above. The variable x parametrizes the string, we assume $x \in \Omega = (a, b)$. We allow $a = -\infty$ and $b = \infty$ as a possibility. The boundary condition at a, b are $u(a, t) = u(b, t) = 0$.³²³ In the case $a = -\infty$ we can simply assume that $u(x, t) = 0$ for sufficiently large negative x . A-priori it is not clear that this is a good assumption - for the heat equation or the Schrödinger equation the assumption would not work,³²⁴ but we will see that

³²²There is not much loss of generality, since the "general vibrations" can be thought of as superpositions of the vibrations in given planes.

³²³Other conditions can also be considered. For example $u(a, t) = 0$ can be replaced by $u_x(a, t) = 0$.

³²⁴When $f = 0$, it would imply that the solution vanishes identically.

it works well for the wave equation. (If we were unaware of this, a reasonable formulation of the boundary condition at $x = \pm\infty$ would be to demand that u decays to 0 “sufficiently fast” as $x \rightarrow \pm\infty$.)

In our derivation below we will not much use that x is one-dimensional, we can also think about $x \in \mathbb{R}^n$ (or a subdomain Ω of \mathbb{R}^n with $u = 0$ at the boundary of Ω , for example). For $n = 2$ we can think of vibrations of a drum. Note that both the string and the drum are under some tension when at rest. This is an important point. One can also consider vibrations of a rod or a plate, but these are not described by the wave equation.³²⁵

Considering a prescribed motion $u(x, t)$ (not necessarily a “real motion”), where ∇u is small, the quadratic part of the kinetic energy at time t is given by

$$E_{\text{kinetic}} = \int_{\Omega} \frac{1}{2} \rho |u_t(x, t)|^2 dx, \quad (1373)$$

where ρ denotes the density of the string/membrane. In principle ρ can be a function of x , but we will mostly consider the case when ρ is constant.

We turn to the potential energy. The terminology in the derivation below will correspond to the case of the string, $n = 1$, but some of the considerations can be made in any dimension, if the terminology is changed accordingly. The potential energy will be coming from the elastic energy stored in the material of the string due to tension. Since there is some tension in the string even when the string is at rest, our potential energy at time t will really be the increase of the potential between the rest state $u = 0$ and a state $u(x, t)$ at time t . In other word, we will take

$$E_{\text{potential}} = \begin{array}{c} \text{Elastic energy} \\ \text{when } u = u(x, t) \end{array} - \begin{array}{c} \text{Elastic energy} \\ \text{when } u = 0 \end{array}. \quad (1374)$$

We imagine that the change in the elastic energy comes from the slight increase of length of the string when in the state $u(x, t)$. If the change of length is δl and the tension of the string is P , we have,

$$E_{\text{potential}} = P \delta l + \text{higher order terms.}^{\text{326}} \quad (1375)$$

We have

$$\delta l = \int_{\Omega} (\sqrt{1 + |\nabla u(x, t)|^2} - 1) dx = \int_{\Omega} \frac{1}{2} |\nabla u(x, t)|^2 dx + \text{higher order terms.} \quad (1376)$$

³²⁵The right equation in this case is $u_{tt} + \Delta^2 u = 0$.

³²⁶If $n \geq 2$, then P should be thought of as a kind of stress tensor. If the tension is isotropic (no preferred directions) then P can be still considered as a scalar (representing the tensor $P\delta_{ij}$) and δl in the formula should be interpreted as the increase of area. If the tension is not isotropic, then P must be considered as a tensor and the formulae become more complicated. In the isotropic case when P can be considered as a scalar, its dimension is energy per unit area.

Hence

$$E_{\text{potential}} = P \int_{\Omega} \frac{1}{2} |\nabla u(x, t)|^2 dx + \text{higher order terms}. \quad (1377)$$

When the spatial derivatives of u are small, the higher order terms can be neglected, and we can take

$$E_{\text{potential}} = P \int_{\Omega} \frac{1}{2} |\nabla u(x, t)|^2 dx. \quad (1378)$$

By Hamilton's principle, the real motion will correspond to $u(x, t)$ for which the integral

$$I(u) = \int_{t_1}^{t_2} E_{\text{kinetic}} - E_{\text{potential}} dt = \int_{t_1}^{t_2} \int_{\Omega} \frac{1}{2} \rho |u_t(x, t)|^2 - \frac{1}{2} P |\nabla u(x, t)|^2 dx dt \quad (1379)$$

is at a critical point among the functions u with fixed $u(x, t_1)$, $u(x, t_2)$ and the prescribed behavior at $x \rightarrow \infty$ (or at the boundary of Ω , if a bounded domain is considered).³²⁷ In other words, the real motion $u(x, t)$ satisfies

$$I'(u)\varphi = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} I(u + \varepsilon\varphi) = 0, \quad (1380)$$

for each smooth φ compactly supported in $\mathbb{R}^n \times (t_1, t_2)$. It is easy to see, using the same calculation as in lecture 21 (see (283) and (284)) that this means that

$$\rho u_{tt} = P \Delta u. \quad (1381)$$

When ρ is constant, the equation can be brought to the form (1371) by setting $x = \sqrt{\frac{P}{\rho}} x'$. Also, equation (1381) is often written in the form

$$u_{tt} = c^2 \Delta u, \quad (1382)$$

with

$$c = \sqrt{\frac{P}{\rho}}. \quad (1383)$$

When c is constant and (1382) is considered in a bounded domain Ω , with, say, the Dirichlet boundary condition at $\partial\Omega$, one can express the general solution in terms of the eigenfunctions ϕ_k of the laplacian, see (1190), as

$$u(x, t) = \sum_k [a_k \cos(\sqrt{\lambda_k} ct) + \frac{b_k}{c\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k} ct)] \phi_k(x), \quad (1384)$$

where – at least formally –

$$\sum_k a_k \phi_k(x) = u(x, 0), \quad \sum_k b_k \phi_k(x) = u_t(x, 0). \quad (1385)$$

³²⁷We note that $I(u)$ is not bounded below on the class of $u(x, t)$ we are considering, and hence we cannot say that I attains its minimum in that class.

We see that the vibration of a string/membrane can be thought of as a superposition of harmonic oscillations with frequencies

$$\omega_k = \sqrt{\lambda_k} c = \sqrt{\frac{\lambda_k P}{\rho}}. \quad (1386)$$

For the case of a string of length L , we know $\lambda_k = \frac{\pi^2 k^2}{L^2}$ and hence

$$\omega_k = \sqrt{\frac{P}{\rho}} \frac{\pi k}{L}. \quad (1387)$$

We can also see from (1384) that a general solution almost periodic, similar to what we have seen for the solutions of the Schrödinger equation in lecture 71.

The formulae (1384) and (1385), together with basic results about convergence in L^2 and in H_0^1 for the series (1385) also give a solution of the boundary-initial-value problem consisting of finding a solution of (1382) with $u|_{\partial\Omega} = 0$ and satisfying the initial conditions $u(x, 0) = u_0(x)$, $u_t(x, 0) = u_1(x)$. We leave the details for the reader as an exercise.

We could also consider the Neumann boundary condition $\frac{\partial u}{\partial \nu} = 0$ at $\partial\Omega$ (in which case we use the eigenfunction with the Neumann boundary condition), or the more general mixed conditions considered in the first semester for the elliptic problems. The results will be quite similar to the case of the Dirichlet boundary conditions.

We see that one of the main features of the solutions of (1382) with constant c in a bounded domain, under quite general boundary conditions, is the almost periodicity. This is similar to what we saw for the Schrödinger equation in lecture 71.

The above calculations do not reveal the finite speed of propagation of disturbances for the solutions of (1371). This is one of the most important features of the wave equation which we have not seen in other equations we have considered, with the exception of the first order equations (such as the transport equations considered in lecture 73, or the Hamilton-Jacobi equations considered in lecture 75).

The finite speed of propagation is easily seen in dimension $n = 1$ for $\Omega = \mathbb{R}$, when we can find in a simple way the general solution. We work with (1371), when c is normalized to 1. Letting $\xi = x - t$ and $\eta = x + t$, and recalling that $n = 1$, equation (1371) becomes

$$u_{\xi\eta} = 0. \quad (1388)$$

Assume that u is a distribution satisfying (1388). Write the equation as

$$\frac{\partial}{\partial \eta} u_\xi = 0, \quad (1389)$$

we see that there exists a distribution $a(\xi)$ in the variable ξ so that

$$u_\xi = a(\xi), \quad (1390)$$

where we also denote by $a(\xi)$ the distribution on \mathbb{R}^2 defined by

$$\langle a, \varphi \rangle = \int_{\mathbb{R}} \langle a, \varphi(\cdot, \eta) \rangle d\eta. \quad (1391)$$

Choosing any distribution A of one variable such that $A' = a$, we see that

$$\frac{\partial}{\partial \xi}(u - A(\xi)) = 0. \quad (1392)$$

This means that there is a distribution $B(\eta)$ of one variable such that

$$u - A(\xi) = B(\eta). \quad (1393)$$

Hence the distribution satisfying (1388) is of the form

$$u = A(\xi) + B(\eta), \quad (1394)$$

where A, B are distributions in one variable. It is clear that if u is given by a locally integrable function, then both A and B are given by locally integrable functions. Going back to the original variables x, t , we see that the general solution of

$$u_{tt} - u_{xx} = 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R} \quad (1395)$$

is given by

$$u(x, t) = A(x - t) + B(x + t), \quad (1396)$$

where A, B are arbitrary functions or distribution (depending on our regularity requirements) in one dimension.

Let us now consider the natural initial-value problem (“Cauchy problem”) for (1395), which is the following: given $u_0: \mathbb{R} \rightarrow \mathbb{R}$ and $u_1: \mathbb{R} \rightarrow \mathbb{R}$, find $u: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying (1395), with

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x). \quad (1397)$$

This problem is an analogue of the finite-dimensional problem of finding the trajectory of a particle with given initial position and velocity.

We have not specified any regularity requirements concerning u, u_0, u_1 . It is enough to assume that all quantities involved are distributions. Substituting (1396) into (1438), we obtain

$$u_0(x) = A(x) + B(x), \quad u_1(x) = -A'(x) + B'(x). \quad (1398)$$

Let U_1 be a distribution in \mathbb{R} with $U_1' = u_1$ (which is determined modulo a constant). Then (1439) gives

$$A = \frac{1}{2}(u_0 - U_1), \quad B = \frac{1}{2}(u_0 + U_1), \quad (1399)$$

where U_1 is only given up to a constant. Hence

$$u(x, t) = \frac{1}{2}(u_0(x - t) + u_0(x + t)) + \frac{1}{2}(-U_1(x - t) + U_1(x + t)), \quad (1400)$$

and we see that the ambiguity of the constant involved in the definition of U_1 does not cause any problems, as the expression does not change if we change U_1 be a constant.

Formula (1400) shows that the “initial disturbance” u_0, u_1 propagates with speed 1. If both u_0 and u_1 are supported in an interval $[\alpha, \beta]$, then the solution $u(x, t)$ at time t is supported in the interval $[\alpha - |t|, \beta + |t|]$. The solutions of the form $A(x - t)$ and $B(x + t)$ are called traveling waves. They represent profiles moving at speed 1 or -1 respectively. Unlike the solutions of the Schrödinger equation, the solutions of the 1-dimensional wave equation exhibit no dispersion. The “wave packets” represented by $A(x - t)$ and $B(x + t)$ preserve their form, they do not disintegrate. In terms of the Fourier variables we have

$$A(x - t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{A}(\xi) e^{i(\xi x - \xi t)} dx, \quad (1401)$$

and recalling the discussion of the wave packet in lecture 68, we see that the dispersion relation is $\omega(\xi) = \xi$. Hence the group velocity is $d\omega/d\xi = 1$, independent of ξ . In a similar way, the dispersion relation for the wave packet $B(x + t)$ is $\omega(\xi) = -\xi$ and the group velocity is -1 , independent of ξ . We emphasize that the “no dispersion” picture applies only to spatial dimension $n = 1$. In dimensions $n \geq 2$ the solutions of the wave equation do exhibit some dispersion.

The finite speed of propagation can also be proved by working with energy and its flux in space-time. The proof is the same for all $n \geq 1$. We consider solutions $u(x, t)$ of (1371) in \mathbb{R}^n . We assume that $u(x, t)$ is as regular as needed for the calculations below. We set

$$e = e(x, t) = \frac{1}{2}(u_t^2 + |\nabla u|^2), \quad q_j = q_j(x, t) = -u_t u_{x_j}. \quad (1402)$$

A direct calculation shows that $u_{tt} - \Delta u = 0$ implies

$$e_t + q_{j,j} = 0. \quad (1403)$$

The quantity e represents energy density, and the vector q represents energy flux. Identity (1403) describes how the the energy density evolves with time.

Lemma

Assume that u satisfies $u_{tt} - \Delta u = 0$ and that $u(x, 0), u_t(x, 0)$ vanish in $B_{x_0, R} = \{x, |x - x_0| < R\}$. Then $u(x, t) = 0$ for all (x, t) with $|x - x_0| < R - |t|$.

Remark

Although in our proof we will assume that u is “sufficiently regular”, the statement remains true if we only assume that u is a distribution. The precise formulation is as follows. If u is a distribution satisfying $u_{tt} - \Delta u = 0$, then

- (i) The distributions $u(\cdot, t)$ and $u_t(\cdot, t)$ are well-defined for each t .
(ii) If the distributions $u(\cdot, 0)$ and $u_t(\cdot, 0)$ vanish in $B_{x_0, R}$, then the distribution u vanishes in the set $\{(x, t), |x - x_0| < R - |t|\}$.

One can compare this statement to the uniqueness theorem for the Schrödinger equation from lecture 69, except that we can only assume that u is a distribution (and not necessarily a tempered distribution).

Proof in the case when u is sufficiently regular

For regular u the statement can be proved by a simple application of (1403). Assume $x_0 = 0$ without loss of generality and define for $0 < t_1 < R$ define

$$\mathcal{O} = \{(x, t), 0 < t < t_1, |x| < R - t\} \subset \mathbb{R}^n \times \mathbb{R} \quad (1404)$$

Let $\nu = (\nu_1, \dots, \nu_n, \nu_0)$ be the outward unit normal at $\partial\mathcal{O}$ (defined everywhere except for the $(n - 1)$ -spheres $\{(x, t_1), |x| = R - t_1\}$ and $\{(x, 0), |x| = R\}$). The key point is the inequality

$$e\nu_0 + q_j\nu_j \geq 0 \quad (1405)$$

on the “slanted boundary” $\{(x, t) \in \partial\mathcal{O}, |x| = R - t\}$, which is easily seen from the Cauchy-Schwartz inequality. In addition to (1405), we also have $(q, e) = 0$ on the “lower lid” $\{(x, t) \in \partial\mathcal{O}, t = 0\}$ of \mathcal{O} and $(q, e) \cdot \nu = e$ on the “upper lid” $\{(x, t) \in \partial\mathcal{O}, t = t_1\}$ of \mathcal{O} .

Identity (1403) shows that

$$\int_{\partial\mathcal{O}} q_j\nu_j + e\nu_0 = 0, \quad (1406)$$

and by the above observations about the integrand we see that $(q, e) \cdot \nu$ vanishes at the boundary, implying that $e = 0$ on the “upper lid” $\{(x, t) \in \partial\mathcal{O}, t = t_1\}$. This is true for any $t_1 \in (0, R)$, and we see that the first derivatives of u in \mathcal{O} vanish. Hence u is constant in \mathcal{O} and since $u = 0$ on the “lower lid” $\{(x, t) \in \partial\mathcal{O}, t = 0\}$ of \mathcal{O} , it has to vanish everywhere in \mathcal{O} . The proof for $t < 0$ is analogous. This finishes the proof of the statement when u is “sufficiently regular”.

When we only know that u is a distribution, we can proceed as follows. Let $\phi = \phi(x)$ be a mollifier in x , with $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(x/\varepsilon)$ as usual. We define

$$u_\varepsilon = u *' \phi_\varepsilon, \quad (1407)$$

where $*'$ denotes the convolution in x only. As an exercise, you can show that u_ε is smooth. Moreover, assuming that $u(\cdot, 0)$ and $u_t(\cdot, 0)$ are well-defined, it is clear that $u_\varepsilon(\cdot, 0)$ and $u_{\varepsilon t}(\cdot, 0)$ vanish in $B_{x_0, R-\varepsilon}$. Applying the above calculation to u_ε with R replaces by $R - \varepsilon$, we obtain the result.

The only remaining issue is to show that $u(\cdot, t)$ and $u_t(\cdot, t)$ are well-defined under our assumptions as distributions. This can be seen from the identity

$$\int_{\mathbb{R}^n} u_\varepsilon(x, t_0)\varphi_t(x, t_0) - u_{\varepsilon t}(x, t_0)\varphi(x, t_0) dx = \int_{t_0}^{\infty} \int_{\mathbb{R}^n} u_\varepsilon(\varphi_{tt} - \Delta\varphi) dx dt \quad (1408)$$

once we know that we have enough test functions φ compactly supported in $\mathbb{R}^n \times [t_0, \infty)$ such that $\varphi_t(x, t_0)$ and $\varphi(x, t_0)$ can be chosen as arbitrary smooth compactly supported functions of x , with $\varphi_{tt} - \Delta\varphi$ compactly supported in $\mathbb{R}^n \times (t_0, \infty)$. The existence of such test function will follow from our consideration about the Cauchy problem for general n . (The case $n = 1$ follows quite easily from (1400)).

Lecture 82, 5/6/2011

In lecture 13 we saw that the equation

$$\Delta u = f \quad (1409)$$

in \mathbb{R}^n is invariant under the symmetry group of the Euclidean space. This means that for any symmetry

$$T: x \in \mathbb{R}^n \rightarrow Tx = Qx + b \quad (1410)$$

where Q is an orthogonal matrix and $b \in \mathbb{R}^n$, the laplacian commutes with the natural action of T on functions. The action of T on functions is usually defined by

$$(Tu)(x) = u(T^{-1}x). \quad (1411)$$

The statement that the laplacian commutes with this action means that

$$\Delta Tu = T\Delta u \quad (1412)$$

for any sufficiently regular function (or, in fact, any distribution). We can also write simply

$$\Delta T = T\Delta. \quad (1413)$$

Let us now consider the situation for the wave equation, which we will write as

$$\frac{\partial^2 u}{c^2 \partial t^2} - \Delta u = f \quad (1414)$$

One sets

$$x_0 = ct \quad (1415)$$

and

$$\square = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2}. \quad (1416)$$

so that we can write (1414) as

$$\square u = f. \quad (1417)$$

We will use the notation $x = (x_0, x_1, \dots, x_n)$. (In (1414), when we write $u = u(x, t)$, we use x for (x_1, \dots, x_n) and so, strictly speaking, we have a slight conflict of notation, but we will see that it will not create problems.)

We are interested in finding the transformations

$$T: x \in \mathbb{R}^{n+1} \rightarrow Qx + b \in \mathbb{R}^{n+1}, \quad \text{with } Q \text{ a regular } (n+1) \times (n+1) \text{ matrix} \quad (1418)$$

which commute with the wave operator (1416).

$$\square T = T \square. \quad (1419)$$

It is clear that the condition (1419) will impose restriction only on Q , the translation b can be arbitrary.

Let us look at this question in a slightly more general context. We consider an operator

$$L = a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}, \quad (1420)$$

where $\{a_{ij}\}$ is a regular symmetric matrix and summation over repeated indices is understood. Let us consider a regular matrix Q and the corresponding linear mapping $x \rightarrow Qx$. For a function $u = u(x)$ we will denote Qu the function $x \rightarrow u(Q^{-1}x)$. We are interested in characterizing the matrices Q which commute with L in the sense that

$$LQu = QLu \quad (1421)$$

for each sufficiently regular function u . (We could also demand this for each distribution u , it would not make a difference.) As in (1413) we will write

$$LQ = QL \quad (1422)$$

when (1421) is true for each u .

We will use the standard notation (x, y) for the canonical scalar product, i. e.

$$(x, y) = x_i y_i. \quad (1423)$$

Lemma

With the notation introduced above, the following conditions are equivalent

(i) $LQ = QL$

(ii) *The quadratic form $x \rightarrow (A^{-1}x, x)$ is invariant under Q , in the sense that $(A^{-1}Qx, Qx) = (A^{-1}x, x)$ for each x .*

(iii) *The quadratic form $x \rightarrow (Ax, x)$ is invariant under Q^t (the transpose matrix), in the sense that $(AQ^t x, Q^t x) = (Ax, x)$ for each x .*

(iv) $QAQ^t = A$.

The proof is elementary and is left to the reader as an exercise. The reason that A^{-1} appears in a natural way is obvious if we start distinguishing the upper and lower indices, writing vector coordinates with upper indices, co-vector coordinates with lower indices, and summing over repeated indices only when one of them is an upper index and the other is a lower index. (This is the full Einstein convention.) In this notation we should write

$$x = (x^0, x^1, \dots, x_n), \quad (1424)$$

so that coordinates have upper indices. The partial derivatives of a function

$$\frac{\partial u}{\partial x^i} = u_i \quad (1425)$$

are co-vectors, and should be treated as lower indices. The operator L should be written as

$$Lu = a^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} = a^{ij} u_{ij}, \quad (1426)$$

with upper indices in a^{ij} . This correspond to the quadratic form on co-vectors $(\xi_0, \xi_1, \dots, \xi_n)$ given by

$$L(\xi) = a^{ij} \xi_i \xi_j \quad (1427)$$

which appears naturally when we take the Fourier transformation

$$\widehat{Lu}(\xi) = -a^{ij} \xi_i \xi_j \hat{u}(\xi) = -L(\xi) \hat{u}(\xi). \quad (1428)$$

(The Fourier variables ξ are “dual” to x . As x are vectors, ξ must be considered as co-vectors, so that the expression $\xi \cdot x$ in $e^{i\xi \cdot x}$ is dimension-free.)

On the other hand, a^{ij} does not define any natural form on the x^i as these have upper indices. To be able to act on x^i in accordance with the full Einstein convention rules, we must “lower the indices” of a^{ij} to a_{ij} , which corresponds to taking the inverse matrix. The dual role of the Fourier variables is also seen from the formulae

$$\widehat{Qu} = Q^t \hat{u}, \quad \widehat{LQu}(\xi) = -L(\xi) \hat{u}(Q^t \xi), \quad \widehat{QLu}(\xi) = -L(Q^t \xi) \hat{u}(Q^t \xi) \quad (1429)$$

which immediately imply the equivalence of (i) and (iii) in the above lemma, although this proof probably cannot be considered as elementary proof. (The elementary proof mentioned above is based only on the chain rule and a simple manipulation of matrices.)

For any given A as above the set of all Q satisfying one of the equivalent conditions of the lemma form a group. These groups have been studied in detail in geometry. The A relevant for the wave equation is

$$A = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -1 \end{pmatrix} \quad (1430)$$

and the corresponding group of matrices Q is called the *Lorentz group*. (Note that in this case we have $A = A^{-1}$.)

Let us look at the Lorentz group in more detail in the 1 + 1 dimensional space-time \mathbb{R}^{1+1} , in which case it is usually denoted by $O(1, 1)$. We have four obvious elements of $O(1, 1)$, the matrices

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}. \quad (1431)$$

From the point of view of the wave equation these transformations tell us that that if $u(x_0, x_1)$ solves the wave equation, then $u(\pm x_0, \pm x_1)$ also solves the wave equation. This is not a surprise. Less obvious transformations in $O(1, 1)$ are given by the following matrices

$$Q(\alpha) = \begin{pmatrix} \cosh \alpha & \sinh \alpha \\ \sinh \alpha & \cosh \alpha \end{pmatrix}, \quad \alpha \in \mathbb{R}. \quad (1432)$$

We note that

$$Q(\alpha_1)Q(\alpha_2) = Q(\alpha_1 + \alpha_2) \quad (1433)$$

and we see that the matrices $Q(\alpha)$, $\alpha \in \mathbb{R}$ form a group, often denoted by $SO^+(1, 1)$. As an exercise you can show that the group $O(1, 1)$ has four connected components, one of them being $SO^+(1, 1)$ and the other three being

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} SO^+(1, 1), \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} SO^+(1, 1), \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} SO^+(1, 1). \quad (1434)$$

The situation is somewhat analogous with the orthogonal group $O(2)$, which has two components: the component $SO(2)$ parametrized by

$$R(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}, \quad \alpha \in \mathbb{R}. \quad (1435)$$

and the component

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} SO(2). \quad (1436)$$

The group $SO^+(1, 1)$ is from many points of view the most interesting part of $O(1, 1)$.

The reader can check that writing $x = Q(\alpha)x'$, going back to the original variables (x_1, t) and (x'_1, t') , and setting $v = c \tanh \alpha$ we obtain the classical expressions

$$x_1 = \frac{x'_1 + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad t = \frac{t' + \frac{vx'_1}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (1437)$$

Returning to the notation $x = (x_0, x_1) = (ct, x_1)$, and defining $x \rightarrow x'$ by $x = Q(\alpha)x'$, the reader can also verify that the action of $Q(\alpha)$ on the general solution (see (1396))

$$u(x) = A(x_1 - x_0) + B(x_1 + x_0) \quad (1438)$$

is

$$Qu(x) = A(x'_1 - x'_0) + B(x'_1 + x'_0) = A((x_1 - x_0)e^\alpha) + B((x_1 + x_0)e^{-\alpha}) \quad (1439)$$

We see that the new solution is still a composition of the right-moving and left-moving wave packets (as it must), but the energy of the right-moving wave packet increased (and it became more focused, with higher frequencies), whereas the energy of the left-moving wave packet decreased (and it became de-focused, with lower frequencies). At the same time, both wave packets still move at speed c .

The Lorentz group of the space \mathbb{R}^{n+1} is denoted by $O(1, n)$. It has again four connected components. The connected component containing identity is denoted by $SO^+(1, n)$. It obviously contains the rotations in the (x_1, \dots, x_n) variables (which do not “mix” x_0 with (x_1, \dots, x_n)) and the transformations generated by matrices (1432) in the (x_0, x_1) plane, which “mix” x_0, x_1 while leaving x_2, \dots, x_n unchanged. (The latter symmetries are sometimes called Lorentz boosts.) These two types of transformations generate the whole group $SO^+(1, n)$, and hence, in some sense, there is no new type of symmetries in comparison to what we have seen in dimension $1 + 1$ and rotations in \mathbb{R}^n . However, the two types of symmetries - rotations and boosts - interact in a non-trivial way, and therefore the group structure of $SO^+(1, n)$ is interesting.³²⁸

So far we have viewed the Lorentz group as a convenient tool to generate new solutions of the wave equation from the old ones. Historically, the significance of the Lorentz group for the Theory of Relativity and our understanding of the notions of space and time appeared in a context which is related to the above considerations, although the main point was not in the mathematics itself, but rather in its interpretation in the context of experimental observations of phenomena described by Maxwell’s equations (which are very closely related to the wave equation).

The equations of classical mechanics are invariant under galilean coordinate transformations

$$t = t', \quad x = x' + vt'. \quad (1440)$$

Here we think of (x, t) as a “stationary” coordinate system,³²⁹ and (x', t') as a system which moves with respect to the stationary system at a constant speed v . The notions of time and space are at first assumed to be the same “obvious” notions which have been used already by Newton. Assume we have some scalar physical quantity $u(x, t)$ which satisfies the wave equation

$$\frac{\partial^2 u}{c^2 \partial t^2} - \Delta u = 0 \quad (1441)$$

in the stationary system (x, t) , with c a constant independent of (x, t) . The observer in the system (x', t') can follow the same quantity u , but u will now

³²⁸In fact, as a group $SO^+(1, n)$ can be identified with the identity component of the group of symmetries of the n -dimensional hyperbolic space. This can be seen by considering the action of $SO^+(1, n)$ on the hyperboloid $x_0^2 - x_1^2 - \dots - x_n^2 = 1$.

³²⁹The notion of “stationary” is only relative, even in classical mechanics, and we use this term only to make the description easier.

depend on the coordinates (x', t') . The quantity u expressed in the coordinates (x', t') will be a new function u' of (x', t') , which is related to the original function $u(x, t)$ by

$$u'(x', t') = u(x' + vt', t'). \quad (1442)$$

The equation satisfied by u' in the (x', t') coordinates will be

$$\left(\frac{\partial}{c\partial t'} - \frac{v_k}{c} \frac{\partial}{\partial x'_k} \right)^2 u' - \Delta_{x'} u' = 0, \quad (1443)$$

which is different from the wave equation (1441). By observing the quantity u in the system (x', t') , we will be able to determine v and see that the coordinate system (x', t') is in motion. For example, in dimension $n = 1$ the general solution of (1443) will be

$$u'(x', t') = A(x' - (c - v)t') + B(x' + (c + v)t'), \quad (1444)$$

and we see that when $v \neq 0$ the wave packets traveling in one direction have a different speed from the wave packets traveling in the opposite direction. The stationary system (x, t) will have a special status among all the systems obtained from it by galilean transformations, in that it will be the only system where u satisfies the wave equation. (In dimension $n = 1$, it will be the only system where the wave packets have the same speed in both directions.) The situation just described arises for example when we observe oscillations of a long string while moving along the string at constant speed. In that case we will see exactly the solutions described by (1444).

Not so long after Maxwell found the complete equations of electrodynamics in 1865, it was realized that a similar situation should occur with Maxwell's equations. By watching carefully various electromagnetic phenomena, one should be able to identify among all the systems related by galilean transformations the one system which is "truly at rest". The experiments are not easy due to the smallness of the ratio v/c for velocities v available for the experiments and the speed of light c . During 1880s physicists designed ingenious experiments which were sufficiently accurate. However, the conclusion from these experiments contradicted the prediction about the existence of a special system at rest. There were no terms equivalent to the term with v_k/c in equation (1443). It appeared as if the right equation in any (inertial) system was exactly the wave equation, as if the transformation between the systems (x, t) and (x', t') was not given by (1440), but rather by (1437) (assuming the motion was along the x_1 -axis and the transformation of the coordinates x_2, x_3 was trivial: $x_2 = x'_2, x_3 = x'_3$.) As the coordinate system is not intrinsic (it is always somewhat artificially chosen by the observer), this required a careful re-examination of many notions about coordinate systems and time, which had been up to then taken for granted without much analysis, based on some intuitively plausible expectations. The conclusion from these investigation³³⁰ is that the correct transformation be-

³³⁰formulated in a definite form in 1905 in the famous paper "On the Electrodynamics of Moving Bodies" by A. Einstein, available online at <http://www.fourmilab.ch/etexts/einstein/specrel/specrel.pdf>

tween inertial coordinate system is indeed (1437). Moreover, the laws of the Classical Mechanics have to be adjusted so that they become invariant under these transformations too, rather than under (1440).

Our next task will be to derive the fundamental solution of the wave equation and representation formulae for general solutions.

We start our study of the fundamental solutions of the wave equation by establishing some useful technical fact about distributions. We will see that in higher dimensions the fundamental solutions of the wave equation are distributions which are not given by locally integrable functions, and the technical points which we now establish give a quite transparent way to express the various formulae we will be dealing with.

Let $\Omega \subset \mathbb{R}^n$ be an open set and

$$\phi: \Omega \rightarrow \mathbb{R} \tag{1445}$$

be a smooth function. Let $u \in \mathcal{D}'(\mathbb{R})$ be a distribution. When can

$$u \circ \phi \in \mathcal{D}'(\Omega) \tag{1446}$$

be well-defined in a natural way? Some assumptions are clearly needed. For example, when ϕ takes on a constant value c on a set $E \subset \Omega$ of non-zero measure and u is a locally integrable function which is not well-defined at c , then $u \circ \phi$ is not well-defined.

Assume for a moment that u is sufficiently regular and let $\varphi \in \mathcal{D}(\Omega)$. We have the following formula, which is a special case of the more general *co-area formula*³³¹.

$$\int_{\Omega} u(\phi(x)) |\nabla \phi(x)| \varphi(x) dx = \int_{\mathbb{R}} u(y) \int_{\phi^{-1}(y)} \varphi(x) d\mathcal{H}^{n-1}(x) dy \tag{1447}$$

where \mathcal{H}^{n-1} is the $(n-1)$ -dimensional Hausdorff measure, and $\phi^{-1}(y)$ denotes the set $\{x \in \Omega, \phi(x) = y\}$. If $\nabla \phi$ does not vanish in Ω , we can write

$$\int_{\Omega} u(\phi(x)) \varphi(x) dx = \int_{\mathbb{R}} u(y) \int_{\phi^{-1}(y)} \varphi(x) \frac{1}{|\nabla \phi(x)|} d\mathcal{H}^{n-1}(x) dy. \tag{1448}$$

Here and below we use the convention that an integral of any function over an empty set is always 0. Letting

$$\tilde{\varphi}(y) = \int_{\phi^{-1}(y)} \varphi(x) \frac{1}{|\nabla \phi(x)|} d\mathcal{H}^{n-1}(x), \tag{1449}$$

and considering the map

$$\varphi \rightarrow \tilde{\varphi}, \tag{1450}$$

formula (1448) suggests defining $u \circ \phi$ by

$$\langle u \circ \phi, \varphi \rangle = \langle u, \tilde{\varphi} \rangle \tag{1451}$$

for any distribution $u \in \mathcal{D}'(\mathbb{R})$. For this definition to work, we need to verify that $\varphi \rightarrow \tilde{\varphi}$ is a continuous map from $\mathcal{D}(\Omega)$ into $\mathcal{D}(\mathbb{R})$. This can be seen from the following formula

³³¹See for example the book “Geometric measure theory” by H. Federer, Theorem 3.2.12

Lemma

In the situation above, let $L: \mathcal{D}(\Omega) \rightarrow \mathcal{D}(\Omega)$ be defined by

$$L\varphi = \operatorname{div} \left(\frac{\nabla\phi}{|\nabla\phi|^2} \varphi \right). \quad (1452)$$

Then

$$\frac{d}{dy} \tilde{\varphi} = \tilde{L}\varphi. \quad (1453)$$

Proof

For any smooth $v: \mathbb{R} \rightarrow \mathbb{R}$ we have

$$\int_{\mathbb{R}} -v' \tilde{\varphi} = \int_{\Omega} -\nabla v(\phi(x)) \cdot \frac{\nabla\phi(x)}{|\nabla\phi(x)|^2} \varphi(x) dx = \int_{\Omega} v(\phi(x)) L\varphi(x) dx, \quad (1454)$$

which proves the statement of the lemma.

Using the lemma it is not hard to show that under our assumptions on ϕ the map $\varphi \rightarrow \tilde{\varphi}$ continuously maps $\mathcal{D}(\Omega)$ into $\mathcal{D}(\mathbb{R})$. We leave this to the reader as an exercise.

Remark: Formulae (1448) and (1449) do not look very “intrinsic”, in the following sense: the left-hand side of (1448) depends on Ω with the Lebesgue measure and the map ϕ . However, the right-hand looks much more “metric” - the definition of both \mathcal{H}^{n-1} and $\nabla\phi$ uses quite more than the Lebesgue measure and ϕ . The function $\tilde{\varphi}$ can be defined in a more intrinsic way as

$$\tilde{\varphi}(y) = \frac{d}{dy} \int_{\Omega \cap \{\phi < y\}} \varphi(x) dx. \quad (1455)$$

This definition involves only the same objects as on the left-hand side of (1447). An alternative way³³² to show that (1455) is (under our assumptions) a smooth function of y is by a suitable change of variables. First, we note that writing ϕ as a finite sum of functions with small support, we can assume without loss of generality that φ is supported in some ball B where the gradient $\nabla\phi$ is “almost constant”. In B we can introduce new coordinates x'_1, \dots, x'_n so that $x'_1 = \phi$ (in B). In the new coordinates we obtain

$$\tilde{\varphi}(y) = \int_B \varphi(y, x'_2, \dots, x'_n) A(y, x'_2, \dots, x'_n) dx'_2 \dots dx'_n, \quad (1456)$$

for some suitable smooth function A (which is related to the Jacobian of the transformation $x \rightarrow x'$).

³³²See for example Section 6.1 of the book “The Analysis of Linear Partial Differential Operators I” by L. Hörmander, Springer 1983.

We see that formula (1451) indeed gives a good definition of the distribution $u \circ \phi$. Moreover, it is easy to see that the definition agrees with the usual meaning of $u \circ \phi$ when u is a locally integrable function. It is also easy to check that the usual formulae remain valid. For example, one has

$$\nabla(u \circ \phi) = (\nabla u)(\phi(x))\nabla\phi. \quad (1457)$$

Example 1

Let us use the usual notation δ for the Dirac mass on \mathbb{R} (located at 0). Then $\delta \circ \phi$, which also will be written simply as $\delta(\phi(x))$ is the measure $d\mathcal{H}^{n-1}/|\nabla\phi|$ restricted to the surface $\phi^{-1}(0)$.

Example 2

Using the same notation as in Example 1, the distribution $\delta'(\phi(x))$ can be identified with the distribution $\frac{\nabla\phi}{|\nabla\phi|^2}\nabla(\delta(\phi(x)))$. It can also be identified with $\frac{\partial}{(\phi_{x_1})\partial x_1}\delta(\phi(x))$ as long as $\phi_{x_1} \neq 0$.

Let us now return to the study of the fundamental solutions of the wave equation. We will use the notation

$$\square = \frac{\partial^2}{\partial t^2} - \Delta = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2}. \quad (1458)$$

The equation for the fundamental solution is

$$\square u(x, t) = \delta(x, t), \quad (\text{in } \mathbb{R}^n \times \mathbb{R}) \quad (1459)$$

where $\delta = \delta_{n+1}$ is the Dirac at 0 mass in $\mathbb{R}^n \times \mathbb{R}$. One can also consider the initial-value problem

$$\begin{aligned} \square u(x, t) &= 0 && \text{in } \mathbb{R}^n \times \mathbb{R}, \\ u(x, 0) &= 0 \\ u_t(x, 0) &= \delta(x), && \text{with } \delta = \delta_n, \text{ the Dirac mass in } \mathbb{R}^n. \end{aligned} \quad (1460)$$

which – as we will see – is closely related to (1459), although the solutions obviously cannot be identical (with one giving δ and one giving 0 as the right-hand side). We recall the remark in lecture 81 that for a distribution $u(x, t)$ satisfying $\square u = 0$ in $\mathbb{R}^n \times \mathbb{R}$ the distributions $u(x, 0)$ and $u_t(x, 0)$ are well-defined, see (1408).

Another natural initial-value problem is

$$\begin{aligned} \square u(x, t) &= 0 && \text{in } \mathbb{R}^n \times \mathbb{R}, \\ u(x, 0) &= \delta(x) && \text{with } \delta = \delta_n, \text{ the Dirac mass in } \mathbb{R}^n. \\ u_t(x, 0) &= 0, \end{aligned} \quad (1461)$$

The Fourier transform of a tempered distribution in the space-time $\mathbb{R}^n \times \mathbb{R}$ will be denoted by \hat{u} , and the Fourier variables will be denoted by (ξ, τ) , with $\xi = (\xi_1, \dots, \xi_n)$, so that we have

$$\hat{u}(\xi, \tau) = \int_{\mathbb{R}^n \times \mathbb{R}} u(x, t) e^{-i\xi x - i\tau t} dx dt . \quad (1462)$$

For distributions this identity is only formal, and we define \hat{u} by formula (686) from lecture 47.

By the uniqueness lemma in lecture 82 and the remark after it we know that the solutions of problems (1460) and (1461) respectively are unique. On the other hand, equation (1459) has many different solutions, as the equation $\square u = 0$ in $\mathbb{R}^n \times \mathbb{R}$ has many different solutions. This can be seen for example by looking at the equation for the Fourier transform. In the Fourier variable the equation $\square u = 0$ becomes

$$(-\tau^2 + \xi_1^2 + \dots + \xi_n^2)\hat{u} = 0 . \quad (1463)$$

Any measure μ supported on the “light cone”

$$C = \{(\xi, \tau), \tau^2 - \xi_1^2 - \dots - \xi_n^2 = 0\} \quad (1464)$$

is provides a solution $\hat{u} = \mu$ of (1463) and if the distribution μ is tempered, its inverse Fourier Transform gives a solution of the free equation $\square u = 0$.

Above we considered Lorentz transforms Q in $\mathbb{R}^n \times \mathbb{R}$ and use the notation Qx with $x = (x_0, x_1, \dots, x_n)$. We now slightly change our convention and will write

$$Q: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n \times \mathbb{R}, \quad (x, t) \rightarrow Q(x, t) . \quad (1465)$$

The action of Q on functions is defined again by

$$Tu(x, t) = u(Q^{-1}(x, t)) . \quad (1466)$$

This extends naturally to distributions by

$$\langle Tu, \varphi \rangle = \langle u, T^{-1}\varphi \rangle , \quad (1467)$$

where we have used $|\det Q| = 1$, which is satisfied for any Lorentz transformation. We will also use the term Lorentz transformation for the transformation T on the distributions and can identify it with the matrix Q .

It is easy to see that for the Dirac mass $\delta = \delta(x, t)$ in $\mathbb{R}^n \times \mathbb{R}$ we have

$$T\delta = \delta \quad (1468)$$

for any Lorentz transformation, and since we also have $T\square = \square T$, the solutions of (1459) are invariant under Lorentz transformations: if u is a solution, so is Tu .

One can also consider the question whether the problem (1460) or the problem (1461) are invariant under Lorentz transformations. It is quite easy to see that they cannot be invariant under the full Lorentz group, but it makes sense to consider the invariance under the subgroup $O^+(1, n)$ of Lorentz transformations consisting of the transformations preserving the direction of time. As an exercise, you can consider this problem, which is not as obvious as the invariance of (1459). We will return to this question later.

We can add additional conditions on the solution which will enforce uniqueness for (1459). Natural cases to consider are

(i) u vanishes for $t < 0$. (More precisely, the distribution u vanishes in the open set $\mathbb{R}^n \times (-\infty, 0)$.)

In this case the solution u is uniquely determined by the uniqueness lemma in lecture 81 and the remark following it. We will denote this unique solution by G^+ . Strictly speaking, we have not yet proved that G^+ exists, but we will see shortly that this is indeed the case. We note that since (1432) is invariant under the Lorentz group and condition (i) is invariant under its subgroup $O^+(1, n)$ consisting of the Lorentz transformation which preserve the direction of time, we must have

$$TG^+ = G^+, \quad T \in O^+(1, n). \quad (1469)$$

Note that the uniqueness lemma also implies that G^+ must vanish outside the convex hull of the positive light cone

$$C^+ = \{(x, t), t^2 - x_1^2 - \dots - x_n^2 = 0, t \geq 0\}. \quad (1470)$$

We will use the notation

$$\tilde{C}^+ = \text{convex hull of } C^+ = \{(x, t), t^2 - x_1^2 - \dots - x_n^2 \geq 0, t \geq 0\}. \quad (1471)$$

We will see later that for $n = 3, 5, \dots$ the support of G^+ is exactly C^+ , which is not obvious at this point. For $n = 1, 2, 4, 6, \dots$ the support of G^+ is \tilde{C}^+ .

(ii) u vanishes for $t > 0$.

This situation can be mapped into (i) by using the transformation $u(x, t) \rightarrow u(x, -t)$. The solution (if it exists) must be unique and will be denoted by G^- . We have

$$G^-(x, t) = G^+(x, -t), \quad TG^- = G^-, \quad T \in O^+(1, n). \quad (1472)$$

The solutions G^+ and G^- can be combined to obtain additional natural solutions.

(iii)

$$G = \frac{1}{2}(G^+ + G^-) \quad (1473)$$

This is a solution of (1459) which is invariant under the full Lorentz group and is supported in $\tilde{C} = \tilde{C}^+ \cup \tilde{C}^-$. One can show that these properties determine G uniquely.³³³

One can also consider the function

$$K = G^+ - G^- . \quad (1474)$$

As $\square G^+ = \square G^-$, we have

$$\square K = 0 . \quad (1475)$$

Also, K is supported in the \tilde{C} and is invariant under $O^+(1, n)$. We will see later that K is the (unique) solution of (1460). Once we know that K is a solution of (1460), it is easy to see that the time derivative K_t of K is a solution of (1461).³³⁴

In addition to the four functions G^+, G^-, G, K there are other distinguished solutions of (1459), such as the so-called Feynman propagators used in the quantum field theory. We will encounter them later, they will appear naturally in connection with some of the methods used to calculate the fundamental solutions above.

We now turn to the task of calculating some the above solutions. The calculations are all classical and can be found in a number of textbooks. There are several methods which can be used. We mention the following.

- (a) Fourier transformation in x .
- (b) Fourier transformation in (x, t) .
- (c) Analytical continuation from fundamental solution of the euclidean laplacian $\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$.
- (d) Reduction to an ODE by the use of the Lorentz invariance.
- (e) The method of spherical averages (in odd dimensions) combined with the method of descent for even dimensions.

We will first briefly consider the Fourier transform in x . For this method we will write

$$\delta(x, t) = \delta(x)\delta(t) \quad (1476)$$

³³³It can be done with the tools which we have, but it is somewhat more difficult than the uniqueness in (i) or (ii).

³³⁴The only part of this statement which is not immediate is $(K_t)_t(x, 0) = 0$. To see this, we note that – assuming K solves (1460) – we have $K_{tt}(x, 0) = \Delta K(x, 0) = \Delta 0 = 0$.

and consider take the Fourier transform of (1459) in x . Let $\tilde{u}(\xi, t)$ be the Fourier transform of u in x . Equation (1459) gives

$$\tilde{u}_{tt}(\xi, t) + |\xi|^2 \tilde{u}(\xi, t) = \delta(t). \quad (1477)$$

To calculate \tilde{G}^+ , we solve (1477) with the “boundary condition” $\tilde{u}(\xi, t) = 0$ for $t < 0$. One can easily find the (unique) solution. It is given by

$$\tilde{u}(\xi, t) = \begin{cases} \frac{\sin(|\xi|t)}{|\xi|} & t \geq 0, \\ 0 & t < 0. \end{cases} \quad (1478)$$

This gives the spatial Fourier transformation of G^+ .

The *space-time* Fourier transformation of G^+ must obviously be equal to $\frac{1}{-\tau^2 + |\xi|^2}$ away from the set $\{-\tau^2 + |\xi|^2 = 0\}$. If one solves the ODE (1477) (taking into account the condition that $\tilde{u}(\xi, t)$ vanishes for $t < 0$) by using Fourier transformation in t , one obtains

$$\widehat{G^+}(\xi, \tau) = \text{p.v.} \frac{1}{-\tau^2 + |\xi|^2} - i\pi \text{sign } \tau \delta(\tau^2 - |\xi|^2). \quad (1479)$$

In a similar way

$$\widehat{G^-}(\xi, \tau) = \text{p.v.} \frac{1}{-\tau^2 + |\xi|^2} + i\pi \text{sign } \tau \delta(\tau^2 - |\xi|^2). \quad (1480)$$

We also note that – taking the spatial Fourier transformation of the problem (1460) – we obtain the initial-value problem

$$\begin{aligned} \tilde{u}_{tt}(\xi, t) + |\xi|^2 \tilde{u}(\xi, t) &= 0, & t \in \mathbb{R}, \\ \tilde{u}(\xi, 0) &= 0, \\ \tilde{u}_t(\xi, 0) &= 1. \end{aligned} \quad (1481)$$

The solution is

$$\tilde{u}(\xi, t) = \frac{\sin(|\xi|t)}{|\xi|}, \quad t \in \mathbb{R}. \quad (1482)$$

We see that the solution of (1460) coincides with G^+ for $t \geq 0$. Denoting by $\theta(t) = \xi_{[0, \infty)}(t)$ the Heaviside function, we can express that by the identities

$$G^+(x, t) = K(x, t)\theta(t), \quad G^- = K(x, t)(-\theta(-t)). \quad (1483)$$

This is an illustration of the Duhamel’s principle which we discussed in some detail in lecture 53 in connection with the heat equation. Relations (1483) also imply

$$G(x, t) = \frac{1}{2}K(x, t)\text{sign}(t). \quad (1484)$$

We see from this that K is invariant under the subgroup $O^+(1, n)$ of the Lorentz group.

One can see this from looking at the Fourier transform of K , which can be calculated from (1479), (1480), and (1474) as

$$\hat{K}(\xi, \tau) = -2\pi i \operatorname{sign} \tau \delta(\tau^2 - |\xi|^2). \quad (1485)$$

We now discuss the Fourier inversion of the distribution (1482). We will assume $t > 0$. Letting $\rho = |\xi|$, we have

$$u(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\sin \rho t}{\rho} e^{i\xi x} d\xi. \quad (1486)$$

Note that Example 1 in lecture 48 shows that for $n = 1$ we have $u(x, t) = \frac{1}{2}\xi_{(-t, t)}(x) = \frac{1}{2}(\theta(x - t) + \theta(x + t))$, as one can also obtain from our discussion of the case $n = 1$ in lecture 81, see (1400). If you have done the optional part of homework 2, you will also notice that the case $n = 3$ is easy: in that case $u(x, t)$ will be the rotationally invariant measure of the sphere of radius $r = t$ whose total mass is t . This can also be written as $u(x, t) = \frac{1}{4\pi t} \delta(t - |x|)$. Independently of the relevant part in the homework assignment 2, you can check the last result simply by calculating the Fourier transform of $u(x, t)$ in the variable x , it is not a hard calculation.

For general $n = 1, 2, \dots$ we can still calculate the inversion (1486) in a relatively straightforward way. If we do not use additional tricks, the calculation is perhaps a little tedious, but we outline the main points, just to illustrate that a direct calculation is possible in this case. The function $\tilde{u}(\xi, t)$ is a smooth function of ξ in \mathbb{R}^n , but it has slow decay as $|\xi| \rightarrow \infty$, and the integral (1486) is not absolutely convergent. We can regularize it by adding a factor $e^{-\varepsilon\rho}$ to the integrand, and calculate the integral

$$u_\varepsilon(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\sin \rho t}{\rho} e^{-\varepsilon\rho} e^{i\xi x} d\xi. \quad (1487)$$

By the symmetry considerations in lecture 50, we know that the function $u_\varepsilon(x, t)$ is radially symmetric in x , and it is enough to calculate it at $x = (r, 0, \dots, 0)$ for $r \geq 0$. By a slight abuse of notation, let us write $u_\varepsilon(r, t)$ for the value of u_ε at such an x (and a given t). Let us write a general $\xi \in \mathbb{R}^n$ as

$$\xi = \rho\eta, \quad \rho \geq 0, \quad \eta \in S^{n-1}. \quad (1488)$$

We have

$$u_\varepsilon(r, t) = \frac{1}{(2\pi)^n} \int_0^\infty \int_{S^{n-1}} \frac{\sin \rho t}{\rho} e^{-\varepsilon\rho} e^{ir\rho\eta_1} \rho^{n-1} d\eta d\rho \quad (1489)$$

We will rewrite this as

$$u_\varepsilon(r, t) = \frac{1}{(2\pi)^n} \int_{S^{n-1}} \int_0^\infty \sin \rho t e^{ir\rho\eta_1 - \varepsilon\rho} \rho^{n-2} d\rho d\eta. \quad (1490)$$

We note that for $n = 2$ the integral ρ can be evaluated explicitly, as the factor ρ^{n-2} then disappears. We can achieve a similar effect for general $n \geq 2$ by writing

$$u_\varepsilon(r, t) = (-1)^{\frac{n-2}{2}} \partial_t^{n-2} \frac{1}{(2\pi)^n} \int_{S^{n-1}} \int_0^\infty \sin \rho t e^{ir\rho\eta_1 - \varepsilon\rho} d\rho d\eta, \quad n = 2, 4, \dots \quad (1491)$$

and

$$u_\varepsilon(r, t) = (-1)^{\frac{(n-3)}{2}} \partial_t^{n-2} \frac{1}{(2\pi)^n} \int_{S^{n-1}} \int_0^\infty \cos \rho t e^{ir\rho\eta_1 - \varepsilon\rho} d\rho d\eta, \quad n = 3, 5, \dots \quad (1492)$$

The integrals over ρ can now be evaluated explicitly by writing $\sin \rho t = \frac{1}{2i}(e^{i\rho t} - e^{-i\rho t})$, and $\cos \rho t = \frac{1}{2}(e^{i\rho t} + e^{-i\rho t})$. We obtain

$$u_\varepsilon(r, t) = (-1)^{\frac{(n-2)}{2}} \partial_t^{n-2} \frac{1}{(2\pi)^n} \int_{S^{n-1}} \frac{1}{2i} \left(\frac{1}{-it - ir\eta_1 + \varepsilon} - \frac{1}{it - ir\eta_1 + \varepsilon} \right) d\eta \quad (1493)$$

for $n = 2, 4, \dots$, and

$$u_\varepsilon(r, t) = (-1)^{\frac{(n-3)}{2}} \partial_t^{n-2} \frac{1}{(2\pi)^n} \int_{S^{n-1}} \frac{1}{2} \left(\frac{1}{-it - ir\eta_1 + \varepsilon} + \frac{1}{it - ir\eta_1 + \varepsilon} \right) d\eta \quad (1494)$$

for $n = 3, 5, \dots$. To evaluate the integral over the sphere, we note that for a general function $f = f(\eta_1)$ we have

$$\int_{S^{n-1}} f(\eta_1) d\eta = |S^{n-2}| \int_{-1}^1 f(\eta_1) (1 - \eta_1^2)^{\frac{(n-3)}{2}} d\eta_1 \quad (1495)$$

Note that when n is odd, then the term $(1 - \eta_1^2)^{\frac{(n-3)}{2}}$ is a polynomial. For n even we can write

$$\int_{S^{n-1}} f(\eta_1) d\eta = |S^{n-2}| \int_0^\pi f(\cos \alpha) (\sin \alpha)^{n-2} d\alpha, \quad (1496)$$

which can be written as a curve integral over the unit circle in the complex plane, which can be evaluated by the residue theorem. Moreover, the integrals in (1494) and (1495) need to be evaluated only modulo polynomials of order $< n - 2$ in t , due to the presence of the time derivative in the formulae. There is still some calculations to be made, and at some point we have to take the limit $\varepsilon \rightarrow 0$, but it is clear that in principle we can obtain the formulae for the solution G^+ from this direct calculation. As an exercise you can finish the calculation in the cases $n = 2$ and $n = 3$, which are quite easy. We will obtain the final formulae for the solutions by a different method, and therefore we will not pursue the above calculations further.

Let us now turn to the method (c) of the analytical continuation from the fundamental solution of the $(n + 1)$ -dimensional laplacian.

Let us consider the operator

$$L_a = -\frac{\partial^2}{a^2 \partial t^2} - \frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2}. \quad (1497)$$

The wave operator corresponds to taking $a = i$, but we will start with a real. We know what the Green's function of L_a is when $a = 1$. A simple change of variables gives the Green's function for general $a \in \mathbb{R}$, $a \neq 0$ as

$$G_a(x, t) = aG_1(x, at) \quad (1498)$$

For $n \geq 2$ we have

$$G_a(x, t) = c_n \frac{a}{(a^2 t^2 + |x|^2)^{\frac{n-1}{2}}}, \quad c_n = \frac{1}{(n-1)|S^n|}. \quad (1499)$$

We have

$$L_a G_a = \delta(x, t), \quad a \in \mathbb{R} \setminus \{0\}. \quad (1500)$$

We note that G_a (considered as distribution) can be analytically continued to the complex domain in the half-plane

$$\mathcal{O} = \{a \in \mathbf{C}, \operatorname{Re} a > 0\}. \quad (1501)$$

The precise meaning of this is that for each test function $\varphi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$ the complex-valued function

$$a \rightarrow \int_{\mathbb{R}^n \times \mathbb{R}} G_a \varphi \quad (1502)$$

can be analytically extended from $(0, \infty)$ to \mathcal{O} . The operator L_a is transparently defined for $a \in \mathbf{C} \setminus \{0\}$, with analytical dependence on a .

We have, by definition,

$$\int_{\mathbb{R}^n \times \mathbb{R}} G_a L_a \varphi = \varphi(0), \quad a \in (0, \infty) \quad (1503)$$

and by analyticity this must be true also for $a \in \mathcal{O}$. For each fixed $a \in \mathcal{O}$ the function G_a is locally integrable, with similar regularity properties and decay at ∞ as G_1 . In the analytic continuation above we of course must not cross the imaginary axis (which is the boundary of \mathcal{O}). However, we can hope to define

$$G_i = \lim_{a \rightarrow i, a \in \mathcal{O}} G_a, \quad (1504)$$

assuming the limit exist in the sense that

$$\langle G_i, \varphi \rangle = \lim_{a \rightarrow i, a \in \mathcal{O}} \langle G_a, \varphi \rangle = \lim_{a \rightarrow i, a \in \mathcal{O}} \int_{\mathbb{R}^n \times \mathbb{R}} G_a \varphi \quad (1505)$$

exists for each test function φ . Moreover, if the map $\varphi \rightarrow \langle G_i, \varphi \rangle$ (which we now assume is well defined) turns out to be continuous on $\mathcal{D}(\mathbb{R}^n \times \mathbb{R})$, it will

have to be a fundamental solution of the wave equation. We recall that the wave equation has many fundamental solutions, and at this stage it is not clear which of them will be singled out by the above procedure, if it works. However, we can easily see that G_i cannot be any of the solutions discussed above, as we have to have

$$G_i(x, 0) = c_n \frac{i}{|x|^{n-1}}. \quad (1506)$$

The calculations are simple for $n = 2$, as in this case the functions G_a , $a \in \mathcal{O}$, $|a - i| \leq 1/2$ are uniformly locally integrable. As an exercise, you can show that for $n = 2$ the distribution G_i is well-defined and is given by a locally integrable function

$$G_i(x, t) = \begin{cases} \frac{1}{4\pi\sqrt{t^2 - |x|^2}}, & t^2 > |x|^2 \\ \frac{i}{4\pi\sqrt{-t^2 + |x|^2}}, & t^2 < |x|^2 \end{cases} \quad (1507)$$

Although we know that it is a fundamental solution of the $(2 + 1)$ -dimensional wave equation by the above procedure of analytic continuation in which we have enough regularity to pass easily to the limit in this case, it is still a good exercise³³⁵ to check directly that $\square G_i = \delta(x, t)$. Since the wave equation is linear and has real coefficients, it is easy to see that, in this case with $n = 2$

$$\square \operatorname{Re} G_i = \delta, \quad \square \operatorname{Im} G_i = 0. \quad (1508)$$

The function $\operatorname{Re} G_i$ coincides with the solution G defined by (1473). From (1483) and (1486) we see that for $n = 2$ in we have

$$G^+(x, t) = \begin{cases} \frac{1}{2\pi\sqrt{t^2 - |x|^2}}, & t \geq |x|, \\ 0 & \text{elsewhere.} \end{cases} \quad (1509)$$

We can also define the function G_{-i} by an analogous procedure. It clear that $G_{-i} = \overline{G_i}$ (complex conjugate) and $G^+ = \frac{1}{2}(G_i + G_{-i})$.

It is interesting to check what is happening with the functions G_a in terms of the space-time Fourier transformation. By taking the Fourier transform, the equation $L_a G_a = \delta$ becomes

$$\left(\frac{\tau^2}{a^2} + \xi_1^2 + \cdots + \xi_n^2\right) \hat{G}_a = 1. \quad (1510)$$

Formally this gives

$$\hat{G}_a(\xi, \tau) = \frac{1}{\frac{\tau^2}{a^2} + \xi_1^2 + \cdots + \xi_n^2}. \quad (1511)$$

For $a = i$ or $a = -i$ this expression is ambiguous, due to the strong singularities near the zero of the denominator, but when $a \in \mathcal{O}$, the formula defines a locally

³³⁵which may not be easy if we do not go about the calculation in the right way

integrable function. For example, we can take $a \in \mathcal{O}$ with $\frac{1}{a^2} = -1 + i\varepsilon$ with a small $\varepsilon > 0$, which gives

$$\hat{G}_a = \frac{1}{i\varepsilon\tau^2 - \tau^2 + \xi_1^2 + \dots + \xi_n^2}. \quad (1512)$$

In the limit $\varepsilon \rightarrow 0_+$ this regularization is equivalent to the so-called Feynman regularization

$$\hat{G}_{F,\varepsilon} = \frac{1}{i\varepsilon - \tau^2 + \xi_1^2 + \dots + \xi_n^2}, \quad (1513)$$

which is used in the quantum field theory. (Note that, unlike G_a for $\operatorname{Re} a > 0$, the approximation $G_{F,\varepsilon}$ is Lorentz invariant.) The solution G_i which we calculated above for $n = 2$ is therefore the so-called Feynman propagator.

The problem of regularization of the expression

$$\frac{1}{-\tau^2 + |\xi|^2} \quad (1514)$$

is related to the composition $u \circ \phi$ of distributions we discussed earlier, see (1451). We define

$$\phi(\xi, \tau) = -\tau^2 + \xi_1^2 + \dots + \xi_n^2. \quad (1515)$$

The function ϕ maps $\mathbb{R}^n \times \mathbb{R}$ onto \mathbb{R} , and its gradient $(\phi_\tau, \phi_{\xi_1}, \dots, \phi_{\xi_n})$ does not vanish in $\mathbb{R}^n \times \mathbb{R} \setminus \{(0, 0)\}$. Hence for any distribution of \mathbb{R} the distribution $u \circ \phi$ is well-defined in $\mathbb{R}^n \times \mathbb{R} \setminus \{(0, 0)\}$. (Eventually we will want to extend $u \circ \phi$ to $\mathbb{R}^n \times \mathbb{R}$, but let us ignore this issue for the moment.) In particular, we can take for u various regularizations of the expression

$$u(y) = \frac{1}{y}, \quad (1516)$$

which we discussed in lecture 46. The natural choices are

$$u_1(y) = \text{p.v.} \frac{1}{y}, \quad u_2(y) = \lim_{\varepsilon \rightarrow 0_+} \frac{1}{y + i\varepsilon}, \quad u_3(y) = \lim_{\varepsilon \rightarrow 0_+} \frac{1}{y - i\varepsilon}. \quad (1517)$$

We can now consider the distributions $A_k = u_k \circ \phi$, $k = 1, 2, 3$. They are at first defined only in $\mathbb{R}^n \times \mathbb{R} \setminus \{(0, 0)\}$, but it is easy to show that they can be extended uniquely to $\mathbb{R}^n \times \mathbb{R}$ when $n \geq 2$. Their inverse Fourier transformations give different fundamental solutions of the wave equation. The solution given by A_1 is the solution G in (1473), and the solutions given by A_2, A_3 correspond to the solutions G_i and G_{-i} (or Feynman propagators) obtained either by the Feynman regularization (1513), or by the analytical continuation above. We will not have the time to prove all these statements, but they are perhaps worth mentioning.

We will now calculate the solution obtained by the analytical continuation for $n = 3$. In this case we have

$$G_a(x, t) = \frac{a}{4\pi^2(a^2t^2 + |x|^2)}, \quad (1518)$$

where we have used (755) to evaluate $(n-2)|S^{n-1}|$ for $n = 3$. Writing $a = i + \varepsilon$ for a small $\varepsilon > 0$, we have

$$G_{i+\varepsilon} = \frac{i + \varepsilon}{4\pi^2((-1 + \varepsilon^2)t^2 + |x|^2 + 2i\varepsilon t^2)}. \quad (1519)$$

Letting $\phi(x, t) = -t^2 + |x|^2$, formula (1519) suggests that G_i (defined by the limit (1504)) is well-defined and that $G_i = A_2 \circ \phi$, where the distribution A_2 is defined in (1517). In other words, we should have

$$G_i(x, t) = \frac{i}{4\pi^2} \text{p.v.} \frac{1}{(-t^2 + |x|^2)} + \frac{1}{4\pi} \delta(t^2 - |x|^2). \quad (1520)$$

It is not hard to show that the distribution (1520), originally defined in $\mathbb{R}^3 \times \mathbb{R} \setminus \{0\}$, extends to $\mathbb{R}^3 \times \mathbb{R}$ and that it indeed coincides with the limit (1504).

As in the case $n = 2$, we have

$$\square \operatorname{Re} G_i = \delta, \quad \square \operatorname{Im} G_i = 0, \quad (1521)$$

and the function $\operatorname{Re} G_i$ coincides with the solution G defined by (1473). From (1483) and (1486) we infer that for $n = 3$ in we have

$$G^+(x, t) = \begin{cases} \frac{1}{2\pi} \delta(t^2 - |x|^2), & t \geq 0, \\ 0 & \text{elsewhere.} \end{cases} \quad (1522)$$

Recall that we have discussed the interpretation of $\delta \circ \phi$ in Example 1 (following (1457)). You can verify that, in the case $n = 3$ we are now considering, we can also write

$$G^+(x, t) = \begin{cases} \frac{\delta(t-|x|)}{4\pi|x|}, & t \geq 0, \\ 0 & \text{elsewhere.} \end{cases} \quad (1523)$$

The distribution G_i^+ defined by (1522) or (1524) can be interpreted as a measure supported on the positive light cone C^+ (defined by (1470)). The measure is invariant under the Lorentz transformations which preserve the time direction (the group $O^+(1, 3)$).

Let f be a smooth function in $\mathbb{R}^3 \times \mathbb{R}$ which vanishes for $t < t_0$. The (unique) solution of the Cauchy problem

$$\square u = f(x, t) \quad \text{in } \mathbb{R}^3 \times \mathbb{R}, \quad u(x, t) = 0 \quad \text{for } t < t_0 \quad (1524)$$

can be written, in the case $n = 3$ we are considering, as

$$u(x, t) = (G^+ * f)(x, t) = \int_{\mathbb{R}^3} \frac{f(x+y, t-|y|)}{4\pi|y|} dy. \quad (1525)$$

The integral (1525) is sometimes called the retarded potential of f .

The Cauchy problem

$$\square u = 0 \quad \text{in } \mathbb{R}^3 \times [t_0, \infty), \quad u(x, t_0) = 0, \quad u_t(x_0, t_0) = u_1(x) \quad (1526)$$

can be solved by using (1525) with

$$f(x, t) = u_1(x)\delta(t - t_0). \quad (1527)$$

The equivalence of (1529) and (1524) with f given by (1530) is a form of Duhamel's principle mentioned earlier. In the case of (1529) we obtain

$$u(x, t) = \frac{1}{4\pi(t - t_0)} \int_{|y|=t-t_0} u_1(x+y) dy = (t-t_0) \frac{1}{|S_{x, (t-t_0)}|} \int_{S_{x, (t-t_0)}} u_1(y) dy. \quad (1528)$$

Finally, the solution of the Cauchy problem

$$\square u = 0 \quad \text{in } \mathbb{R}^3 \times [t_0, \infty), \quad u(x, t_0) = u_0(x), \quad u_t(x_0, t_0) = 0 \quad (1529)$$

can be written as

$$u(x, t) = v_t(x, t), \quad (1530)$$

where $v(x, t)$ is given by replacing u_1 by u_0 in (1531).

Let us look in more detail at the space-time Fourier transform of G_i . When $n = 3$ the operator L_1 defined by (1497) is the four-dimensional laplacian. It has the special property that its fundamental solution is mapped by the Fourier transform on a multiple of itself. More precisely, for $n = 3$ we have (see (743) and (750))

$$\hat{G}_1(\xi, \tau) = (2\pi)^2 G_1(\xi, \tau). \quad (1531)$$

For $a > 0$ this implies

$$\hat{G}_a(\xi, \tau) = (2\pi)^2 a G_{1/a}(\xi, \tau), \quad (1532)$$

and by analytic continuation we have this for $a \in \mathcal{O}$. Taking the limit $a \rightarrow i, a \in \mathcal{O}$, obtain, for the case $n = 3$ we are considering

$$\hat{G}_i(\xi, \tau) = (2\pi)^2 i G_{-i}(\xi, \tau). \quad (1533)$$

From (1486),(1522), and (1485) we also obtain, in the case $n = 3$ we are considering

$$K(x, t) = \frac{1}{2\pi} \text{sign } t \delta(t^2 - |x|^2), \quad (1534)$$

and

$$\hat{K} = -(2\pi)^2 i K. \quad (1535)$$

We have seen that the method of analytical continuation can be quite effective. In fact, one can use it to calculate the fundamental solutions in any dimension. Here we will not carry out the calculations for $n \geq 4$, but we can refer the

reader to Theorem 6.2.1 in the book “The Analysis of Linear Partial Differential Operators I” by L. Hörmander, Springer 1983, where the calculations are done (in a slightly different way) for general n .

Even without calculation we can easily reach the following important conclusion. In dimensions $n = 3, 5, \dots$ the analytical continuation G_i will be purely imaginary everywhere, except possibly at the light cone. This suggests that the fundamental solution G should be supported exactly at the light cone. This is what we have seen above for $n = 3$. On the other hand, in dimensions $n = 2, 4, \dots$ the analytical continuation G_i will be real in the “solid cone” \tilde{C} , suggesting that the support of the fundamental solution G will be all the “solid cone” \tilde{C} . This is what we have seen above for $n = 2$. We will confirm all this by another method.

Let us try to find the formula for the solution G in general dimension by exploiting the Lorentz invariance. Let us set

$$q = t^2 - x_1^2 - \dots - x_n^2. \quad (1536)$$

We know that G must be supported in the “solid light cone” \tilde{C} , and that it is invariant under the full Lorentz group. Moreover, the requirements and the condition $\square G = \delta$ determine G uniquely. If G was a function, the invariance under the Lorentz group and the condition on the support of G would easily imply that

$$G(x, t) = f(t^2 - x_1^2 - \dots - x_n^2) = f(q) \quad (1537)$$

for some function

$$f: \mathbb{R} \rightarrow \mathbb{R}, \quad f = 0 \text{ on } (-\infty, 0). \quad (1538)$$

If we only know that G is a distribution (invariant under the Lorentz group and supported in \tilde{C}), can we infer that there exists a distribution f on \mathbb{R} (vanishing in $(-\infty, 0)$) such that $G = f \circ q$? It is a good (and non-trivial) exercise to answer this question, but we can try to side-step it as follows. We can try to find a distribution f such that $f \circ q$ gives a fundamental solution with the required properties. If such f can be found, then by uniqueness of G , we know that the assumption $G = f \circ q$ is justified.³³⁶

Let us then seek G in the form $G = f \circ q$. We have to have $\square G = 0$ in $\mathbb{R}^n \times \mathbb{R} \setminus \{(0, 0)\}$. An easy calculation shows that for a smooth function f this would mean

$$4f''(q)q + 2(n+1)f'(q) = 0, \quad f = 0 \text{ in } (-\infty, 0) \quad (1539)$$

We can only assume that f is a distribution, but we have seen above that $f \circ q$ is still well-defined. Moreover, the usual chain rule formula remains valid.

³³⁶On the other hand, if it turned out that no function of the form $f \circ q$ can satisfy our requirements, we could not conclude that G does not exist without having a rigorous proof that G has to be of the form $f \circ q$.

Therefore our task is to the distributions in \mathbb{R} satisfying (1539) (where q is considered as a coordinate in \mathbb{R}), keeping in mind that they do not have to be represented by functions. In other words, we have to find the distributions f on \mathbb{R} supported in $[0, \infty)$ such that

$$\int_{\mathbb{R}} f(s)(4(s\varphi(s))'' - 2(n+1)\varphi'(s)) ds = 0, \quad \varphi \in \mathcal{D}(\mathbb{R}). \quad (1540)$$

In the class of distributions f vanishing on $(-\infty, 0)$ we can uniquely define the operation of taking the primitive function $\int f$ of f (also called anti-derivative) by

$$(\int f)' = f. \quad (1541)$$

We will write

$$\int f = f^{(-1)}, \quad \int \int f = f^{(-2)}, \dots \text{ and } f = f^{(0)}, f' = f^{(1)}, f'' = f^{(2)}, \dots \quad (1542)$$

Lemma

Consider the class of the distributions on \mathbb{R} vanishing on $(-\infty, 0)$. Let k be an integer. Then f satisfies (1540) if and only if $f^{(k)}$ satisfies (1540) with n replaced by $n + 2k$.

Proof

By simple integration by parts it is clear that f' satisfies (1540) with n replaced by $n + 2$ if and only if

$$\int_{\mathbb{R}} f(s)((4s\varphi'(s))'' - (2(n+1)\varphi'(s))') ds = 0, \quad \varphi \in \mathcal{D}(\mathbb{R}). \quad (1543)$$

The proof will be finished if we show that this is equivalent to (1540). Clearly (1540) implies (1543). Assume now that (1543) is satisfied and let $\varphi \in \mathcal{D}(\mathbb{R})$ be given. We can find a smooth function ψ supported in $(-\infty, x_0]$ for some $x_0 \in \mathbb{R}$ such that $\psi' = \varphi$. The function ψ may not be in $\mathcal{D}(\mathbb{R})$, but since it vanishes in (x_0, ∞) and f is supported in $[0, \infty)$ we can apply (1543) with ψ replaced by $\psi\eta$, where $\eta \in \mathcal{D}(\mathbb{R})$ is such that $\eta = 1$ in $(-1, x_0 + 1)$. This gives (1540) and the proof is finished.

The solutions of (1540) are easy to determine in $(0, \infty)$. In that interval the equation implies that f is smooth and, by standard ODE methods we obtain that

$$f(s) = as^{-\frac{(n-1)}{2}} + b, \quad s > 0 \quad (1544)$$

for some $a, b \in \mathbb{R}$. A more subtle point is to determine the solutions in a open interval containing 0. This is however easy for large negative n . When n is large negative, then $f(s) = a(s_+)^{-\frac{n-1}{2}}$ (where s_+ is the positive part of s , defined as s for $s \geq 0$ and 0 for $s \leq 0$) will solve (1540) in \mathbb{R} . If solutions of a different form existed, one would also have non-trivial solutions supported at 0. These

would have to be finite combinations of the Dirac mass and its derivatives. It is easy to check that for n large negative such solutions do not exist. (They do exist for n positive!). Therefore, for n large negative the space of solutions is 1– dimensional, with

$$f(s) = a(s_+)^{-\frac{n-1}{2}} \quad (1545)$$

where a is a constant. By the lemma above, the space of solutions is one-dimensional for any n .

If we determine f by the above procedure, we know that the distribution $f(q)$ will be well-defined and $(n - 1)$ – homogeneous in $\Omega_{n+1} = \mathbb{R}^n \times \mathbb{R} \setminus \{(0, 0)\}$, with $\square f(q) = 0$ in that set. One can look more closely at the definition of $f(q)$ and check that it is in fact a well-defined distribution on $\mathbb{R}^n \times \mathbb{R}$. Alternatively, one can use the following lemma.³³⁷

Lemma

Let u be a distribution in $\mathbb{R}^n \setminus \{0\}$. Assume u is a –homogeneous for some $a > -n$. Then u can be uniquely extended to an a –homogeneous distribution in \mathbb{R}^n .

Proof

We first construct a suitable “partition of unity” (often called the Littlewood-Paley partition), which is often used in Harmonic Analysis. Let $\psi \in \mathcal{D}(\mathbb{R}^n)$ with $\psi = 1$ in a neighborhood of 0. Consider the function $\phi(x) = \psi(\frac{x}{2}) - \psi(x)$. (Clearly ϕ vanishes near 0.) For any integer k define $\phi_k(x) = \phi(2^k x)$. Then the series $\sum_k \phi_k(x)$ consists only of a fixed finite number of non-zero terms for each $x \in \mathbb{R}^n \setminus \{0\}$, and $\sum_k \phi_k(x) = 1$ for each $x \in \mathbb{R}^n \setminus \{0\}$. For each $\varphi \in \mathcal{D}(\mathbb{R}^n)$ we define the extension \tilde{u} by

$$\langle \tilde{u}, \varphi \rangle = \sum_k \langle u, \phi_k \varphi \rangle . \quad (1546)$$

The key point is that, using the a –homogeneity of u we can write (with slight abuse of notation)

$$\langle u, \phi_k \varphi \rangle = \int u(x) \phi(2^k x) \varphi(x) dx = \int 2^{-k(n+a)} u(y) \phi(y) \varphi(2^{-k} y) dy, \quad (1547)$$

the function $\phi(y) \varphi(2^{-k} y)$ can be estimated in $\mathcal{D}(\mathbb{R}^n)$ uniformly for $k \geq 0$, and we can take the sum over $k \geq 0$, due to the factor $2^{-k(n+a)}$ and the assumption $n + a > 0$. We leave the rest of the proof to the reader as an exercise.

The lemma shows that $f(q)$ can be considered as a well-defined distribution in $\mathbb{R}^n \times \mathbb{R}$, which moreover is $-(n - 1)$ –homogeneous. This means that $\square f(q)$ is

³³⁷See Theorem 3.2.3 in L.Hörmander’s book “The theory of linear partial differential operators I” for a more general statement (with a different proof).

a well-defined distribution which is $-(n+1)$ -homogeneous. Moreover, $\square f(q)$ is supported at $(0,0)$. It is easy to see that this means that $\square f(q)$ is a multiple of the Dirac mass $\delta = \delta_{n+1}$.

We have determined the solution G up to a multiplicative factor. To determine it exactly, we can proceed for example as follows. It not not hard to see that the solution G^+ satisfies (formally)

$$\int_{\mathbb{R}^n} G^+(x, t) dx = t, \quad t > 0. \quad (1548)$$

In view of (1473) this means that

$$\int_{\mathbb{R}^n} G(x, t) dx = \frac{|t|}{2}, \quad t \in \mathbb{R}. \quad (1549)$$

Let us assume that in $\mathbb{R}^n \times \mathbb{R}$ we have $G(x, t) = f_n(q)$. Formally this means that

$$\int_{\mathbb{R}^n} f_n(t^2 - |x|^2) dx = |S^{n-1}| \int_0^\infty f_n(t^2 - r^2) r^{n-1} dr = t/2 \quad t > 0. \quad (1550)$$

By the lemma above we have

$$f_{n+2} = c_n f'_n. \quad (1551)$$

We can write (for $t > 0$)

$$\begin{aligned} \frac{t}{2} &= |S^{n+1}| \int_0^\infty f_{n+2}(t^2 - r^2) r^{n+1} dr = \\ \frac{|S^{n+1}|}{|S^{n-1}|} |S^{n-1}| \int_0^\infty c_n \left(-\frac{d}{2r} f_n(t^2 - r^2)\right) r^{n+1} dr &= \\ \frac{|S^{n+1}|}{|S^{n-1}|} c_n \frac{n}{2} |S^{n-1}| \int_0^\infty f_n(t^2 - r^2) r^{n-1} dr &= \frac{|S^{n+1}|}{|S^{n-1}|} c_n \frac{n}{2} \frac{t}{2} = \pi c_n \frac{t}{2} \end{aligned} \quad (1552)$$

We infer that

$$f_{n+2} = \frac{1}{\pi} f'_n. \quad (1553)$$

We know that

$$f_1(s) = \frac{1}{4} \theta(s), \quad (1554)$$

where θ is the Heaviside function, defined by $\theta(s) = 1$ for $s > 0$ and $\theta(s) = 0$ for $s < 0$.

We also know that

$$f_2(s) = \begin{cases} \frac{1}{4\pi\sqrt{s}}, & s > 0, \\ 0 & s < 0 \end{cases}. \quad (1555)$$

The three equations (1551),(1553) and (1554) determine $G = f_n(q)$ is any dimension. For example, for odd dimensions we have

$$n = 2k + 1 \implies G(x, t) = \frac{1}{4\pi^k} \delta^{(k-1)}(t^2 - |x|^2). \quad (1556)$$

In a similar way

$$n = 2k \implies G(x, t) = \frac{1}{\pi^{k-1}} f_2^{(k-1)}(t^2 - |x|^2), \quad (1557)$$

where f_2 is given by (1555).

Note that in each case the solution is $-(n-1)$ -homogeneous (and note also that $n-1 = n+1-2$). The Heaviside function $\theta(s)$ can be considered as 0-homogeneous, the Dirac mass $\delta(s)$ as (-1) -homogeneous, its derivative $\delta'(s)$ as (-2) -homogeneous, etc. So, in some sense, the situation is quite similar to the laplace operator: in that case the fundamental solution is (a multiple of) the right power of $|x|^2$ which makes it solution $-(n-2)$ -homogeneous. For the wave operator we have a similar heuristics if we replace $|x|^2$ by $t^2 - |x|^2$ (as is natural in view of the Lorentz invariance). In dimensions $n = 1, 2$ the analogy is straightforward. In dimensions $n \geq 3$ we must interpret the “naive” formula $((t^2 - |x|^2)_+)^{-\frac{n-1}{2}}$ correctly.³³⁸

For practical calculations with the fundamental solutions (1556) and (1557), it is useful to keep in mind formulae such as

$$f^{(k)}(t^2 - r^2) = \left(\frac{\partial}{2t\partial t} \right)^k f(t^2 - r^2) = \left(\frac{-\partial}{2r\partial r} \right)^k f(t^2 - r^2). \quad (1558)$$

As an exercise you can consider the following question about sound propagation in general dimension. Assume that the propagation of sound is described by the linear wave equation $\square u = f(x, t)$, where u is for example the deviation of pressure from some rest value, and $f(x, t)$ describes a “sound source”. Assume we have a sharply localized source of sound which can be modeled as $f(x, t) = \delta(x)g(t)$, where g is smooth, compactly supported in (t_1, t_2) and $\delta = \delta_n$ is the Dirac mass in \mathbb{R}^n . Consider the problem

$$\square u = g(t)\delta(x) \quad \text{in } \mathbb{R}^n \times \mathbb{R}, \quad u = 0 \text{ for } t < t_1, \quad (1559)$$

and determine the function $t \rightarrow u(x, t)$ for a given $x \neq 0$. You will see that only in dimension $n = 3$ the signal at x , given by $u(x, t)$ will be “undistorted”, in the sense that $t \rightarrow u(x, t)$ will be, in some sense, a (scaled and delayed) copy of $g(t)$. In all other dimension $t \rightarrow u(x, t)$ will be a non-trivial transformation of $g(t)$. In dimensions $n = 3, 5, \dots$ the signal will not have “echoes” (which is related to the fact that the fundamental solution G^+ is supported on the light cone C^+). On the other had, we will see echoes in even dimensions, when the support of the fundamental solution G^+ is the full “solid light cone” \tilde{C}^+ , see (1471).

³³⁸There is more than one way to do it - in addition to the heuristics mentioned above, one can also consider analytical continuation in n , for example. For more details see Theorem 6.2.1 in the book of L. Hörmander quoted earlier.

In our discussion above we only very briefly mentioned the method of spherical averages, which enables one to obtain quite quickly good representation formulae for solutions. In fact, it may be the quickest and also the most elementary ways to arrive at the representation formulae for solutions. The method can be found for example of the textbook "Partial Differential Equations" by L. C. Evans, and also in the textbook with the same title by F. John.