THE WITT VECTORS FOR GREEN FUNCTORS

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Abstract. We define twisted Hochschild homology for Green functors. This construction is the algebraic analogue of the relative topological Hochschild homology $\mathrm{THH}_{C_n}(-)$, and it describes the E_2 term of the Künneth spectral sequence for relative THH. Applied to ordinary rings, we obtain new algebraic invariants. Extending Hesselholt's construction of the Witt vectors of noncommutative rings, we interpret our construction as providing Witt vectors for Green functors.

1. Introduction

The definition of topological Hochschild homology (THH) is one of the most successful examples of the program of "brave new algebra", in which classical algebraic constructions are carried out "over the sphere spectrum". In the case of THH, one literally replaces rings with ring spectra and tensor products over $\mathbb{Z}$ with smash products over $\mathbb{S}$ to pass from Hochschild homology to topological Hochschild homology.

One byproduct of the foundational work of Hill-Hopkins-Ravenel has been a new interpretation of THH as the S^1-norm $N_e^{S^1}$ from the trivial group to S^1 [1, 8]. This description immediately suggests that one might consider relative versions of THH which take as input H-equivariant ring spectra for $H \subseteq S^1$; the H-relative THH_H is then defined to be the norm $N_H^{S^1}$ [1, 8.2]. We can also produce H-relative theories TF_H, TR_H, and TC_H (generalizing TF, TR, and TC).

At this point, a natural question arises: what is the algebraic analogue of this equivariant relative THH? The goal of this paper is to answer this question: we define and study twisted Hochschild homology for Green functors. To explain the nature of our construction, it is helpful to recall the role of Mackey functors and Green functors in equivariant homotopy theory.

Mackey functors play the role that abelian groups play in non-equivariant homotopy. In particular, an equivariant spectrum has homotopy Mackey functors, rather than just homotopy groups. The category of Mackey functors can be endowed with a symmetric monoidal structure via the box product, and we can talk about associative or commutative monoids in this category. These are associative or commutative Green functors. The category of Mackey functors (and more generally, the category of $\underline{R}$-modules for a Green functor $\underline{R}$) has enough projectives, and if we consider only finite groups, then these are all flat. One could therefore attempt to directly mimic the classical definition of Hochschild homology, forming a derived tensor product of a Green functor $\underline{R}$ with itself over its enveloping algebra.

Unfortunately, this version of Hochschild homology is not the one that we want, for several reasons. From a philosophical standpoint, it is too general — this definition makes sense for any finite group G, whereas our equivariant THH can only be defined for finite subgroups of S^1. Second, there is no obvious relationship between the K-relative Hochschild homology and the H-relative Hochschild homology for $K \subset H$ finite subgroups of S^1, in contrast to the situation with $N_H^{S^1}$ and $N_K^{S^1}$. Finally and perhaps most importantly, the naive Hochschild homology does not receive an obvious edge map from the topological version.

There is an extension of the Künneth spectral sequence from [1] which can be used to compute the homotopy Mackey functors of $i_H^* N_K^{S^1} R$ for finite subgroups $K \subset H \subset S^1$. Specifically, when Adams indexed, our spectral sequence has the form

$$E_2^{s,t} = \underline{\mathrm{Tor}}_{-s}^{\underline{\pi}_* N_K^H R^e}\left(\underline{\pi}_*(N_K^H R), \Sigma^t\left({}^{\gamma}\underline{\pi}_*(N_K^H R)\right)\right) \Rightarrow \underline{\pi}_{t-s} i_H^* N_K^{S^1}(R),$$

where the superscript γ indicates a particular twist of the bimodule structure and $\underline{\mathrm{Tor}}$ denotes the derived functors of the symmetric monoidal product. In this Künneth spectral sequence, we see that we are not considering the $\underline{\mathrm{Tor}}$ Mackey functors of a fixed module with itself: instead, we are $\underline{\mathrm{Tor}}$-ing together a module with a particular twist of itself.

These observations guide our definition of a twisted Hochschild homology of Green functors for finite subgroups of S^1. Our definition is purely algebraic, building a differential graded Mackey functor out of a Green functor $\underline{R}$ (and building a dg Green functor if $\underline{R}$ is commutative).

Let $K \subset G \subset S^1$ be finite subgroups, and let $\underline{R}$ be an associative Green functor for K. We will define the *twisted Hochschild complex* as a simplicial Mackey functor for G with k-simplices given by

$$\underline{HC}_k^G(\underline{R}) := \left(N_K^G \underline{R}\right)^{\square(k+1)}.$$

The structure maps are the usual Hochschild maps except that the k^{th} face map d_k is given by moving the last factor to the front, acting via a generator g of G on the new first factor, and then multiplying the first two factors.

The *G-twisted Hochschild homology* of $\underline{R}$ relative to K, denoted $\underline{\mathrm{HH}}_i^G(\underline{R})$, is the homology of this complex. These are compatible as G varies, and are first defined and studied in Section 4.

The twisted Hochschild homology of $\underline{R}$ relative to K serves as an algebraic approximation to the K-relative THH of the Eilenberg-MacLane spectrum $H\underline{R}$. Recall that in the classical setting, for a ring A, the topological Hochschild homology of the Eilenberg-MacLane spectrum HA and the Hochschild homology of A are related by a linearization map

$$\pi_k \, \mathrm{THH}(HA) \to \mathrm{HH}_k(A)$$

that factors the Dennis trace from algebraic K-theory to Hochschild homology

$$K_k(A) \to \pi_k \, \mathrm{THH}(HA) \to \mathrm{HH}_k(A).$$

In Section 4 we prove the following.

Theorem 1.1. *For $H \subset G \subset S^1$ finite subgroups, and R a (-1)-connected H-equivariant commutative ring spectrum, we have a natural homomorphism*

$$\underline{\pi}_k^G \, \mathrm{THH}_H(R) \to \underline{\mathrm{HH}}_k^G(\underline{\pi}_0^H R),$$

refining the classical maps.

Since the construction of twisted Hochschild homology is purely algebraic, it also gives us new invariants of associative and commutative rings. For A a commutative ring, the linearization map of 1.1 yields lifts of the Dennis trace to G-twisted Hochschild homology for each finite $G \subset S^1$

$$K_k(A) \to \underline{\mathrm{HH}}_k^G(A)(G/G).$$

The twisted Hochschild complex of a Green functor has a kind of cyclotomic structure. We give a direct definition of the geometric fixed points Φ^N for a simplicial Green functor (see Definition 3.7), and using this we prove the following structural result about $\underline{HC}^G$.

Proposition 1.2. *Let $H \subset G \subset S^1$ be finite subgroups, and let N be a subgroup of G. For $\underline{R}$ a commutative Green functor for H, there is a natural isomorphism of simplicial Green functors*

$$\Phi^N\big(\underline{HC}_*^G(\underline{R})\big) \cong \underline{HC}_*^{G/N}(\Phi^{H\cap N}\underline{R}).$$

Using this cyclotomic structure, in Section 4.5 we define a purely algebraic analogue of $\mathrm{TR}(A;p)$, and compute this algebraic analogue in the case of $A = \mathbb{F}_p$.

A particularly interesting facet of our definition of H-twisted Hochschild homology is that it can be interpreted as defining the *Witt vectors of a Green functor* (See Section 4.6). Work of Hesselholt and of Hesselholt-Madsen shows that

$$\pi_0(\mathrm{THH}(R)^{C_{p^n}}) \cong W_{n+1}(R),$$

where $W_{n+1}(R)$ denotes the length $n+1$ p-typical Witt vectors of R [15]. For H-relative topological Hochschild homology, we prove that $\underline{\pi}_0$ is captured by twisted Hochschild homology.

Theorem 1.3. *Let $H \subset G \subset S^1$ be finite subgroups and let R be a (-1)-connected commutative ring orthogonal H-spectrum. We have a natural isomorphism*

$$\underline{\pi}_0^G\big(\mathrm{THH}_H(R)\big) \cong \underline{\mathrm{HH}}_0^G(\underline{\pi}_0^H R).$$

This suggests the following definition:

Definition 1.4. Let $H \subset G \subset S^1$ be finite subgroups and let $\underline{R}$ be a Green functor for H. The G-Witt vectors for $\underline{R}$ are defined by

$$\underline{\mathbb{W}}_G(\underline{R}) := \underline{\mathrm{HH}}_0^G(\underline{R}).$$

With this definition we have the following analog of the Hesselholt-Madsen result

Corollary 1.5. *For $H \subset G \subset S^1$ finite subgroups, and R a (-1)-connected commutative ring orthogonal H-spectrum,*

$$\underline{\pi}_0^G\,\mathrm{THH}_H(R) \cong \underline{\mathbb{W}}_G(\underline{\pi}_0^H R).$$

One of the key insights in the theory of equivariant homotopical algebra is that Green functors are not the correct algebraic analogues of E_∞ rings. Specifically, the homotopy Mackey functors of an equivariant commutative ring spectrum have more structure than simply a Green functor: they are Tambara functors [7]. These are Green functors with multiplicative transfer maps called norms.

When working with Tambara functors, our equivariant Witt vectors inherit additional structure. When $\underline{R}$ is a Tambara functor, then the norm maps provide Teichmüller style lifts in the C_{mn}-Witt vectors. We study this in Section 5, proving the following theorem there.

Theorem 1.6. *Let $\underline{R}$ be a G-Tambara functor, where $G = C_n$ is a finite cyclic group. For all $H \subset G$, we have a Teichmüller-style, multiplicative map*

$$\underline{R}(G/H) \to \big(\underline{\mathbb{W}}_G(\underline{R})\big)(G/G).$$

When we apply this to an ordinary, non-equivariant commutative ring, then this provides an algebraic explanation of the Teichmüller lifts in terms of constructions of equivariant stable homotopy theory.

In Section 6, we apply our twisted Hochschild homology to compute the homotopy Mackey functors of the relative THH for a pointed monoid ring. Hesselholt and Madsen observed that if M is a pointed monoid, then we have a natural S^1-equivariant equivalence

$$\mathrm{THH}(R[M]) \simeq \mathrm{THH}(R) \wedge N^{\mathrm{cyc}}(M).$$

In our C_n-relative context, there is a generalization of this statement that yields an analogous decomposition for our twisted Hochschild homology. Here we also allow C_n to act on the pointed monoid.

Theorem 1.7. *Let $G = C_n$. If M is a pointed monoid in G-sets and if $\underline{R}$ is a G-Green functor, then we have a natural weak equivalence*

$$\underline{HC}_*^G(\underline{R}[M]) \simeq \underline{HC}_*^G(\underline{R}) \Box \underline{C}_*^{cell}(N_G^{cyc} M; \underline{A}),$$

where $\underline{A}$ is the Burnside Mackey functor and where $\underline{C}_*^{cell}$ is the Mackey extension of the Bredon cellular chains.

We believe there is a natural relationship between the construction of twisted Hochschild homology of Green functors we study in this paper and the definitions of Hochschild-Witt vectors introduced by Kaledin [23]. Kaledin has introduced a functorial Hochschild-Witt complex associated to an associative unital k-algebra A. When A is commutative, finite type (as an algebra) over k, and smooth, a version of the HKR theorem implies that this construction recovers the de Rham-Witt complex of A [23, §6]. When A is noncommutative, the zeroth homology coincides with Hesselholt's Witt vectors [14]. Most interestingly, the higher homology groups of this Hochschild-Witt complex provides a (conjectural) algebraic model of TR. We intend to return to give a more complete treatment in future work.

Our treatment here focuses on the commutative case, and we largely restrict attention to commutative Green functors for our constructions. One could instead ask for the purely associative case, allowing us to build most easily a version of Shukla homology for associative Green functors. We sketch how this works in Section 7, showing that the homotopy invariant versions of our constructions are, in fact, homotopy invariant. The main problem here is that the category of Mackey functors has infinite projective dimension [12], and in practice, even the most basic non-projective Mackey functors have infinite projective dimension. This makes the construction of resolutions much trickier than in the classical case, and we leave this section as mainly a sketch.

2. The derived category of Mackey functors

In this section, we set up the necessary foundations for homotopical algebra in the category of Mackey functors. Throughout we fix a finite group G.

Definition 2.1. Let $\mathcal{A}_{\mathbb{N}}$ denote the Burnside category, the category with objects finite G-sets and with morphisms isomorphism classes of spans:

$$\mathcal{A}_{\mathbb{N}}(S,T) = \Big\{[S \leftarrow U \rightarrow T]\Big\},$$

where two spans $S \leftarrow U \rightarrow T$ and $S \leftarrow U' \rightarrow T$ are isomorphic if there is an isomorphism $U \rightarrow U'$ making the obvious triangles with S and T commute. Composition is via pullback.

Since a span is also just a map $U \rightarrow S \times T$, the Burnside category is self-dual. In particular, the disjoint union of finite G-sets is both the product and coproduct in the Burnside category.

Definition 2.2. A semi-Mackey functor is a product preserving functor

$$\underline{M} \colon \mathcal{A}_{\mathbb{N}} \rightarrow \mathcal{S}et.$$

The category of semi-Mackey functors is the obvious category of functors and natural trasnformations.

Since the Burnside category is pre-additive in the sense that finite coproducts and products exist and agree, the hom objects are all commutative monoids. This is a consequence of the pre-additivity, and it implies that for every finite set T and for every semi-Mackey functor $\underline{M}$, the set $\underline{M}(T)$ inherits the structure of a commutative monoid.

Definition 2.3. A Mackey functor is a semi-Mackey functor $\underline{M}$ such that for each finite G-set T, the commutative monoid $\underline{M}(T)$ is an abelian group. Let Mack_G denote the full subcategory of semi-Mackey functors spanned by the Mackey functors.

Again, we stress that being a group is a property of a commutative monoid, and the commutative monoid structure arises by functoriality. We will find it helpful in what follows to regard Mackey functors as functors from $\mathcal{A}_{\mathbb{N}}$ to the category of sets; in particular, the categorical constructions we use are defined in terms of the underlying functors to sets.

For homological algebra reasons, it is helpful also to build several other equivalent forms of Mackey functors.

Definition 2.4. Let $\mathcal{A}$ denote the category enriched in abelian groups for which the objects are finite G-sets and for which the hom objects $\mathcal{A}(S,T)$ are the group completions of $\mathcal{A}_{\mathbb{N}}(S,T)$.

In the category $\mathcal{A}$, we again have that disjoint union is both the product and the coproduct, but now the category is additive. The inverses on the hom objects allow us to record the inverses in a Mackey functor, as opposed to a semi-Mackey functor.

Proposition 2.5. *A Mackey functor is an additive functor*

$$\underline{M} \colon \mathcal{A} \rightarrow \mathcal{A}b.$$

This shows that the category of Mackey functors is an abelian category. Since $\mathcal{A}$ is self-dual, we can equivalently talk about "co-Mackey functors".

Finally, when discussing the model structure, we will also find it useful to describe Mackey functors as modules over the "Mackey algebra" $\mu_G(\mathbb{Z})$ [32, §3].

We begin by describing the symmetric monoidal and G-symmetric monoidal structures on Mack_G.

2.1. **The box product of Mackey functors.** We now review the closed symmetric monoidal structure on the category of Mackey functors (see also [24, §1] for a review of this structure). The box product of Mackey functors $\underline{M}\square\underline{N}$ is a convolution product, defined by left Kan extension of the tensor product of commutative monoids over the Cartesian product of finite G-sets:

$$\begin{tikzcd} \mathcal{A}\times\mathcal{A} \arrow[r, "\underline{M}\times\underline{N}"] \arrow[d, "-\times-"'] & \mathcal{A}b\times\mathcal{A}b \arrow[r, "-\otimes-"] & \mathcal{A}b \\ \mathcal{A} \arrow[urr, "\underline{M}\square\underline{N}"'] & & \end{tikzcd}$$

Theorem 2.6. *The box product of two Mackey functors is again a Mackey functor, and the category $Mack_G$ is a symmetric monoidal category with product $\square$ and unit $\underline{A}=\mathcal{A}(*,-)$.*

This Kan extension takes place in additive categories, and this makes comparisons with later Kan extensions more difficult. We can, however, work instead entirely in $\mathcal{S}et$-valued functors, noting that a product preserving functor from the Burnside category to $\mathcal{S}et$ is necessarily commutative monoid valued and that being an abelian group is a property of a commutative monoid. Work of Strickland makes this precise.

Proposition 2.7 ([31, Appendix A.3])**.** *The box product of two Mackey functors $\underline{M}$ and $\underline{N}$ is given by the left Kan extension of the Cartesian product of sets over the Cartesian product of finite G-sets:*

$$\begin{tikzcd} \mathcal{A}_{\mathbb{N}}\times\mathcal{A}_{\mathbb{N}} \arrow[r, "\underline{M}\times\underline{N}"] \arrow[d, "-\times-"'] & \mathcal{S}et\times\mathcal{S}et \arrow[r, "-\times-"] & \mathcal{S}et \\ \mathcal{A}_{\mathbb{N}} \arrow[urr, "\underline{M}\square\underline{N}"'] & & \end{tikzcd}$$

In particular, all of the abelian group structure which we want on our putative symmetric monoidal product comes along for free on the ordinary left Kan extension.

To get a more explicit description, it helps to unpack this a little in terms of representable Mackey functors.

Definition 2.8. If T is a finite G-set, then let

$$\underline{A}_T=\mathcal{A}(T,-)$$

be the functor represented by T.

These are projective objects in the category of Mackey functors, and by the Yoneda Lemma, we have a canonical isomorphism

$$\mathrm{Hom}(\underline{A}_T,\underline{M})\cong\underline{M}(T)$$

for any Mackey functor $\underline{M}$. Moreover, by the contravariant Yoneda Lemma, the assignment

$$T\mapsto\underline{A}_T$$

defines a co-Mackey functor object in the category of Mackey functors. This structure will be applied below to determine the internal Hom in the category of Mackey functors.

Since the box product is defined by a Kan extension, the value on representable functors is canonically determined by the following formula:

Proposition 2.9. *If T_1 and T_2 are finite G-sets, then we have a canonical isomorphism*

$$\underline{A}_{T_1} \Box \underline{A}_{T_2} \cong \underline{A}_{T_1 \times T_2}.$$

Since the box product commutes with colimits in each variable by construction, Proposition 2.9 allows us to reduce box product computations to colimits of representables.

Definition 2.10. If $\underline{M}$ is a Mackey functor and T is a finite G-set, then let

$$\underline{M}_T := \underline{A}_T \Box \underline{M}.$$

Generalizing the observation above, the $\underline{M}_T$ fit together as a co-Mackey functor:

Proposition 2.11. *For any Mackey functor $\underline{M}$, the assignment*

$$T \mapsto \underline{M}_T$$

is a co-Mackey functor object in Mackey functors.

Proposition 2.11 allows us to describe an internal hom object for Mack_G.

Definition 2.12. If $\underline{M}$ and $\underline{N}$ are Mackey functors, then let

$$\underline{\mathrm{Hom}}(\underline{M}, \underline{N})(T) := \mathrm{Hom}(\underline{M}_T, \underline{N}).$$

It is straightforward to verify that $\underline{\mathrm{Hom}}$ satisfies the expected adjunctions with the box product.

Theorem 2.13. *The category Mack_G is a closed symmetric monoidal category with $\underline{\mathrm{Hom}}$ as the internal mapping object.*

The category $\mathcal{S}p_G$ of orthogonal G-spectra for a complete universe U is also a symmetric monoidal category under the smash product, and this is compatible with the symmetric monoidal structure on Mackey functors; the following compatibility result follows from [24, 1.3].

Proposition 2.14. *If E and F are cofibrant (-1)-connected genuine G-spectra, then we have a natural isomorphism*

$$\underline{\pi}_0 E \Box \underline{\pi}_0 F \cong \underline{\pi}_0(E \wedge F).$$

Remark 2.15. This isomorphism can be used to define the symmetric monoidal structure on Mackey functors, using that the category of Mackey functors is the heart of the category of genuine G-spectra:

$$\underline{M} \Box \underline{N} \cong \underline{\pi}_0(H\underline{M} \wedge H\underline{N}).$$

Here H denotes an Eilenberg-Mac Lane functor

$$H \colon \mathrm{Mack}_G \to \mathcal{S}p_G$$

that takes a Mackey functor to (a cofibrant version of) the associated Eilenberg-Mac Lane G-spectrum.

2.2. **G-symmetric monoidal structure on Mack_G.** Both the categories $\mathcal{S}p_G$ and Mack_G have an equivariant enrichment of the symmetric monoidal product, a G-symmetric monoidal category structure. Roughly speaking, a G-symmetric monoidal structure consists of coherent multiplicative norms for all subgroups $H \subset G$. The G-symmetric monoidal structure on $\mathcal{S}p_G$-spectra was implicitly introduced by Hill-Hopkins-Ravenel [17] and then later explicitly codified and studied in Hill-Hopkins [16]. See also [4] for a discussion for incomplete universes and [2] for an ∞-categorical treatment.

In analogy with Remark 2.15, the G-symmetric monoidal structure on $\mathcal{S}p_G$ induces a G-symmetric monoidal category structure on Mack_G. Specifically, we can define the norms in Mack_G in terms of the norms in $\mathcal{S}p_G$.

Definition 2.16 ([16, Definition 5.9])**.** Let $K \subset G$ be a subgroup. If $\underline{M}$ is a K-Mackey functor, then let

$$N_K^G \underline{M} := \underline{\pi}_0^G(N_K^G H\underline{M}).$$

Here the superscript on $\underline{\pi}_0$ is simply to help the reader keep track of the domain and range of the functor.

With this definition, the functor $\underline{\pi}_0$ becomes a G-symmetric monoidal functor from the category of (-1)-connected G-spectra to Mackey functors. We can extend the norms to arbitrary finite G-sets using the decomposition of a finite G-set into a disjoint union of orbits:

Definition 2.17. If $H_1, \ldots, H_n$ are subgroups of G and $\underline{M}$ is a G-Mackey functor, then let

$$N^{G/H_1 \amalg \cdots \amalg G/H_n} \underline{M} \cong \big(N_{H_1}^G i_{H_1}^* \underline{M}\big) \Box \ldots \Box \big(N_{H_n}^G i_{H_n}^* \underline{M}\big).$$

Proposition 2.18. *For any finite K-set T, we have an isomorphism*

$$N_K^G(\underline{A}_T^K) \cong \underline{A}_{F_K(G,T)}^G.$$

Proof. We have a natural equivalence of K-spectra

$$H\underline{A}_T^K \simeq \Sigma^\infty T_+ \wedge H\underline{A}^K.$$

Since the norm is strong symmetric monoidal, we have equivalences

$$N_K^G(H\underline{A}_T^K) \simeq N_K^G(\Sigma^\infty T_+ \wedge H\underline{A}^K) \cong N_K^G(\Sigma^\infty T_+) \wedge N_K^G H\underline{A}^K.$$

By construction, we have an equivariant isomorphism

$$N_K^G(\Sigma^\infty T_+) \cong \Sigma^\infty F_K(G,T)_+.$$

The result follows once we identify $N_K^G \underline{A}^K$. For this, we recall that the Burnside Mackey functor $\underline{A}$ is the zeroth homotopy group of the zero sphere, so the fiber of the Postnikov section $S^0 \to H\underline{A}$ is 0-connected. In particular, we know that

$$\underline{\pi}_0 N_K^G S^0 \cong \underline{\pi}_0 N_K^G H\underline{A}^K.$$

By construction $N_K^G S^0 \simeq S^0$, from which the desired result follows. $\square$

Since the norm in spectra commutes with sifted colimits, Proposition 2.18 can be used (very inefficiently) to compute the norm of any Mackey functor. However, a direct expression for the norm of Mackey functors is often more useful. The thesis work of Mazur and Hoyer produced purely algebraic constructions of the norm on Mackey functors [25, 20]. Mazur's construction is analogous to Lewis' description of the box product of Mackey functors and gives an explicit construction of the norm for cyclic p-groups. Hoyer generalized this, giving a functorial description of the norm for any finite group G.

Definition 2.19 ([20, 2.3.2])**.** If $\underline{M}$ is an H semi-Mackey functor, then let $N_H^G\underline{M}$ be the left Kan extension of $\underline{M}$ along the coinduction functor $F_H(G,-)$ on finite H-sets:

$$\begin{tikzcd} \mathcal{A}^H_{\mathbb{N}} \arrow[r, "\underline{M}"] \arrow[d, "F_H(G{,}-)"'] & \mathcal{S}et. \\ \mathcal{A}^G_{\mathbb{N}} \arrow[ur, "N_H^G\underline{M}"'] & \end{tikzcd}$$

Proposition 2.20 ([20, 2.3.6])**.** *If $\underline{M}$ is an H-Mackey functor, then $N_H^G\underline{M}$ is a G-Mackey functor.*

Since this is a left Kan extension, we again have a canonical identification on representable functors. Composing with group completion then gives the following.

Proposition 2.21. *If T is a finite H-set, then we have a canonical isomorphism of Mackey functors*

$$N_H^G\underline{A}_T^H \cong \underline{A}^G_{F_H(G,T)}.$$

This agrees with the construction in terms of the norm on spectra, as both are expressing a distributive law. Hoyer showed that this is true very generally, building on work of Ullman.

Theorem 2.22 ([20, 2.3.7])**.** *There is a canonical isomorphism between N_H^G as defined in Definition 2.16 and Definition 2.19 on Mackey functors and on Green functors.*

Remark 2.23. Although coinduction is the right adjoint to the forgetful functor on finite G-sets, it is not so on the Burnside category. Moreover, it is not a product preserving functor on the Burnside category. In particular, the Hoyer norm is not a left adjoint on Mackey functors.

From this perspective, it is straightforward to verify the properties of the G-symmetric monoidal structure on Mack_G.

Proposition 2.24. *If $K \subset H \subset G$, then we have a natural isomorphism of functors*

$$N_H^G \circ N_K^H \cong N_K^G.$$

Proof. Since we have a natural isomorphism of functors

$$F_H\big(G, F_K(H,-)\big) \cong F_K(G,-),$$

we have a natural isomorphism of the corresponding left Kan extensions. □

We can similarly understand the composition of the restriction functor with the norm. This is closely connected with the external description of a Tambara functor as a G-commutative monoid in the category of Mackey functors [25, 20], as it describes the externalized Weyl action. First we need a standard, categorical definition.

Definition 2.25. If K is a finite group, then let BK be the category with one object $*$ and with the morphisms from $*$ to itself given by K, with composition the multiplication in K.

If $\mathcal{C}$ is a category, then a K-object in $\mathcal{C}$ is a functor $BK \to \mathcal{C}$.

In particular, a K-object in G-Mackey functors is a G-Mackey functor $\underline{M}$ such that for each subgroup H, $\underline{M}(G/H)$ is a K-module, and all restriction, transfer, and conjugation maps are maps of K-modules. A K-object in G-Green functors moreover requires the multiplication be a K-equivariant map.

Proposition 2.26. *If $\underline{M}$ is a G-Mackey functor, then for all $H \subset G$, $N_H^G i_H^* \underline{M}$ is a $W_G(H)$-object in G-Mackey functors.*

If $\underline{R}$ is a G-Green functor, then $N_H^G i_H^* \underline{R}$ is a $W_G(H)$-object in G-Green functors.

Proof. By definition, the functors N_H^G and $-\Box-$ are left Kan extensions. The functor i_H^* is so too: i_H^* is the left Kan extension of the restriction functor i_H^* on the Burnside category. This means we have a natural isomorphism of functors

$$N_H^G i_H^*(-) = \operatorname{Lan}_{F_H(G,-)} \operatorname{Lan}_{i_H^*}(-) \cong \operatorname{Lan}_{F_H(G,i_H^*(-))}(-).$$

We have a natural isomorphism of functors

$$F_H(G, i_H^*(-)) \cong F(G/H, -),$$

and this isomorphism is compatible with the isomorphisms showing that $F_H(G,-)$ and $F(G/H,-)$ are strong symmetric monoidal functors. In particular, the isomorphism of left Kan extensions on Mackey functors gives an isomorphism of left Kan extensions on Green functors.

The automorphism group $Aut_G(G/H) \cong W_G(H)^{op}$ acts on the right on the functor $\operatorname{Lan}_{F(G/H,-)}(-)$ via precomposition, and this gives us a natural action of $W_G(H)$ on the composite functor, as desired. $\square$

Combining Proposition 2.24 and Proposition 2.26 yields the following corollary.

Corollary 2.27. *If $\underline{M}$ is a G-Mackey functor and we have a chain of groups $H \subset G \subset G'$, then the G'-Mackey functor*

$$N_H^{G'} i_H^* \underline{M}$$

is naturally a $W_G(H)$-object. If $\underline{M}$ is a Green functor, then this is again a $W_G(H)$-object in Green functors.

This is conceptually connected to our later construction of Hochschild homology in this context. The following proposition is immediate.

Proposition 2.28. *For any ring R and for any group G, there is an associative G-Green functor $\underline{R}$ for which $i_e^* \underline{R} \cong R$.*

Proof. Endow R with the trivial G-action and let $\underline{R}$ be the fixed-point Mackey functor. Since R is an associative ring, this is an associative Green functor. $\square$

Corollary 2.29. *For any associative ring R, the associative Green functor $N_e^G R$ is a G-object in Green functors.*

Proof. Proposition 2.28 shows that there is an associative Green functor $\underline{R}$ such that $i_e^* \underline{R} \cong R$ as associative rings (which are the same as associative Green functors for the trivial groups). In particular, we know that

$$N_e^G R \cong N_e^G i_e^* \underline{R},$$

so by Proposition 2.26, this is a $W_G(e) = G$-object in G-Green functors. $\square$

Remark 2.30. If R is an associative ring orthogonal spectrum, then $N_e^G R$ is naturally an associative ring orthogonal G-spectrum. The proof is essentially the same: the role of the fixed point Mackey functor is played here by the push-forward functor $\mathcal{S}p \to \mathcal{S}p^G$.

The other composite of the restriction and norm is also easy to understand. For simplicity, we restrict attention here to abelian groups, where all subgroups are normal. In this case, if $J, H \subset G$ are subgroups, then the double cosets are the same as the cosets of JH.

Theorem 2.31. *Let G be a finite, abelian group and let H and J be subgroups. Then we have a natural isomorphism*

$$i_J^* N_H^G(\text{-}) \cong \left(N_{H\cap J}^J i_{H\cap J}^*(\text{-})\right)^{\Box[G:JH]}.$$

The action of G on this is via tensor induction.

Proof. The restriction in Mackey functors can also be modeled by a left Kan extension: it is the left Kan extension along the ordinary restriction functor from $\mathcal{A}^G$ to $\mathcal{A}^J$. It therefore suffices to show that we have a natural isomorphism of functors

$$i_J^* F_H(G,\text{-}) \cong \prod_{g\in J\backslash G/H} F_{H\cap J}\big(J, i_{H\cap J}^*(\text{-})\big).$$

Here we observe that we are forming the indexed product in the sense of [17, §A.3], so it suffices to show this at the level of diagrams. The result follows from the observation that

$$i_J^* G/H \cong \coprod_{g\in J\backslash G/H} J/(J\cap H).$$

$\Box$

Corollary 2.32. *Let G be an abelian group.*

(1) *For $H \subset J \subset G$, we have a natural isomorphism of functors*

$$i_J^* N_H^G \cong \left(N_H^J\right)^{\Box[G:J]}.$$

(2) *For $J \subset H \subset G$, we have a natural isomorphism of functors*

$$i_J^* N_H^G \cong \left(i_J^*\right)^{\Box[G:H]}.$$

We include a small computation that is of independent interest. We point the reader to Section 4.5 for further discussion of the computation here.

Proposition 2.33. *Let $G = C_{p^n}$ and let M be a module over an $\mathbb{F}_p$-algebra R. Then the Mackey functor $N_e^{C_{p^n}} M$ is a module over the constant Mackey functor $\underline{\mathbb{Z}}$: for any $1 \leq k \leq n$, we have*

$$tr_{C_{p^{k-1}}}^{C_{p^k}} \circ res_{C_{p^{k-1}}}^{C_{p^k}} = p.$$

Proof. Since the norm functor is strong symmetric monoidal, it suffices to show this for $M = R = \mathbb{F}_p$. Here Mazur's original construction and her example, [25, Example 3.12], show that the norm to C_{p^n} of $\mathbb{F}_p$ has

$$(N_e^{C_{p^n}} \mathbb{F}_p)(C_{p^n}/C_{p^k}) = \mathbb{Z}/p^{k+1},$$

the restriction maps are all surjective, and the transfer maps are the obvious inclusions. $\Box$

2.3. **Differential graded and simplicial Mackey functors.** When discussing the homotopical nature of our various functors, we use model structures on the categories of dg and simplicial Mackey functors. Regarding the category of Mackey functors as modules over a ring, establishing the existence of such model structures is standard; we take as a convenient reference [11], although of course these results go back to Quillen [28, §2].

Theorem 2.34. *The category $Ch_*^+(Mack_G)$ of non-negatively graded dg Mackey functors has a model structure in which a map $f\colon \underline{B}_\bullet \to \underline{C}_\bullet$ is*

1. *a weak equivalence if it induces a quasi-isomorphism (i.e., an isomorphism of homology Mackey functors),*
2. *a fibration if it is a surjection in positive degrees, and*
3. *a cofibration if it is an injection with projective levelwise cokernel.*

We can still obtain a model structure on the category $\mathrm{Ch}_*(\mathrm{Mack}_G)$ of unbounded complexes of Mackey functors [19, 2.3.11], but in that case the description of the cofibrations becomes more complicated. For our purposes, however, it suffices that cofibrant objects in $\mathrm{Ch}_*(\mathrm{Mack}_G)$ are in particular levelwise projective [19, 2.3.9].

Remark 2.35. Generalizing the comparison between $H\mathbb{Z}$-module spectra and dg modules over $\mathbb{Z}$ (e.g., see [29, 30]), there is a Quillen equivalence between the category $\mathrm{Ch}_*(\mathrm{Mack}_G)$ and the category of modules in $\mathcal{S}p^G$ over $H\underline{A}$. However, a careful proof of this result has not yet appeared in the literature. Note also that there is some subtlety to the multiplicative story, as recorded in [33]: if $\underline{R}$ is a commutative Green functor that has no Tambara functor structure, then $H\underline{R}$ cannot be a commutative ring spectrum.

We now turn to the category sMack_G of simplicial Mackey functors. The Dold-Kan correspondence shows that the category sMack_G is equivalent to the category $\mathrm{Ch}_*^+(\mathrm{Mack}_G)$ via the normalized chain complex construction; this provides a lifted model structure on sMack_G [11, 4.4.2].

Theorem 2.36. *The category $sMack_G$ of simplicial Mackey functors has a simplicial model structure in which a map $f\colon \underline{M}_\bullet \to \underline{N}_\bullet$ is*

1. *a weak equivalence if it induces a quasi-isomorphism of associated normalized chain complexes,*
2. *a fibration if it is a fibration of simplicial sets, and*
3. *a cofibration if the cokernel of the underlying degeneracy diagrams (i.e., the objects obtained by forgetting the face maps) is of the form*

$$\bigoplus_k \bigoplus_{\phi\colon [n]\to[k]} \phi^* \underline{P}_k$$

in degree n, where the second sum runs over the surjections and each $\underline{P}_k$ is projective.

We also have several easy but very useful observations about this model structure; the first is immediate, and the second follows from the usual Dold-Kan analysis.

Proposition 2.37.

1. *Every simplicial Mackey functor is fibrant.*

(2) *A simplicial Mackey functor is cofibrant if and only if it is levelwise projective.*

As suggested by the description of the weak equivalences in Theorem 2.36, we make the following definition.

Definition 2.38. Let $\underline{M}_\bullet$ be a simplicial Mackey functor. The homology $H_*(\underline{M}_\bullet)$ is defined to be the homology of the associated normalized dg Mackey functor.

Remark 2.39. Note that in contrast to the notion of dg Mackey functor that is relevant in the context of Kaledin's work [21, 22], here we are simply using dg objects in the category of Mackey functors. Kaledin instead studies dg functors out of a dg model of the Burnside category, i.e., the algebraic analogue of the Guillou–May and Barwick "spectral presheaves" approach to the equivariant stable category [13] [3].

We now turn to investigate the interaction of the box product and the norms with these model structures.

2.3.1. *The derived box product.*

Definition 2.40. If $\underline{M}_\bullet$ and $\underline{N}_\bullet$ are simplicial Mackey functors, then let $\underline{M}_\bullet \Box \underline{N}_\bullet$ be the simplicial Mackey functor with

$$(\underline{M}_\bullet \Box \underline{N}_\bullet)_k = \underline{M}_k \Box \underline{N}_k,$$

and where the structure maps are just the box products of the corresponding structure maps.

Since cofibrant objects in the model structure of Theorem 2.36 are levelwise projective, the following proposition is immediate.

Proposition 2.41. *Let $\underline{P}$ be a cofibrant simplicial Mackey functor. Then the functor $\underline{P}\Box(-)$ preserves weak equivalences and cofibrant objects.*

Similarly, the box product induces a symmetric monoidal product on $\mathrm{Ch}_*^+(\mathrm{Mack}_G)$ and $\mathrm{Ch}_*(\mathrm{Mack}_G)$ in the usual fashion; again we use the description of the cofibrant objects in the model structures to prove the following result.

Proposition 2.42. *Let $\underline{P}_.$ be a cofibrant dg Mackey functor. Then the functor $\underline{P}_.\Box(-)$ preserves weak equivalences.*

2.3.2. *The derived norm.* We can similarly define the norm levelwise and check that it is derivable. We will use this in § 7 to build a homotopy invariant form of our complexes. Of course, if we step through spectra, then we know that the norm is derivable, since the norm in spectra is a left Quillen functor. We give a more self-contained treatment here.

Definition 2.43. If $\underline{M}_\bullet$ is a simplicial H-Mackey functor, then let $N_H^G \underline{M}_\bullet$ be the simplicial Mackey functor for G defined by applying the norm functor levelwise.

We being with a definition motivated by Bredon homology. Using the transfer, the maps

$$\underline{A}_{(\text{-})} \colon T \mapsto \underline{A}_T$$

extend to a functor from the category of finite G-sets to Mackey functors. This gives a functor from simplicial G-sets with finitely many k-simplices for all k to simplicial Mackey functors.

Definition 2.44. Let $T_\bullet$ be a simplicial G-set such that for each k, the set T_k is finite. Composing with $\underline{A}_{(\text{-})}$ then gives a simplicial Mackey functor, which we will denote $\underline{A} \cdot T_\bullet$. This gives a functor from the full subcategory of finite simplicial G-sets to sMack_G.

Remark 2.45. For $T_\bullet$ as above, the complex associated to $\underline{A} \cdot T_\bullet$ is the ordinary cellular chain complex computing Bredon homology of the geometric realization of $T_\bullet$ with coefficients in $\underline{A}$.

Coinduction provides a kind of norm from simplicial H-sets to simplicial G-sets.

Definition 2.46. If $T_\bullet$ is a finite simplicial H-set, then let

$$\mathrm{Map}_H(G, T_\bullet)$$

be the finite simplicial G-set which arises by composing with the coinduction functor.

Since the norm on a representable Mackey functor is easy to compute, the following is immediate.

Proposition 2.47. *For any finite simplicial H-set $T_\bullet$, we have a natural isomorphism of simplicial G-Mackey functors*

$$N_H^G\big(\underline{A}^H \cdot T_\bullet\big) \cong \underline{A}^G \cdot Map_H(G, T_\bullet).$$

Homotopy invariance of the norm then follows from two observations.

Proposition 2.48. *Let $\underline{M}_\bullet$ and $\underline{N}_\bullet$ be simplicial Mackey functors and let*

$$f, f' \colon \underline{M}_\bullet \to \underline{N}_\bullet$$

be two maps of simplicial Mackey functors. Then a simplicial homotopy F from f to f' is a map of simplicial Mackey functors

$$\underline{M}_\bullet \Box \big(\underline{A} \cdot \Delta^1_\bullet\big) \to \underline{N}_\bullet$$

which restricts to $f \oplus f'$ on

$$\underline{M}_\bullet \oplus \underline{M}_\bullet \cong \underline{M} \Box \big(\underline{A} \cdot \partial\Delta^1_\bullet\big).$$

Lemma 2.49. *The norm preserves simplicial homotopies.*

Proof. The norm functor, being strong symmetric monoidal, gives a map

$$N_H^G(F) \colon N_H^G \underline{M}_\bullet \Box N_H^G\big(\underline{A} \cdot \Delta^1_\bullet\big) \to N_H^G \underline{N}_\bullet.$$

We can rewrite the source using Proposition 2.47:

$$N_H^G\big(\underline{A} \cdot \Delta^1_\bullet\big) \cong \underline{A} \cdot \mathrm{Map}_H(G, \Delta^1_\bullet).$$

Restricting along the diagonal map

$$\Delta^1_\bullet \hookrightarrow \mathrm{Map}_H(G, \Delta^1_\bullet)$$

then gives the desired simplicial homotopy. $\Box$

Corollary 2.50. *The norm on simplicial Mackey functors takes cofibrant objects to cofibrant objects and preserves weak equivalences between cofibrant objects.*

Proof. Proposition 2.37 shows that a simplicial Mackey functor is cofibrant if and only if it is levelwise projective, and Proposition 2.21 shows that the norm preserves the generating projectives. This shows that the norm on simplicial Mackey functors preserves cofibrant objects.

For the second part, any weak equivalence between cofibrant simplicial Mackey functors is a homotopy equivalence. Lemma 2.49 shows that the norm preserves these. □

3. Geometric Fixed Points in Mackey Functors

Our construction of the twisted Hochschild complex of a Green functor will have a "cyclotomic" structure, relating the geometric fixed points to the original complex. To explain what we mean by cyclotomic here, we must first describe geometric fixed points in the derived category of Mackey functors. This is related to Remark 6.5 of [18] and B.7 of [3]. We will define our version in terms of a particular choice of point-set model for the derived functor.

The material in this section works for all finite groups; we let $\underline{A}$ denote the Burnside Mackey functor for the finite group G.

Definition 3.1. Fix a finite group G and let $N \subset G$ be a normal subgroup. Let $E\mathcal{F}_N(\underline{A})$ denote the subMackey functor of $\underline{A}$ generated by $\underline{A}(G/H)$ for all subgroups H which do not contain N. Finally, let

$$\tilde{E}\mathcal{F}_N(\underline{A}) = \underline{A}/E\mathcal{F}_N(\underline{A}).$$

If H does not contain N, then by construction $\tilde{E}\mathcal{F}_N(\underline{A})(G/H) = 0$, while if H does contain N, then $\tilde{E}\mathcal{F}_N(\underline{A})(G/H)$ is the quotient of $\underline{A}(G/H)$ by the transfers from any subgroup that does not contain N. In other words,

$$\tilde{E}\mathcal{F}_N(\underline{A})(G/H) = \begin{cases} 0 & N \not\subset H \\ \underline{A}\big((G/N)/(H/N)\big) & N \subset H. \end{cases}$$

Our definition of the geometric fixed points will be a composite; we begin by taking the box product with $\tilde{E}\mathcal{F}_N$.

Definition 3.2. If $\underline{M}$ is a G-Mackey functor and N is a normal subgroup of G, then let

$$\tilde{E}\mathcal{F}_N\underline{M} = \underline{M}\Box\tilde{E}\mathcal{F}_N\underline{A}.$$

If $\underline{M}_\bullet$ is a dg-G-Mackey functor, then let

$$(\tilde{E}\mathcal{F}_N\underline{M})_n = \underline{M}_n\Box\tilde{E}\mathcal{F}_N\underline{A},$$

with the obvious differential.

This is closely connected to the topological object by the same name. Associated to N is a smashing localization which nullifies all spectra induced up from subgroups that do not contain N, the value of which on the sphere is commonly denoted $\tilde{E}\mathcal{F}_N$. The following is immediate from direct considerations of the nullification functor.

Proposition 3.3. *For any Mackey functor $\underline{M}$, we have*

$$\underline{\pi}_0(\tilde{E}\mathcal{F}_N \wedge H\underline{M}) \cong \tilde{E}\mathcal{F}_N\underline{M},$$

where here $\tilde{E}\mathcal{F}_N$ on the left denotes a cofibrant model of the universal space for the family of subgroups not containing N.

Proof. By Proposition 2.14, we know that we have an isomorphism

$$\underline{\pi}_0(\tilde{E}\mathcal{F}_N \wedge H\underline{M}) \cong \underline{\pi}_0(\tilde{E}\mathcal{F}_N)\Box\underline{M}.$$

The space $\tilde{E}\mathcal{F}_N$ is the localization of the zero sphere where we kill the localizing subcategory generated by G/H_+ for H a subgroup not containing N. The zeroth homotopy Mackey functor of $\Sigma^\infty G/H_+$ is by construction

$$\underline{A}^G_{G/H}.$$

Since G/H_+ agrees with $Ind_H^G S^0$, we know also that the set of equivariant homotopy classes of maps

$$G/H_+ \to X$$

is the group $\underline{\pi}_0(X)(G/H)$. In particular, we see that $\underline{\pi}_0$ of the nullification of S^0 killing these G/H_+ agrees with the nullification in Mackey functors. This is exactly the quotient. □

Remark 3.4. One might think that a good model of $\tilde{E}\mathcal{F}_N$ as dg Mackey functor could be obtained by taking the cellular chains on the G-space $S^{\infty\bar{\rho}}$, where $\bar{\rho}$ is the reduced regular representation of G/N. However, although there is an isomorphism

$$\underline{\pi}_0(\underline{C}_* S^{\infty\bar{\rho}}) \cong \tilde{E}\mathcal{F}_N,$$

the canonical map is *not* in general a quasi-isomorphism. For example, if $G = C_p$, then

$$\underline{\pi}_2\big(\underline{C}_* S^{\infty\bar{\rho}}\big) = \underline{H}_2\big(S^{\infty\bar{\rho}}\big) \cong \tilde{E}\mathcal{F}_G/p$$

is the reduction modulo p of $\tilde{E}\mathcal{F}_G$.

The proof of Proposition 3.3 explains how we should compute our derived geometric fixed points. Specifically, we want the nullification in the category of dg Mackey functors which kills the localizing, triangulated subcategory generated by $\underline{A}_{G/H}$ for H not containing N. From the definition of $\tilde{E}F_N\underline{A}$, we immediately deduce the following proposition which lets us compute this nullification.

Proposition 3.5. *For any normal subgroup N and for any dg-Mackey functor $\underline{M}_\bullet$, the Mackey functor $\tilde{E}\mathcal{F}_N(\underline{M})_\bullet$ is in the image of the (fully faithful) pullback π_N^* from G/N-dg-Mackey functors to G-dg-Mackey functors.*

Remark 3.6. The preceding proposition is closely related to an observation of Kaledin that homology is not sufficient to detect the image of the pullback (see the introduction to [21]).

Using Proposition 3.5 and the fact that the pullback is an isomorphism onto its image, we can now define the geometric fixed points.

Definition 3.7. For a normal subgroup N, let the *geometric fixed points* of a Mackey functor $\underline{M}$ be defined as

$$\Phi^N(\underline{M}) = {\pi_N^*}^{-1}\big(\tilde{E}F_N(\underline{M})\big).$$

If $\underline{M}_\bullet$ is a dg-Mackey functor, then $\Phi^N\underline{M}_\bullet$ is obtained by applying Φ^N levelwise.

Combining the definition with Proposition 3.3 connects this with the ordinary geometric fixed points.

Corollary 3.8. *Given any G-Mackey functor $\underline{M}$, we have a natural isomorphism of G/N-Mackey functors*

$$\underline{\pi}_0^{G/N}(\Phi^N H\underline{M}) \cong \Phi^N \underline{M}.$$

The geometric fixed-points can be derived by cofibrantly replacing $\underline{M}$ or $\underline{M}_\bullet$. Notice however that we are not using a cofibrant replacement of $\tilde{E}\mathcal{F}_N\underline{A}$ here; although the result would have quasi-isomorphic box product if we did this, Proposition 3.5 would not hold.

Remark 3.9. The functor Φ^N is in fact a left Quillen functor. If $\underline{P}_\bullet$ is a dg-Mackey functor such that $\underline{P}_n = \underline{A}^G_{T_n}$ for some G-set T, then

$$\Phi^N(\underline{P})_n = \underline{A}^{G/N}_{T_n^N}$$

is again a cofibrant object.

The image of the transfer is an ideal in Green functors (and in fact, Frobenius reciprocity says that the transfer is the analogue of an ideal for any bilinear map), and so the Mackey functor $\tilde{E}\mathcal{F}_N\underline{A}$ is canonically a Green functor. Since pullback is strong symmetric monoidal, the following proposition is immediate.

Proposition 3.10. *The functor Φ^N is a strong symmetric monoidal functor.*

As a consequence, the geometric fixed points functor preserves multiplicative structures.

Corollary 3.11. *The geometric fixed points functor lifts to a functor*

$$\Phi^N \colon \mathcal{G}reen_G \to \mathcal{G}reen_{G/N}.$$

To understand the analogue of the cyclotomic structure on our twisted Hochschild homology, we need to understand the interaction of the geometric fixed points functor and the norm.

Theorem 3.12. *For any H-Mackey functor $\underline{M}$, we have an isomorphism of G/N-Mackey functors*

$$\Phi^N N_H^G \underline{M} \cong N_{HN/N}^{G/N} \Phi^{H\cap N} \underline{M}.$$

Proof. The diagonal map in spectra induces a canonical weak equivalence for any cofibrant H-spectrum E:

$$N_{HN/N}^{G/N}\Phi^{H\cap N}E \xrightarrow{\simeq} \Phi^N N_H^G E.$$

Applying $\underline{\pi}_0$ gives an isomorphism. By Corollary 3.8 and the definition of the norm in Mackey functors, this gives us an isomorphism

$$N_{HN/N}^{G/N}\Phi^{H\cap N}\underline{M} \cong \underline{\pi}_0 N_{HN/N}^{G/N}\Phi^{H\cap N}H\underline{M} \cong \underline{\pi}_0\Phi^N N_H^G H\underline{M} \cong \Phi^N N_H^G \underline{M},$$

where for the inner isomorphisms, we have used that the difference between a spectrum and its zeroth Postnikov truncation is 0-connected. □

Remark 3.13. The "diagonal" map in spectra used in the proof of Theorem 3.12 has a direct Mackey version that is significantly harder to describe. On representables, however, it is easy to write down an explicit description. If $\underline{M} = \underline{A}^H_T$, then we are looking at the map

$$\underline{A}^{G/N}_{F_{HN/N}(G/N, T^{H\cap N})} \to \tilde{E}\mathcal{F}_N \underline{A}^G_{F_H(G,T)}$$

which is induced by the inclusion

$$F_{HN/N}(G/N, T^{H\cap N}) \cong F_H(G,T)^N \hookrightarrow F_H(G,T).$$

Although transfer maps are in general not multiplicative, the failure of this to be such is exactly contained in the image of the transfer maps that are killed for $\tilde{E}\mathcal{F}_N$.

4. Twisted Hochschild Homology

We now turn to our main definition, the construction of the twisted Hochschild complex in Mackey functors. The definition relies on the fact that in Mackey functors for cyclic groups (or more generally, a group in which all subgroups are normal), the Weyl group of H is always G/H. This easy observation has the consequence that $\underline{M}(G/H)$ is actually a G-module for all subgroups H, and the restriction and transfer maps are actually maps of G-modules. In particular, objects of Mack_G have an action of G.

4.1. **The twisted cyclic nerve.**

Our basic construction is a cyclic nerve which takes as input a Green functor $\underline{R}$, an $\underline{R}$-bimodule $\underline{M}$, and an element $g \in G$ which is used to twist the action on $\underline{M}$.

Definition 4.1. Let $\underline{R}$ be a Green functor for C_n, and let $\underline{M}$ be a left $\underline{R}$-module. If $g \in C_n$, then let ${}^g\underline{M}$ denote $\underline{M}$ with the module structure twisted by g: if μ is the action map for $\underline{M}$, then the action map ${}^g\mu$ for ${}^g\underline{M}$ is given by

$$\begin{tikzcd} \underline{R}\Box\underline{M} \arrow[d, "g\Box 1"] \arrow[dr, "{}^g\mu"] & \\ \underline{R}\Box\underline{M} \arrow[r, "\mu"] & \underline{M}. \end{tikzcd}$$

If $\underline{M}$ is instead a right $\underline{R}$-module, then we will denote the obvious analogous structure by $\underline{M}^g$.

We call these induced module structures the twisted module structure.

The (twisted) cyclic bar constructions we consider are simply the evident generalization of the ordinary cyclic bar constructions where we allow various twistings on the bimodule coordinate. The (twisted) cyclic cobar construction is defined analogously; see Remark 4.13 for further discussion.

Definition 4.2. Let $G \subset S^1$ be a finite subgroup, let $g \in G$, and let $\underline{R}$ be an associative Green functor for G, and let $\underline{M}$ be an $\underline{R}$-bimodule. Define the *twisted cyclic nerve* of $\underline{R}$ with coefficients in ${}^g\underline{M}$ as the simplicial Green functor with k-simplices given by

$$[k] \mapsto \underline{HC}_k^G(\underline{R}; {}^g\underline{M}) := {}^g\underline{M}\Box\underline{R}^{\Box k}.$$

For $1 \le i \le k-1$, the face map d_i is simply the multiplication between the i^{th} and $(i+1)^{\text{st}}$ box factors. The 0^{th} face map d_0 is the ordinary (right) action map. The k^{th} face map d_k is given by the composite:

$${}^g\underline{M}\Box\underline{R}^{\Box k} \xrightarrow{\cong} \underline{R}\Box{}^g\underline{M}\Box\underline{R}^{\Box(k-1)} \xrightarrow{{}^g\mu\Box Id} {}^g\underline{M}\Box\underline{R}^{\Box(k-1)}.$$

For $0 \le i \le k-1$, the degeneracy map s_i is induced by the unit in the $(i+1)^{\text{st}}$ factor.

It is straightforward to verify that this is in fact a simplicial object. All identities that do not include the kth face map follow from the standard arguments, relying only on the associativity of the product and the definition of the unit. The identities involving the kth face map follow from the observation that the G-action is via associative ring maps.

The standard homological algebra argument shows the following, since g in our definition only changed the bimodule, rather than changing the fundamental category.

Lemma 4.3. *For any finite $G \subset S^1$, any choice of $g \in G$, any associative Green functor $\underline{R}$ that is flat over the Burnside Mackey functor $\underline{A}$, and any $\underline{R}$-bimodule $\underline{M}$, the homology of $\underline{HC}_*^G(\underline{R}; {}^g\underline{M})$ is*

$$\underline{\mathrm{Tor}}_*^{\underline{R}\Box\underline{R}^{op}}(\underline{R}, {}^g\underline{M}).$$

Remark 4.4. Flatness here is a harsher condition than classically, since even localizations need not preserve flatness. For example, for $G = C_2$, inverting 2 in the underlying ring of the Burnside Mackey functor produces a localization which is not flat, splitting into a copy of the augmentation ideal of the Burnside Mackey functor (which is not flat) and the constant Mackey functor $\mathbb{Z}[\frac{1}{2}]$.

Remark 4.5. It is obvious that we have similar constructions wherein we perturb both the bimodule structures on $\underline{R}$ and on $\underline{M}$; we ignore these, since they do not seem to play a role in topological Hochschild homology.

4.2. **Twisted Hochschild Homology.** We now define a twisted version of Hochschild homology for Green functors in terms of the twisted cyclic nerve.

Definition 4.6. Let $G \subset S^1$ be a finite subgroup. The *generator* of G is the element $g = e^{2\pi i/|G|} \in S^1$.

For a cyclic group G, a choice of generator is equivalent to a choice of embedding $G \to S^1$.

Definition 4.7. Let g be the generator of $G \subset S^1$. Let $\underline{R}$ be an associative Green functor for G. The *G-twisted Hochschild homology*, $\underline{\mathrm{HH}}_i^G(\underline{R})$, is the homology of the twisted cyclic nerve $\underline{HC}_*^G(\underline{R}; {}^g\underline{R})$.

We also have a relative version of the twisted cyclic nerve and correspondingly a relative version of the Hochschild homology of Green functors.

Definition 4.8. Let $G \subset G' \subset S^1$ be finite subgroups, let $g \in G'$ be the generator, and let $\underline{R}$ be an associative Green functor for G. Define the G'-twisted cyclic nerve relative to G as

$$\underline{HC}_*^{G'}(\underline{R}) := \underline{HC}_*^{G'}\big(N_G^{G'}\underline{R}; {}^g(N_G^{G'}\underline{R})\big).$$

The G'-twisted Hochschild homology relative to G, $\underline{\mathrm{HH}}_i^{G'}(\underline{R})$, is the homology of $\underline{HC}^{G'}(\underline{R})$.

When $\underline{R}$ is a commutative Green functor, the twisted cyclic nerve inherits that structure.

Proposition 4.9. *If $\underline{R}$ is a commutative Green functor, then $\underline{HC}_*^G(\underline{R})$ is a simplicial Green functor.*

Proof. Since Green functors are the commutative monoids for the box product of Mackey functors, all of the face maps except the last one are obviously maps of Green functors. The degeneracy maps are just boxing with the unit, and hence are also maps of Green functors. Finally, since all of the structure maps in a Green functor are G-equivariant, the action by an element g is a map of Green functors; the last face map is also a map of Green functors. □

Applying Lemma 4.3 gives the following immediate proposition.

Proposition 4.10. *Let $H \subset G \subset S^1$ be finite subgroups and $g \in G$ the generator. For any H-Green functor $\underline{R}$ which can be written as a filtered colimit of projectives, we have a natural isomorphism*

$$\underline{\mathrm{HH}}_i^G(\underline{R}) \cong \underline{\mathrm{Tor}}_i^{N_H^G \underline{R} \Box N_H^G \underline{R}^{op}}(N_H^G \underline{R}, {}^g(N_H^G \underline{R})),$$

where $\underline{\mathrm{Tor}}_i$ is the ith derived functor of the box product.

Proof. The norm preserves projective objects and filtered colimits, so in particular, the norm of $\underline{R}$ is again a filtered colimit of projectives. These are flat. □

The twisting by a choice of generator can be somewhat confusing. We find it here most helpful to recast this in a more transparent way for $\underline{R}$ a commutative Green functor.

Proposition 4.11. *Let $\underline{R}$ be a commutative Green functor for G. Let $g \in G$ be the generator, and let $\mathbb{Z} \to G$ be the associated homomorphism. Then the natural map*

$$\underline{R} \to \underline{HC}_0^G \underline{R}$$

given by the inclusion of the zero cells extends to a natural weak equivalence

$$\underline{R}_{h\mathbb{Z}} \xrightarrow{\simeq} \underline{HC}_*^G \underline{R}$$

between the homotopy orbits of $\underline{R}$ in commutative Green functors to the twisted Hochschild homology.

Proof. The formula for the boundary maps of the twisted Hochschild chains shows that $\underline{HC}_*^G(\underline{R})$ is exactly the homotopy coequalizer in commutative Green functors of the identity with the generator g of G. By definition, this is also the homotopy orbits. □

Remark 4.12. The topological statement is also true, namely the same argument shows that there is a natural equivalence between $R_{h\mathbb{Z}}$ and $S^1 \otimes_G R$ for R a commutative ring orthogonal G-spectrum. Choosing different generators corresponds to tensoring up over different embeddings $G \to S^1$.

Remark 4.13. In the equivariant context, what we might mean by topological Hochschild cohomology becomes more subtle. Classically, this is the endomorphism spectrum of R as an (R, R)-bimodule, and the action on THH is obvious. The equivariant structure of THH, however, has an asymmetry in the roles of R in the two smash factors. In particular, we see *both* an action of $\mathrm{End}_{R^e}(R)$ and of $\mathrm{End}_{R^e}({}^gR)$. We expect that the most relevant definition will be guided by the applications; we leave further investigation of this construction for future work.

4.3. **Cyclotomic Structure.** The twisted Hochschild complex enjoys a kind of cyclotomic structure.

Proposition 4.14. *Let $H \subset G \subset S^1$ be finite subgroups, and let N be a normal subgroup of G. For $\underline{R}$ a commutative Green functor for H, there is a natural isomorphism of simplicial Green functors*

$$\Phi^N\big(\underline{HC}^G_*(\underline{R})\big) \cong \underline{HC}^{G/N}_*(\Phi^{H\cap N}\underline{R}).$$

Proof. We apply the geometric fixed points functor Φ^N levelwise in the simplicial set. Since by Proposition 3.10 we know that Φ^N is a strong symmetric monoidal functor, Theorem 3.12 yields a natural isomorphism

$$\Phi^N\big(\underline{HC}^G_k(\underline{R})\big) \cong (\Phi^N N_H^G\underline{R})^{\Box k+1} \cong (N_{HN/N}^{G/N}\Phi^{H\cap N}\underline{R})^{\Box k+1} \cong \underline{HC}^{G/N}_k(\Phi^{H\cap N}\underline{R}).$$

Additionally, multiplication by a generator g induces multiplication by g on the geometric fixed points, so by naturality, we see that these isomorphisms commute with all of the simplicial structure maps. □

Corollary 4.15. *If G is a finite subgroup of S^1, $H, N \subset G$, and $\underline{R}$ a commutative Green functor for H, then we have a "geometric fixed points" map of simplicial commutative rings*

$$\underline{HC}^G(\underline{R})(G/G) \to \underline{HC}^{G/N}\big(\Phi^{H\cap N}\underline{R}\big)\big((G/N)/(G/N)\big).$$

Proof. For any Mackey functor $\underline{M}$, we have a natural isomorphism of abelian groups

$$\tilde{E}\mathcal{F}_N\underline{M}(G/G) \cong \Phi^N\underline{M}\big((G/N)/(G/N)\big).$$

The result follows from considering the natural map of simplicial Green functors

$$\underline{HC}^G(\underline{R}) \to \tilde{E}\mathcal{F}_N\underline{HC}^G(\underline{R})$$

and evaluating at G/G. □

In this context, however, passage to fixed points is more difficult. The restriction to H of the G-Hochschild complex for $\underline{R}$ is *not* equal to the H-Hochschild complex for $\underline{R}$ or even for the restriction to H of $\underline{R}$. However, they are always quasi-isomorphic. To see this, we use the edgewise subdivision functor applied to simplicial Green functors. (See [6, §1] for a review of the properties of the edgewise subdivision.)

Proposition 4.16. *If $H \subset J \subset G$, and $\underline{R}$ is a commutative Green functor for H, then we have a natural quasi-isomorphism*

$$i_J^*\big(\underline{HC}^G_*(\underline{R})\big) \simeq \mathrm{sd}_{[G:J]}\left(\underline{HC}^J_*(\underline{R})\right).$$

Proof. The argument here is a classical one (see, for instance, [26, Section 5]). We include it here for completeness. Since the restriction is a strong symmetric monoidal functor on Mackey functors, we have

$$i_J^*\big(\underline{HC}^G_k(\underline{R})\big) \cong {}^g\big(i_J^*N_H^G\underline{R}\big)\Box\big(i_J^*N_H^G\underline{R}\big)^{\Box k}.$$

Theorem 2.31 shows that we have an isomorphism of G-objects in J-Green functors

$$i_J^*N_H^G\underline{R} \cong \big(N_H^J\underline{R}\big)^{\Box|G/J|},$$

where the action on the right hand side is via tensor induction. This shows that the k-simplices are given by

$$i_J^*\big(\underline{HC}^G_k(\underline{R})\big) \cong {}^g\big(N_H^J\underline{R}\big)^{\Box|G/J|}\Box\big(N_H^J\underline{R}^{\Box|G/J|}\big)^{\Box k} \cong \big(N_H^J\underline{R}\big)^{\Box(|G/J|(k+1))}.$$

The face maps besides the last are induced by the multiplications on $N_H^G\underline{R}$, which when restricted to J, give the component-wise multiplications

$$\big(N_H^J\underline{R}\big)^{\Box|G/J|}\Box\big(N_H^J\underline{R}\big)^{\Box|G/J|} \to \big(N_H^J\underline{R}\big)^{\Box|G/J|}.$$

This is exactly the iterated face map which arises in the edgewise subdivision. Similarly, the degeneracy maps are induced by inserting the unit for $N_H^G\underline{R}$, which restricts to inserting a collection of units for each of the tensor factors. For the last face map, we observe that the map "act by g and multiply" realizes the rotation of the tensor factors, and then the action of $g^{|G/J|}$, a generator of J, taking the last to the first. This is exactly the value of the edgewise subdivision on the last face. $\square$

Using Proposition 4.11, we can also describe the restriction to other subgroups. Since any G-Green functor is a G-object in G-Green functors, the restriction to a subgroup K is automatically a G-object in K-Green functors. This describes our restriction to other subgroups.

Proposition 4.17. *Let $K \subset H \subset G \subset S^1$ and let $h \in H$ be the generator. For $\underline{R}$ a commutative Green functor for H, we have a natural quasi-isomorphism*

$$i_K^*\big(\underline{HC}_*^G(\underline{R})\big) \simeq \big(i_K^*\underline{R}\big)_{h\mathbb{Z}},$$

where the $\mathbb{Z}$-action on $i_K^*\underline{R}$ is via the quotient $\mathbb{Z} \to H$ sending 1 to h.

Proof. Proposition 4.16 and naturality of the restriction allow us to reduce to the case that $H = G$ (since otherwise, we first restrict to $J = H$ in Proposition 4.16 and then continue restricting down). Here the result is obvious, since the two homotopical left adjoints commute. $\square$

Remark 4.18. When we restrict to subgroups that neither contain H nor are contained in H, then the answer is somewhat trickier to describe. If $\underline{R}$ is a commutative H-Green functor, then $i_{H\cap J}^*\underline{R}$ has an H-action that extends the natural $H \cap J$-action. By functoriality, the norm

$$N_{H\cap J}^J i_{H\cap J}^*\underline{R}$$

has an action of both H and J and they agree on $H \cap J$. This gives an action of HJ, and the restriction of $\underline{HC}_*^G(\underline{R})$ is

$$\big(N_{H\cap J}^J i_{H\cap J}^*\underline{R}\big)_{h\mathbb{Z}},$$

via a generator of HJ. We do not pursue this further here.

4.4. **Comparison to topological Hochschild homology.** Our algebraic constructions are approximations to the structure seen in the usual TR tower approximating algebraic K-theory. We can make this assertion precise using the connection between the algebraic norm in Mackey functors and the Hill-Hopkins-Ravenel norm in spectra (recall Theorem 2.22). This correspondence allows us to establish a tight relationship between the twisted cyclic nerve and the H-relative topological Hochschild homology THH_H considered in [1].

Theorem 4.19. *Let $H \subset G \subset S^1$ be finite subgroups and let R be a (-1)-connected commutative ring orthogonal H-spectrum. We have a natural isomorphism*

$$\underline{\pi}_0^G\big(\,\mathrm{THH}_H(R)\big) \cong \underline{\mathrm{HH}}_0^G(\underline{\pi}_0^H R).$$

Proof. For any H-commutative ring spectrum R, we have a natural weak equivalence

$$i_G^* \operatorname{THH}_H(R) \simeq (N_H^G R) \underset{N_H^G R^e}{\wedge} {}^g(N_H^G R),$$

where just as above, ${}^g(N_H^G R)$ is just $N_H^G R$ with a bimodule structure twisted on one side by the Weyl action. If R is (-1)-connected, then so is $N_H^G R$, and for degree reasons, the Künneth spectral sequence collapses, giving

$$\underline{\pi}_0^G\Big((N_H^G R) \underset{N_H^G R^e}{\wedge} {}^g(N_H^G R)\Big) \cong \big(\underline{\pi}_0^G N_H^G R\big) \underset{\left(\underline{\pi}_0^G N_H^G R\right)^e}{\Box} {}^g\big(\underline{\pi}_0^G N_H^G R\big),$$

which is exactly $\underline{\operatorname{HH}}_0^G\big(\underline{\pi}_0 R\big)$. □

More generally, we have a natural homomorphism from the C_n-relative TR groups to our twisted Hochschild homology, refining the classical trace map.

Theorem 4.20. *For $H \subset G \subset S^1$ finite subgroups, and R a (-1)-connected H-equivariant commutative ring spectrum, we have a natural homomorphism*

$$\underline{\pi}_k^G \operatorname{THH}_H(R) \to \underline{\operatorname{HH}}_k^G(\underline{\pi}_0^H R).$$

Proof. Recall that for any simplicial spectrum $E_\bullet$, we have a spectral sequence

$$E_{p,q}^2 = H_p\big(\pi_q(E_\bullet)\big) \Rightarrow \pi_{p+q}\big(|E_\bullet|\big),$$

where here we have simply applied π_q to $E_\bullet$ to get a simplicial abelian group [10, X.2.9]. The differentials are homology Serre type, and hence if for all k, E_k is (-1)-connected, then we have an edge homomorphism

$$\pi_p\big(|E_\bullet|\big) \to H_p\big(\pi_0(E_\bullet)\big).$$

The proof is by the filtration by the skeleton, and goes through without change equivariantly. Applied to our context, we then have a spectral sequence with

$$E_{p,q}^2 = H_p\Big(\underline{\pi}_q\big((N_H^G R)^{\wedge(\bullet+1)}\big)\Big) \Rightarrow \underline{\pi}_{p+q} \operatorname{THH}_H(R).$$

The edge homomorphism is

$$\underline{\pi}_p \operatorname{THH}_H(R) \to H_p\Big(\underline{\pi}_0\big((N_H^G R)^{\wedge(\bullet+1)}\big)\Big).$$

The collapse of the Künneth spectral sequence in degree zero, together with the definition of the norm in Mackey functors, then shows that we have an isomorphism of simplicial Green functors

$$\underline{\pi}_0\big((N_H^G R)^{\wedge(\bullet+1)}\big) \cong \underline{HC}_\bullet^G(\underline{\pi}_0 R),$$

as desired. □

In the case where H is the trivial group, and hence R is a ring, the linearization map of Theorem 4.20 yields new lifts of the Dennis trace.

Corollary 4.21. *For R a commutative ring, and any finite $G \subset S^1$, we have a lift of the Dennis trace*

$$K_q(R) \to \underline{\operatorname{HH}}_q^G(R)(G/G).$$

Proof. The following diagram commutes, where the top is the composite of the Bökstedt-Hsiang-Madsen trace map and the edge map from Theorem 4.20. The bottom composite is the classical Dennis trace.

$$\begin{tikzcd}
 & \pi_q(\mathrm{THH}(R)^G) \cong \underline{\pi}_q^G \,\mathrm{THH}(R)(G/G) \arrow[r] \arrow[d, "res_e^G"] & \underline{\mathrm{HH}}_k^G(R)(G/G) \arrow[d, "res_e^G"] \\
\pi_q K(R) \arrow[ur, "trc"] \arrow[dr] & & \\
 & \pi_q \,\mathrm{THH}(R) \cong \underline{\pi}_q^G \,\mathrm{THH}(R)(G/e) \arrow[r] & \mathrm{HH}_q(R) \cong \underline{\mathrm{HH}}_q^G(R)(G/e)
\end{tikzcd}$$

□

4.5. **Algebraic approximation to TR.** In the classical theory, to compute algebraic K-theory using trace methods one wants to understand fixed points of topological Hochschild homology, or TR-theory. For a ring A, $\mathrm{TR}^{n+1}(A;p) := \mathrm{THH}(A)^{C_{p^n}}$. We can consider TR as a C_{p^n}-Mackey functor

$$\underline{\mathrm{TR}}_q(A;p) := \underline{\pi}_q^{C_{p^n}} \,\mathrm{THH}(A)$$

Recall that there are two operators on TR, the Frobenius and the Restriction, which are used classically to defined topological cyclic homology. These are operators

$$F, R : \mathrm{TR}^{n+1}(A;p) \to \mathrm{TR}^n(A;p).$$

There is an unfortunate clash of notation here, as the Frobenius map on TR is the restriction map $res_{C_{p^{n-1}}}^{C_{p^n}}$ in the Mackey functor $\underline{\mathrm{TR}}$. The map R on TR is not part of the Mackey functor structure. This map R is defined using the cyclotomic structure on THH. The spectrum $\mathrm{TR}(A;p)$ is then defined to be

$$\mathrm{TR}(A;p) := \operatorname{holim}_{\overleftarrow{R}} \mathrm{TR}^n(A;p).$$

In this section we define an algebraic analog of $\mathrm{TR}(A;p)$ using twisted Hochschild homology.

We first need an analogue of the Restriction map R from the topological setting. When $H = e$, for a commutative ring A the "geometric fixed points" map of Corollary 4.15 gives a map

$$\underline{\mathrm{HH}}_q^{C_{p^n}}(A)(C_{p^n}/C_{p^n}) \to \underline{\mathrm{HH}}_q^{C_{p^{n-1}}}(A)(C_{p^{n-1}}/C_{p^{n-1}})$$

which serves as an algebraic analogue of the restriction map R on $\mathrm{TR}^{n+1}(A;p)$. We use these geometric fixed point maps to define an algebraic analogue of TR.

Definition 4.22. For a commutative ring A, the algebraic TR groups of A are

$$tr_k(A;p) := \lim_{\longleftarrow} \underline{\mathrm{HH}}_k^{C_{p^n}}(A)(C_{p^n}/C_{p^n}).$$

Below we include an explicit computation for $A = \mathbb{F}_p$ of the algebraic TR-groups $tr_k(\mathbb{F}_p;p)$. Recall that $\underline{\mathrm{HH}}_k^{C_{p^n}}(\mathbb{F}_p)$ is the homology of the twisted cyclic nerve on $N_e^{C_{p^n}}\mathbb{F}_p$. This Mackey functor norm can be computed following the work of Mazur [25].

Proposition 4.23. *For $G = C_{p^n}$,*

$$N_e^G \mathbb{F}_p(G/C_{p^k}) = \mathbb{Z}/p^{k+1},$$

all restriction maps are the canonical quotients, all transfer maps are multiplication by the index (so the obvious injections) and all norms are determined by where they send 1*. The Weyl action is trivial.*

Since the norm is symmetric monoidal, we know that

$$(N_e^G \mathbb{F}_p)^{\square(k+1)} \cong N_e^G \mathbb{F}_p.$$

Since the Weyl action is trivial, there is no difference between the G-twisted Hochschild complex and the ordinary Hochschild complex in Green functors, so we conclude the following.

Proposition 4.24. *For all $n \geq 1$,*

$$\underline{\mathrm{HH}}_k^{C_{p^n}}(\mathbb{F}_p) = \begin{cases} N_e^{C_{p^n}} \mathbb{F}_p & k = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Note that this agrees with the classical computation, where we use the Frobenius and Verschiebung to assemble the groups $TR_0^*(\mathbb{F}_p; p)$ for $1 \leq * \leq (n+1)$ into a Mackey functor. Our cyclotomic structure (Proposition 4.14) connects these via the geometric fixed points. When evaluated on a point (i.e. G/G), we have geometric fixed points maps

$$\underline{\mathrm{HH}}_0^{C_{p^n}}(\mathbb{F}_p)(C_{p^n}/C_{p^n}) \to \underline{\mathrm{HH}}_0^{C_{p^n}/C_p}(\mathbb{F}_p)\big((C_{p^n}/C_p)/(C_{p^n}/C_p)\big).$$

The target is just

$$\underline{\mathrm{HH}}_0^{C_{p^{n-1}}}(\mathbb{F}_p)(C_{p^{n-1}}/C_{p^{n-1}}),$$

and the map we get is just the canonical quotient map

$$\mathbb{Z}/p^n = \underline{\mathrm{HH}}_0^{C_{p^n}}(\mathbb{F}_p)(C_{p^n}/C_{p^n}) \to \underline{\mathrm{HH}}_0^{C_{p^{n-1}}}(\mathbb{F}_p)(C_{p^{n-1}}/C_{p^{n-1}}) = \mathbb{Z}/p^{n-1}.$$

This gives us the purely algebraic analogue of $TR(\text{-}; p)$.

Proposition 4.25. *The algebraic TR groups of $\mathbb{F}_p$ are*

$$tr_k(\mathbb{F}_p; p) := \varprojlim \underline{\mathrm{HH}}_k^{C_{p^n}}(\mathbb{F}_p)(C_{p^n}/C_{p^n}) = \mathbb{Z}_p.$$

In the case of $\mathbb{F}_p$, algebraic TR is an excellent approximation for the topological theory. Indeed, the result for $tr_k(\mathbb{F}_p; p)$ agrees with $\mathrm{TR}_k(\mathbb{F}_p; p)$, and is the p-completion of $K_k(\mathbb{F}_p)$.

4.6. **Green Witt vectors.** Classically, Hesselholt and Madsen [15] show that the fixed points of topological Hochschild homology are closely related to Witt vectors. In particular, for a commutative ring R

$$\pi_0(\mathrm{THH}(R)^{C_n}) \cong \mathbb{W}_{\langle n \rangle}(R)$$

where $\langle n \rangle$ denotes the truncation set of natural numbers which divide n. Using this result and the isomorphism in Theorem 4.19, it follows that twisted Hochschild homology also has a close relationship to classical Witt vectors.

Proposition 4.26. *If R is a ring, then we have a natural isomorphism*

$$\underline{\mathrm{HH}}_0^{C_n}(R)(C_n/C_n) \cong \mathbb{W}_{\langle n \rangle}(R),$$

where $\langle n \rangle$ denotes the truncation set of natural numbers which divide n.

Theorem 4.19 and the isomorphism of Proposition 4.26 provide motivation for a definition of Witt vectors for Green functors.

Definition 4.27. Let $\underline{R}$ be a Green functor for $C_n \subset S^1$. The C_{mn}-Witt vectors for $\underline{R}$ are defined by

$$\underline{\mathbb{W}}_{C_{mn}}(\underline{R}) := \underline{\mathrm{HH}}_0^{C_{mn}}(\underline{R}).$$

When $m = p^k$ for some prime p, then we can also describe a version of the "length $(k+1)$ p-typical Witt vectors".

Definition 4.28. Let $\underline{R}$ be a Green functor for $C_n \subset S^1$. Then the "length $(k+1)$ p-typical Witt vectors" for $\underline{R}$ are defined by

$$\underline{\mathbb{W}}_{k+1}(\underline{R}) := \underline{\mathbb{W}}_{C_{np^k}}(\underline{R}).$$

With this definition, we have the following H-relative analog of the classical relationship between Witt vectors and fixed points of topological Hochschild homology.

Proposition 4.29. *For $H \subset G \subset S^1$ finite subgroups, and R a (-1)-connected commutative ring orthogonal H-spectrum,*

$$\underline{\pi}_0^G \,\mathrm{THH}_H(R) \cong \underline{\mathbb{W}}_G(\underline{\pi}_0^H R)$$

When $\underline{R}$ is a commutative Green functor, we can describe the Witt vectors via a universal property. First observe that by construction, we have an isomorphism of C_{np^k}-Green functors

$$\underline{\mathbb{W}}_{k+1}(\underline{R}) \cong \big(N_{C_n}^{C_{np^k}} \underline{R}\big)_{\mathbb{Z}},$$

where $\mathbb{Z}$ acts on the norm via the quotient $\mathbb{Z} \to C_{np^k}$ given by specifying a generator (this is zeroth homology of Proposition 4.11). In particular, since we are considering the actual quotient rather than the homotopy one, we have a further identification

$$\underline{\mathbb{W}}_{k+1}(\underline{R}) \cong \big(N_{C_n}^{C_{np^k}} \underline{R}\big)_{C_{np^k}}.$$

In general, the orbits are the left adjoint to the inclusions of some kind of "trivial" objects in the category. The same is true here.

Definition 4.30. Let $\mathcal{O}$ be an indexing system, and let $\mathcal{O}$-$\mathcal{T}amb^G$ be the category of $\mathcal{O}$-Tambara functors. Let $\mathcal{O}$-$\mathcal{T}amb^G_{tr}$ denote the full subcategory of $\mathcal{O}$-$\mathcal{T}amb^G$ spanned by the $\mathcal{O}$-Tambara functors $\underline{R}$ such that the Weyl action on $\underline{R}(G/H)$ is trivial for all H. Let

$$i_* \colon \mathcal{O}\text{-}\mathcal{T}amb^G_{tr} \to \mathcal{O}\text{-}\mathcal{T}amb^G$$

be the inclusion.

The following is immediate.

Proposition 4.31. *The orbits functor is left-adjoint to the functor i_*.*

For Tambara functors, Mazur and Hoyer showed that norm is the left adjoint to the forgetful functor. This hold more generally for $\mathcal{O}$-Tambara functors [5, Theorem 6.15], provided the C_{np^k}-set C_{np^k}/C_n is an admissible set for $\mathcal{O}$. There is a preferred one, arising topologically.

Let $\mathcal{O}_n$ be the indexing system for $G = C_{np^k}$ where the admissible sets for a subgroup H are all of those H-sets fixed by $C_n \cap H$ (The topological operad giving rise to this is the little disks operad for the universe given by infinitely many copies of the permutation representation on G/C_n). An $\mathcal{O}_n$-algebra has an underlying Green functor for C_n (with no norms), and it has norms connecting any two subgroups that contain C_n.

Proposition 4.32 ([5, Theorem 6.5]). *The functor $N_{C_n}^{C_{np^k}}$ is left adjoint to the forgetful functor*

$$i_{C_n}^*\colon \mathcal{O}_n\text{-}\mathcal{T}amb^{C_{np^k}} \to \mathcal{O}_n\text{-}\mathcal{T}amb^{C_n} \cong \mathcal{G}reen^{C_n}.$$

Putting this together, we see that the Green Witt vector construction is a left adjoint.

Theorem 4.33. *The length $(k+1)$ p-typical Green Witt vectors functor is left adjoint to the functor which takes an $\mathcal{O}_n$-Tambara functor with trivial G-action and sends it to the underlying C_n-Green functor.*

We also have a notion of Ghost coordinates for the Green Witt vectors given by the geometric fixed points. We give two versions, one internal to Green functors and one describing the evaluation.

Definition 4.34. Let $\underline{R}$ be a Green functor for C_n. Then the ghost coordinates for $\underline{\mathbb{W}}_{C_{nm}}(\underline{R})$ are the maps of Green functors, parameterized by the subgroups H of C_{nm},

$$\underline{\mathbb{W}}_{C_{nm}}(\underline{R}) \xrightarrow{\phi_H} \mathrm{CoInd}_H^{C_{nm}}\left(\tilde{E}\mathcal{F}_H i_H^*\underline{\mathbb{W}}_{C_{nm}}(\underline{R})\right)$$

adjoint to the canonical maps

$$i_H^*\underline{\mathbb{W}}_{C_{nm}}(\underline{R}) \xrightarrow{\hat{\phi}_H} \tilde{E}\mathcal{F}_H i_H^*\underline{\mathbb{W}}_{C_{nm}}(\underline{R}).$$

Put more conceptually, the map ϕ_H is simply the restriction to H followed by the H-geometric fixed points. We unpack this slightly in the p-typical case to get a more nuanced understanding of these maps, describing in more detail what the ghost coordinates look like for subgroups H between C_n and C_{np^k}. In Mackey functors, induction and coinduction agree and are both the right adjoint to the forgetful functor. The map ϕ_H is then the composite

$$\underline{\mathbb{W}}_{C_{np^k}}(\underline{R}) \xrightarrow{\underline{res}} \mathrm{CoInd}_H^{C_{np^k}}\, i_H^*\underline{\mathbb{W}}_{C_{np^k}}(\underline{R}) \xrightarrow{\mathrm{CoInd}\,\hat{\phi}_H} \mathrm{CoInd}_H^{C_{np^k}}\left(\tilde{E}\mathcal{F}_H i_H^*\underline{\mathbb{W}}_{C_{np^k}}(\underline{R})\right),$$

where the first map, the unit of the adjunction, is a kind of extension to Mackey functors of the restriction map

$$\underline{\mathbb{W}}_{C_{np^k}}(\underline{R})(C_{np^k}/C_{np^k}) \to \underline{\mathbb{W}}_{C_{np^k}}(\underline{R})(C_{np^k}/H).$$

Now if $C_n \subset H \subset C_{np^k}$, then we have a natural isomorphism

$$i_H^*\underline{\mathbb{W}}_{C_{np^k}}(\underline{R}) \cong \underline{\mathbb{W}}_H(\underline{R})$$

by Proposition 4.16, and hence the target of the map is the coinduced H-Green functor

$$\tilde{E}\mathcal{F}_H\underline{\mathbb{W}}_H(\underline{R}) = \pi_H^*\Phi^H\underline{\mathbb{W}}_H(\underline{R}).$$

Proposition 4.14 shows then that the geometric fixed points in question are just the ordinary Hochschild homology HH_0 of the C_n-geometric fixed points of $\underline{R}$.

Remark 4.35. When C_n is the trivial group, this reproduces the foundational work of Brun [9] building Witt vectors via the adjoints to the forgetful functor in Tambara functors. Our Ghost coordinates also are reflected in Brun's work, and we have chosen notation which matches his. The most transparent connection is via the following dictionary: an element

$$(r_H) \in \prod_{(H)} R,$$

where H ranges over the conjugacy classes of subgroups, corresponds to the element

$$\sum_{(H)} tr_H^G n_e^H(r) \in N_e^G(R)(G/G).$$

This is apparent from Brun's treatment; we include it only to help orient the reader.

5. Tambara Functors and Teichmüller Lifts

Classically, the Teichmüller maps are (compatible) multiplicative maps

$$R \to W_n(R)$$

from R to the length n Witt vectors. In this section, we construct analogous maps in the setting of the twisted Hochschild complex for the C_{mn}-Witt vectors. The existence of these maps is closely connected to the existence of a multiplicative structure on $\underline{R}$; although we restrict attention to the case where $\underline{R}$ is a Tambara functor, analogues of our results hold for the incomplete Tambara functors of [5] (i.e., $\mathcal{O}$-commutative monoid objects in Mackey functors).

The basic building block for our work in this section is the following theorem, which constructs precursors that are maps of dg Green functors.

Theorem 5.1. *If $\underline{R}$ is a G-Tambara functor, then for all $H \subset G \subset G' \subsetneq S^1$, we have maps of chain complexes of Green functors*

$$\underline{HC}_*^{G'}\big((N_H^G i_H^* \underline{R})_{W_G(H)}\big) \to \underline{HC}_*^{G'}(\underline{R}).$$

Precomposing with the canonical quotient map gives us maps of chain complexes

$$\underline{HC}_*^{G'}(i_H^* \underline{R}) = \underline{HC}_*^{G'}(N_H^G i_H^* \underline{R}) \to \underline{HC}_*^{G'}(\underline{R}).$$

Proof. Mazur and Hoyer showed that Tambara functors are the G-commutative monoids in the standard G-symmetric monoidal structure on Mackey functors [25, 20]. In particular, this means that we have an extension of the functor

$$T \mapsto N^T \underline{R}$$

from the category of finite G-sets and isomorphisms to the full category of finite G-sets and moreover that this extension is compatible with restriction to subgroups. Restricting attention to $T = G/H$, we have as part of this data a map (necessarily of commutative Green functors)

$$N^{G/H}\underline{R} \xrightarrow{n_H^G} \underline{R},$$

such that for all $g \in W_G(H)$, we have a commutative diagram

$$\begin{tikzcd} N^{G/H}\underline{R} \arrow[r, "n_H^G"] \arrow[d, "g"'] & \underline{R} \\ N^{G/H}\underline{R}. \arrow[ur, "n_H^G"'] & \end{tikzcd}$$

(Note that here we have implicitly used Proposition 2.26 to understand the external action of the Weyl group on the whole Mackey functor). Therefore, we have induced maps

$$N_H^G i_H^* \underline{R} \to \left(N_H^G i_H^* \underline{R}\right)_{W_G(H)} \to \underline{R}$$

of G-Green functors, and the theorem follows by applying $\underline{HC}_*^{G'}(\text{-})$ to these maps. □

The maps of Theorem 5.1 are the antecedent to the Teichmüller maps on the ordinary Witt vectors, linking the values of the dg Green functor together for different orbits G/H. To apply this, we need to recall how the classical norms on a Tambara functor fit into the externalized version.

Proposition 5.2 ([25, 20])**.** *For any $H \subset G$ and for any Tambara functor $\underline{R}$, there is a multiplicative map*

$$N_H^G \colon \underline{R}(G/H) \to N^{G/H}\underline{R}(G/G).$$

If $g \in W_G(H)$, then we have a commutative square

$$\begin{tikzcd} \underline{R}(G/H) \arrow[r, "N_H^G"] \arrow[d, "g"'] & N^{G/H}\underline{R}(G/G) \arrow[d, "g"] \\ \underline{R}(G/H) \arrow[r, "N_H^G"'] & N^{G/H}\underline{R}(G/G). \end{tikzcd}$$

These give the norm maps in the usual definition of a Tambara functor.

We now construct the Teichmüller maps on the C_{mn}-Witt vectors.

Theorem 5.3. *Let $\underline{R}$ be a G-Tambara functor for $G = C_{nm}$ a cyclic group. For all $H \subset G$, we have a natural Teichmüller-style, multiplicative map*

$$\underline{R}(G/H) \to \big(\underline{R}(G/H)\big)_{W_G(H)} \to \big(\underline{\mathrm{HH}}_0^G(\underline{R})\big)(G/G) = \underline{\mathbb{W}}_G(\underline{R})(G/G)$$

lifting the $|G/H|$th power map

$$\underline{R}(G/H) \xrightarrow{r \mapsto r^{|G/H|}} \underline{R}(G/H)_{W_G(H)} = \underline{\mathbb{W}}_G(\underline{R})(G/H).$$

Proof. Since evaluation at a finite G-set T is an exact functor on the category of Mackey functors, and since the Weyl action on $\underline{R}(G/G)$ is always trivial, we have an isomorphism of complexes

$$\big(\underline{HC}_*^G \underline{R}\big)(G/G) \cong HC_*\big(\underline{R}(G/G)\big)$$

and hence

$$\big(\underline{\mathrm{HH}}_0^G(\underline{R})\big)(G/G) \cong \mathrm{HH}_0\left(\underline{R}(G/G)\right).$$

Theorem 5.1 gives us a map of complexes

$$HC_*\big(N^{G/H}\underline{R}(G/G)\big) \to HC_*\big(\underline{R}(G/G)\big),$$

and passing to the zeroth homology group gives a map of rings

$$\mathrm{HH}_0\left(N^{G/H}\underline{R}(G/G)\right) \to \mathrm{HH}_0\left(\underline{R}(G/G)\right).$$

Composing this with the canonical map

$$N^{G/H}\underline{R}(G/G) \to \mathrm{HH}_0\left(N^{G/H}\underline{R}(G/G)\right)$$

and with the norm map from Proposition 5.2 gives the desired map. Since for any $r \in \underline{R}(G/H)$, we have

$$i_H^* N_H^G(r) = \prod_{g \in W_G(H)} r,$$

and since the values of the Witt Green $\underline{\mathbb{W}}_G(\underline{R})$ on G/H is the Weyl orbits in commutative rings, we conclude the desired formula. □

To justify calling these Teichmüller maps, we connect them to the classical construction in Witt vectors. We first need an elementary proposition.

Proposition 5.4. *Let R be a commutative ring and G a finite group, then the commutative Green functor $N_e^G(R)$ has a canonical Tambara functor structure.*

Proof. For a commutative ring R, there is a model of the Eilenberg-MacLane functor HR that is a commutative ring spectrum. Since the norm functor is the left adjoint to the forgetful functor on commutative ring spectra, $N_e^G HR$ is a G-commutative ring orthogonal G-spectrum. Since $\underline{\pi}_0$ is lax G-symmetric monoidal, the result now follows. □

As an immediate corollary, we obtain a natural "Teichmüller lift".

Corollary 5.5. *For any commutative ring R and for any cyclic group $G = C_n$, we have a natural, multiplicative map*

$$R \mapsto \underline{\mathrm{HH}}_0^G(R)(G/G) = \mathbb{W}_{\langle n \rangle}(R).$$

This is a Teichmüller lift in the sense that it provides a section of the G^{th} ghost map from $\mathrm{HH}_0^G(R) \to R$. Moreover, the H-ghost map for any subgroup $H \subset G$ composed with our Teichmüller lift is simply the $[G : H]^{\mathrm{th}}$ power map; this is the defining property of the usual Teichmüller maps.

6. Twisted Hochschild homology and the relative cyclic nerve of C_n-sets

On the way to the computation of the topological cyclic homology of the dual numbers, Hesselholt and Madsen [15] established that if M is a pointed monoid, there is a natural equivalence of cyclotomic spectra

$$\mathrm{THH}(R[M]) \simeq \mathrm{THH}(R) \wedge N^{\mathrm{cyc}}(M).$$

In this section, we prove an analogous statement in the C_n-relative context, giving a similar decomposition for twisted Hochschild homology.

Recall that a pointed monoid in based G-sets is simply a monoid in the category of based G-sets. An important example of pointed G-monoids come from norms of non-equivariant pointed monoids.

Lemma 6.1. *Let M be a (non-equivariant) pointed monoid. Then $N_e^{C_n} M$ is a pointed monoid in C_n-sets.*

For a pointed monoid M and a G-Green functor $\underline{R}$, the pointed monoid algebra $\underline{R}[M]$ is defined as follows.

Definition 6.2. If M is a pointed monoid in C_n-sets, then for any Green functor $\underline{R}$, we can build a new Green functor

$$\underline{R}[M] = \underline{R}_M/\underline{R}_* := \left(\bigoplus_{C_n \cdot m \in M/C_n} \underline{R}_{C_n \cdot m} \right) / \underline{R}_*,$$

and where the multiplication is specified by

$$\underline{R}_{C_n \cdot m} \Box \underline{R}_{C_n \cdot m'} \cong \underline{R}_{C_n \cdot m \times C_n \cdot m'} \to \underline{R}_M/\underline{R}_*,$$

where the last map is simply induced by the multiplication in M.

Remark 6.3. If M is a pointed monoid in C_n-sets, then for any Green functor $\underline{R}$, have a natural isomorphism of Green functors

$$\underline{R}[M] \cong \underline{R} \Box \underline{A}[M].$$

In particular, $\underline{R}[M]$ is also describing the $\underline{R}$-valued Bredon chains on M.

If M is a pointed monoid in C_n-sets, then we can define a relative variant of the cyclic nerve (c.f. [1, 8.1]). As in the topological setting, the relative cyclic nerve is not a cyclic set but has the structure of the n-fold subdivision of a cyclic set; it is a functor over Λ_n^{op} [6, 1.5].

Definition 6.4. Let M be a pointed monoid in C_n-sets. We write $N_{C_n}^{\mathrm{cyc}} M$ to denote the Λ_n^{op}-set that has k-simplices

$$\left(N_{C_n}^{\mathrm{cyc}} M \right)_k = M^{\wedge(k+1)} = N_{C_n}^{C_{n(k+1)}}(M).$$

Here the face maps are given by the unit, the structure maps d_i for $0 \leq i < k$ are the pairwise multiplications, and the last structure map uses the cyclic permutation and then applies the action of γ before multiplying. The additional operator τ_k is specified by the action of $C_{n(k+1)}$ on the indexed smash product.

A Λ_n^{op}-set forgets to a simplicial C_n-set (with the action in degree k generated by τ_k^{k+1}), and so the geometric realization has a C_n-action. Moreover, the geometric realization of a Λ_n^{op}-set is endowed with an S^1-action that extends the C_n action.

When M is non-equivariant, it follows by inspection that we can describe the relative cyclic bar construction of the norm in terms of the edgewise subdivision.

Proposition 6.5. *Let M be a (non-equivariant) pointed monoid. Then there is a natural isomorphism of Λ_n^{op}-sets*

$$N_{C_n}^{cyc} N_e^{C_n} M \cong sd_n N^{cyc} M.$$

The following theorem describes the twisted Hochschild homology of $\underline{R}[M]$.

Theorem 6.6. *Fix a cyclic group $G = C_n$. Let M be a pointed monoid in G-sets and $\underline{R}$ a G-Green functor. Then we have a natural weak equivalence*

$$\underline{HC}_*^G(\underline{R}) \Box \underline{C}_*^{cell}(N_G^{cyc} M; \underline{A}) \to \underline{HC}_*^G(\underline{R}[M]),$$

where here we are regarding $N_G^{cyc} M$ as a simplicial G-set and $\underline{C}_*^{cell}$ denotes the Mackey extension of the Bredon cellular chains.

Proof. The key point is that there is a natural equivalence

$$C_*^{cell}(N_G^{\mathrm{cyc}} M; \underline{A}) \simeq \underline{HC}_*^G(\underline{A}[M]),$$

since the cellular chains takes the cartesian product to the box product of Mackey functors and $C_*^{cell}(M; \underline{A}) \cong \underline{A}[M]$. Remark 6.3 implies that the natural map

$$\underline{HC}_*^G(\underline{R}) \Box C_*^{cell}(N_G^{\mathrm{cyc}} M; \underline{A}) \cong \underline{HC}_*^G(\underline{R}) \Box \underline{HC}_*^G(\underline{A}[M]) \to \underline{HC}_*^G(\underline{R}[M]),$$

where the last map is induced by the levelwise box product, is a weak equivalence. □

7. Shukla Homology

Whereas THH is constructed as a derived functor, the classical definition of Hochschild homology is not derived. As a consequence, the hypothesis of flatness of the input over the base ring is often necessary; for example, such hypotheses are standard when considering the identification of the E_2 term of the Bökstedt spectral sequences in terms of Hochschild homology. It is sometimes convenient to work with the derived analogue of Hochschild homology; here we take a simplicial free resolution of the input ring and apply Hochschild homology. This construction is traditionally called Shukla homology. For example, see [7] and [27] for applications of Shukla homology.

There is a natural extension of the twisted Hochschild homology of Green functors to a twisted analogue of Shukla homology. Unfortunately, there are complications arising from difficulties in the homological algebra of Green functors. Specifically, we need to drop the assumption that we are working with commutative rings or Green functors, as the free commutative ring objects fail to be flat over the Burnside ring.

Proposition 7.1. *The symmetric powers of a projective Mackey functor need not be flat.*

Proof. Let $G = C_2$, and let $\underline{P} = \underline{A}_{C_2}$. Then

$$\mathrm{Sym}^2(\underline{P}) = \underline{A}_{C_2} \oplus \underline{\mathbb{Z}}^*.$$

The module $\underline{\mathbb{Z}}^*$ has infinite Tor dimension. □

Remark 7.2. For a general finite group G, if our projective is of the form $\underline{P} = \underline{A}_{G/H}$ for H a proper subgroup, then symmetric powers $\mathrm{Sym}^n \underline{P}$ might not be projective for n dividing $[G : H]$. This is exactly the issue that the G-symmetric monoidal structure in equivariant stable homotopy fixes: the non-projective summands are essentially just "missing" the norm classes which we would see if we computed $\underline{\pi}_0 \mathrm{Sym}^n(G/H_+)$.

To avoid the difficulties that arise in the commutative case, we instead simply work with associative Green functors.

Proposition 7.3. *If $\underline{P}$ is a projective Mackey functor, then the free associative Green functor on $\underline{P}$, the box-tensor algebra $T(\underline{P})$, is a projective object in Mackey functors.*

Proof. The tensor algebra in Mackey functors is simply

$$T(\underline{M}) = \bigoplus_{n \in \mathbb{N}} \underline{M}^{\Box n}.$$

By functoriality, if $\underline{P}$ is a direct summand of $\bigoplus_i \underline{A}_{S_i}$, with each S_i a finite G-set, then $T(\underline{P})$ is a direct summand of $T(\bigoplus_i \underline{A}_{S_i})$. It suffices therefore to show that $T(\underline{A}_S)$ is projective for any finite G-set S. Since these are representable Mackey functors, we have natural isomorphisms

$$\underline{A}_S^{\Box n} \cong \underline{A}_{n\cdot S},$$

where $n \cdot S$ is the disjoint union of n copies of S. $\Box$

Corollary 7.4. *Let $\underline{R}$ be an associative Green functor and let $\tilde{\underline{R}}_\bullet$ be a simplicial resolution of $\underline{R}$ by free associative Green functors. Then the underlying simplicial Mackey functor for $\tilde{\underline{R}}_\bullet$ is cofibrant.*

Since Corollary 2.50 shows that the norm of Mackey functors preserves weak equivalences between cofibrant objects, this also allows us to build homotopically meaningful resolutions of $N_H^G \underline{R}$.

Corollary 7.5. *Let $\underline{R}$ and $\underline{R}'$ be associative Green functors for H such that there is a weak equivalence $\underline{R} \to \underline{R}'$. Then for any two simplicial resolutions $\tilde{\underline{R}}_\bullet$ and $\tilde{\underline{R}}'_\bullet$ of $\underline{R}$ and $\underline{R}'$ by free associative Green functors, there is a zigzag of weak equivalences*

$$N_H^G \tilde{\underline{R}}_\bullet \simeq N_H^G \tilde{\underline{R}}'_\bullet.$$

Remark 7.6. Since the norm commutes with sifted colimits, we know that the zeroth homology group of $N_H^G \tilde{\underline{R}}_\bullet$ is actually $N_H^G \underline{R}$. We make no claim, however, that the higher homology groups vanish. Moreover, this is almost never a resolution by free associative Green functors.

The cofibrancy of the underlying simplicial Mackey functor for the simplicial Green functor $N_H^G \tilde{\underline{R}}_\bullet$ is all we need to define our Shukla complex, however.

Definition 7.7. Let $\underline{R}$ be an associative Green functor for $H \subset G$, a finite subgroup of S^1, and let $\tilde{\underline{R}}_\bullet$ be a simplicial resolution of $\underline{R}$ by free associative Green functors. Then the G-Shukla complex of $\underline{R}$ is the complex associated to the simplicial Mackey functor

$$\big(N_H^G \tilde{\underline{R}}_\bullet\big) \underset{\left(N_H^G \tilde{\underline{R}}_\bullet\right)^e}{\Box} {}^{\gamma}\big(N_H^G \tilde{\underline{R}}_\bullet\big),$$

where γ is the generator of G.

Recall that Definition 2.16 describes the norm in Mackey functors via the norm in spectra. We can describe relative norms of Mackey functor modules over Green functors in this way as well. Since $H\underline{A}$ is a genuine equivariant commutative ring spectrum, the category of $H\underline{A}$ modules has internal norms (e.g., see [16, 4] and [1, 2.20, §6]): if E is an $i_H^* H\underline{A}$-module,

$${}_{H\underline{A}}N_H^G E := H\underline{A} \underset{N_H^G i_H^* H\underline{A}}{\wedge} N_H^G E,$$

where the map $N_H^G i_H^* H\underline{A} \to H\underline{A}$ is the counit of the adjunction. Choosing a cofibrant model for $H\underline{A}$, this internal norm is homotopical (see [1, 6.11] for analogous arguments) in the sense that it preserves weak equivalences between cofibrant objects. Moreover, a straightforward extension of the work of [25, 20] establishes a correspondence between π_0 of the relative norm and the algebraic construction. This structure gives another way to interpret the Shukla homology of $\underline{R}$.

Proposition 7.8. *The Shukla homology groups of $\underline{R}$ are isomorphic to the homotopy groups of the derived smash product*

$$H\underline{R} \underset{H\underline{R} \wedge_{H\underline{A}} H\underline{R}}{\wedge} H\underline{R}.$$

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