# An introduction to Bousfield localization

Tyler Lawson

March 11, 2019

## 1 Introduction

Bousfield localization encodes a wide variety of constructions in homotopy theory, analogous to localization and completion in algebra. Our goal in this chapter is to give an overview of Bousfield localization, sketch how basic results in this area are proved, and illustrate some applications of these techniques. Near the end we will give more details about how localizations are constructed using the small object argument. The underlying methods apply in many contexts, and we have attempted to provide a variety of examples that exhibit different behavior.

We will begin by discussing categorical localizations. Given a collection of maps in a category, the corresponding localization of that category is formed by making these maps invertible in a universal way; this technique is often applied to discard irrelevant information and focus on a particular type of phenomenon. In certain cases, localization can be carried out internally to the category itself: this happens when there is a sufficiently ample collection of objects that already see these maps as isomorphisms. This leads naturally to the study of reflective localizations.

Bousfield localization generalizes this by taking place in a category where there are *spaces* of functions, rather than sets, with uniqueness only being true up to contractible choice. Bousfield codified these properties, for spaces in [Bou75] and for spectra in [Bou79]. The definitions are straightforward, but proving that localizations exist takes work, some of it of a set-theoretic nature.

Our presentation is close in spirit to Bousfield's work, but the reader should go to the books of Farjoun [Far96] and Hirschhorn [Hir03] for more advanced information on this material. We will focus, for the most part, on *left* Bousfield localization, since the techniques there are easier and is where most of our applications lie. In [Bar10] right Bousfield localization is discussed at more length.

### 1.1 Historical background

The story of localization techniques in algebraic topology probably begins with Serre classes of abelian groups [Ser53]. After choosing a class $\mathcal{C}$ of abelian groups that is closed under subobjects, quotients, and extensions, Serre showed that one could effectively ignore groups in $\mathcal{C}$ when studying the homology and homotopy of a simply-connected space $X$. In particular, he proved mod-$\mathcal{C}$ versions of the Hurewicz and Whitehead theorems, showed the equivalence between finite generation of homology and homotopy groups, determined the rational homotopy groups of spheres, and significantly reduced the technical overhead in computing the torsion in homotopy groups by allowing one to work with only one prime at a time. His techniques for computing rational homotopy groups only require rational homology groups; $p$-local homotopy groups only require $p$-local homology groups; $p$-completed homotopy groups only require mod-$p$ homology groups.

These techniques received a significant technical upgrade in the late 1960's and early 1970's, starting with the work of Quillen on rational homotopy theory [Qui69b] and work of Sullivan and Bousfield–Kan on localization and completion of spaces [Sul05, Sul74, BK72]. Rather than using Serre's algebraic techniques to break up the homotopy groups $\pi_* X$ and homology groups $H_* X$ into localizations and completions, their insight was that *space-level* versions of these constructions provided a more robust theory. For example, a simply-connected space $X$ has an associated space $X_{\mathbb{Q}}$ whose homotopy groups and (positive-degree) homology groups are, themselves, rational homotopy and homology groups of $X$; similarly for Sullivan's $p$-localization $X_{(p)}$ and $p$-completion $X_p^\wedge$. Without this, each topological tool requires a proof that it is compatible with Serre's mod-$\mathcal{C}$-theory, such as Serre's mod-$\mathcal{C}$ Hurewicz and Whitehead theorems or mod-$\mathcal{C}$ cup products. Now these are simply consequences of the Hurewicz and Whitehead theorems applied to $X_{\mathbb{Q}}$, and any subsequent developments will automatically come along. Moreover, Sullivan pioneered arithmetic fracture techniques that allowed $X$ to be recovered from its rationalization $X_{\mathbb{Q}}$ and its $p$-adic completions $X_p^\wedge$ via a homotopy pullback diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod_p X_p^\wedge \arrow[d] \\ X_{\mathbb{Q}} \arrow[r, "\alpha"'] & (\prod_p X_p^\wedge)_{\mathbb{Q}} \end{tikzcd}$$

This allows us to reinterpret homotopy theory. We are no longer using rationalization and completion just to understand algebraic invariants of $X$: instead, knowledge of $X$ is equivalent to knowledge of its localizations, completions, and an "arithmetic attaching map" $\alpha$. This entirely changed both the way theorems are proved and the way that we think about the subject. Later, work of Morava, Ravenel, and others made extensive use of localization techniques [Mor85, Rav84], which today gives an explicit decomposition of the stable homotopy category into layers determined by Quillen's relation to the structure theory of formal group laws [Qui69a].

Many of the initial definitions of localization and completion were constructive. One can build $X_{\mathbb{Q}}$ from $X$ by showing that one can replace the basic cells $S^n$ in a CW-decomposition with rationalized spheres $S^n_{\mathbb{Q}}$, or by showing that the Eilenberg–Mac Lane spaces $K(A, n)$ in a Postnikov decomposition can be replaced by rationalized versions $K(A \otimes \mathbb{Q}, n)$. One can instead use Bousfield and Kan's more functorial, but also more technical, construction as the homotopy limit of a cosimplicial space. Quillen's work gives more, in the form of a model structure whose weak equivalences are isomorphisms on rational homology groups. In his work, the map $X \to X_{\mathbb{Q}}$ is a fibrant replacement, and the essential uniqueness of fibrant replacements means that

$X_{\mathbb{Q}}$ has a form of universality. It is this universal property that Bousfield localization makes into a definition.

### 1.2 Notation

We will use $\mathcal{S}$ to denote an appropriately convenient category of spaces (one can use simplicial sets, but with appropriate modifications throughout) with internal function objects. We similarly write $\mathcal{S}p$ for a category of spectra.

Throughout this paper we will often be working in categories enriched in spaces: for any $X$ and $Y$ in $\mathcal{C}$ we will write $\mathrm{Map}_{\mathcal{C}}(X,Y)$ for the mapping space, or just $\mathrm{Map}(X,Y)$ if the ambient category is understood. Letting $[X,Y] = \pi_0 \mathrm{Map}_{\mathcal{C}}(X,Y)$, we obtain an ordinary category called the ***homotopy category*** $h\mathcal{C}$. Two objects in $\mathcal{C}$ are ***homotopy equivalent*** if and only if they become isomorphic in $h\mathcal{C}$.

For us, homotopy limits and colimits in the category of spaces are given by the descriptions of Vogt or Bousfield–Kan [Vog73, BK72]. A homotopy limit or homotopy colimit in $\mathcal{C}$ is characterized by having a natural weak equivalence of spaces:

$$\mathrm{Map}_{\mathcal{C}}(X, \operatorname*{holim}_{J} Y_j) \simeq \operatorname*{holim}_{J} \mathrm{Map}_{\mathcal{C}}(X, Y_j)$$

$$\mathrm{Map}_{\mathcal{C}}(\operatorname*{hocolim}_{I} X_i, Y) \simeq \operatorname*{holim}_{I} \mathrm{Map}_{\mathcal{C}}(X_i, Y)$$

In particular, since homotopy limit constructions on spaces preserve objectwise weak equivalences of diagrams, homotopy limits and colimits also preserve objectwise homotopy equivalences in $\mathcal{C}$.

Some set theory is unavoidable, but we will not spend a great deal of time with it. For us, a *collection* or *family* may be a proper class, rather than a set. Categories will be what are sometimes called *locally small* categories: the collection of objects may be large, but there is a set of maps between any pair of objects.


### 1.3 Acknowlegements

The author would like to thank Clark Barwick and Thomas Nikolaus for discussions related to localizations of spaces.

Th author was partially supported by NSF grant 1610408 and a grant from the Simons Foundation. The author would like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the programme HHH when work on this paper was undertaken. This work was supported by: EPSRC grant numbers EP/K032208/1 and EP/R014604/1.


## 2 Motivation from categorical localization

In general, we recall that for an ordinary category $\mathcal{A}$ and a class $\mathcal{W}$ of the maps called *weak equivalences* (or simply *equivalences*), we can attempt to construct a categorical localization $\mathcal{A} \to \mathcal{A}[\mathcal{W}^{-1}]$. This localization is universal among functors $\mathcal{A} \to \mathcal{D}$

that send the maps in $\mathcal{W}$ to isomorphisms. The category $\mathcal{A}[\mathcal{W}^{-1}]$ is unique up to isomorphism if it exists.[1]

*Example* 2.1. We will begin by remembering the case of the category $\mathcal{S}$ of spaces, with $\mathcal{W}$ the class of weak homotopy equivalences. The projection $p\colon X \times [0,1] \to X$ is always a weak equivalence with homotopy inverses $i_t$ given by $i_t(x) = (x,t)$. In the localization, we find that homotopic maps are equal: for a homotopy $H$ from $f$ to $g$, we have $f = Hi_0 = Hp^{-1} = Hi_1 = g$. Therefore, localization factors through the homotopy category $h\mathcal{S}$.

However, within the category of spaces we have a collection with special properties: the subcategory $\mathcal{S}^{CW}$ of CW-complexes. For any CW-complex $K$, weak equivalences $X \to Y$ induce bijections $[K,X] \to [K,Y]$—this can be proved, for example, inductively on the cells of $K$—and any space $X$ has a CW-complex $K$ with a weak homotopy equivalence $K \to X$. These two properties show, respectively, that the composite

$$h\mathcal{S}^{CW} \to h\mathcal{S} \to \mathcal{S}[\mathcal{W}^{-1}]$$

is fully faithful and essentially surjective. Within the homotopy category $h\mathcal{S}$ we have found a large enough library of special objects, and localization can be done by forcibly moving objects into this subcategory.[2]

*Example* 2.2. A similar example to the above occurs in the category $\mathcal{K}_R$ of nonnegatively graded cochain complexes of modules over a commutative ring $R$, with $\mathcal{W}$ the class of quasi-isomorphisms. Within $\mathcal{K}_R$ there is a subcategory $\mathcal{K}_R^{Inj}$ of complexes of injective modules. Fundamental results of homological algebra show that for a quasi-isomorphism $A \to B$ and a complex $Q$ of injectives, there is a bijection $[B,Q] \to [A,Q]$ of chain homotopy classes of maps, and that any complex $A$ has a quasi-isomorphism $A \to Q$ to a complex of injectives. This similarly shows that the composite functor

$$h\mathcal{K}_R^{Inj} \to h\mathcal{K}_R \to \mathcal{K}_R[\mathcal{W}^{-1}]$$

is an equivalence of categories.

These examples are at the foundation of Quillen's theory of model categories, and we will return to examples like them when we discuss localization of model categories.

# 3 Local objects in categories

In this section we will fix an ordinary category $\mathcal{A}$.

[1]For the record, this category also satisfies a 2-categorical universal property: for any $\mathcal{D}$, the map of functor categories

$$\mathrm{Fun}(\mathcal{A}[\mathcal{W}^{-1}],\mathcal{D}) \to \mathrm{Fun}(\mathcal{A},\mathcal{D})$$

is fully faithful, and the image consists of those functors sending $\mathcal{W}$ to isomorphisms. If we replace "image" with "essential image" in this description, we recover a universal property characterizing $\mathcal{A} \to \mathcal{A}[\mathcal{W}^{-1}]$ up to equivalence of categories rather than up to isomorphism.

[2]Technically speaking, we often use a result like this to actually show that $\mathcal{S}[\mathcal{W}^{-1}]$ exists.

**Definition 3.1.** Let $S$ be a class of morphisms in $\mathcal{A}$. An object $Y \in \mathcal{A}$ is $S$*-local* if, for all $f : A \to B$ in $S$, the map

$$\mathrm{Hom}_{\mathcal{A}}(B, Y) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{A}}(A, Y)$$

is a bijection. We write $L^S(\mathcal{A})$ for the full subcategory of $S$-local objects.

If $S = \{f : A \to B\}$ consists of just one map, we simply refer to this property as being $f$*-local* and write $L^f(\mathcal{A})$ for the category of $f$-local objects.

*Remark* 3.2. If $S = \{f_\alpha : A_\alpha \to B_\alpha\}$ is a set and $\mathcal{A}$ has coproducts indexed by $S$, then by defining $f = \coprod_\alpha f_\alpha : \coprod A_\alpha \to \coprod B_\alpha$ we find that $S$-local objects are equivalent to $f$-local objects.

A special case of localization is when our maps in $S$ are maps to a terminal object.

**Definition 3.3.** Suppose $S$ is a class of maps $\{W_\alpha \to *\}$, where $*$ is a terminal object. In this case, we refer to such a localization as a ***nullification*** of the objects $W_\alpha$.

*Remark* 3.4. Nullification often takes place when $\mathcal{A}$ is pointed. If $S$ is a set, $\mathcal{A}$ is pointed, and $\mathcal{A}$ has coproducts, then any coproduct of copies of $*$ is again $*$ and we can again replace nullification of a set of objects with nullification of an individual object.

**Definition 3.5.** A map $A \to B$ in $\mathcal{A}$ is an $S$*-equivalence* if, for all $S$-local objects $Y$, the map

$$\mathrm{Hom}_{\mathcal{A}}(B, Y) \to \mathrm{Hom}_{\mathcal{A}}(A, Y)$$

is a bijection.

The class of $S$-equivalences contains $S$ by definition.

**Definition 3.6.** A map $X \to Y$ is an $S$*-localization* if it is an $S$-equivalence and $Y$ is $S$-local, and under these conditions we say that $X$ *has an* $S$*-localization.* If all objects in $\mathcal{A}$ have $S$-localizations, we say that $\mathcal{A}$ *has* $S$*-localizations.*

**Proposition 3.7.** *Any two* $S$*-localizations* $f_1 : X \to Y_1$ *and* $f_2 : X \to Y_2$ *are isomorphic under* $X$ *in* $\mathcal{A}$.

*Proof.* Because $Y_i$ are $S$-local, $\mathrm{Hom}(B, Y_i) \to \mathrm{Hom}(A, Y_i)$ is always an isomorphism for any $S$-equivalence $A \to B$. Applying this to the $S$-equivalences $X \to Y_j$, we get isomorphisms $\mathrm{Hom}(Y_j, Y_i) \to \mathrm{Hom}(X, Y_i)$ in $\mathcal{A}$: any map $X \to Y_i$ has a unique extension to a map $Y_j \to Y_i$. Existence allows us to find maps $Y_1 \to Y_2$ and $Y_2 \to Y_1$ under $X$, and uniqueness allows us to conclude that these two maps are inverse to each other in $\mathcal{A}$.

More concisely, $Y_1$ and $Y_2$ are both initial objects in the comma category of $S$-local objects under $X$ in $\mathcal{A}$, and this universal property forces them to be isomorphic. □

As a result, it is reasonable to call such an object *the* $S$-localization of $X$ and write it as $L^S X$ (or simply $LX$ if $S$ is understood). More generally than this, if $X \to LX$ and $X' \to LX'$ are $S$-localization maps, any map $X \to X'$ in $\mathcal{A}$ extends uniquely to a commutative square. This is encoded by the following result.

**Proposition 3.8.** *Let $Loc^S(\mathcal{A})$ be the category of* localization morphisms, *whose objects are $S$-localization maps $X \to LX$ in $\mathcal{A}$ and whose morphisms are commuting squares. Then the forgetful functor*

$$Loc^S(\mathcal{A}) \to \mathcal{A},$$

*sending $(X \to LX)$ to $X$, is fully faithful. The image consists of those objects $X$ that have $S$-localizations.*

**Proposition 3.9.** *The collection of $S$-local objects is closed under limits, and the collection of $S$-equivalences is closed under colimits.*

*Proof.* If $f: A \to B$ is in $S$ and $\{Y_j\}$ is a diagram of $S$-local objects, then

$$\mathrm{Hom}(B, Y_j) \to \mathrm{Hom}(A, Y_j)$$

is a diagram of isomorphisms, and taking limits we find that we have an isomorphism

$$\mathrm{Hom}(B, \lim_J Y_j) \to \mathrm{Hom}(A, \lim_J Y_j).$$

Since $A \to B$ was an arbitrary map in $S$, this shows that $\lim_J Y_j$ is $S$-local.

Similarly, if $\{A_i \to B_i\}$ is a diagram of $S$-equivalences and $Y$ is $S$-local, then

$$\mathrm{Hom}(B_i, Y) \to \mathrm{Hom}(A_i, Y)$$

is a diagram of isomorphisms, and taking limits we find that

$$\mathrm{Hom}(\operatorname{colim}_I B_i, Y) \to \mathrm{Hom}(\operatorname{colim}_I A_i, Y)$$

is also an isomorphism. Since $Y$ was an arbitrary local object, this shows that the map $\operatorname{colim}_I A_i \to \operatorname{colim}_I B_i$ is an $S$-equivalence. □

*Example* 3.10. Consider the map $f: \mathbb{N} \to \mathbb{Z}$ in the category of monoids. A monoid $M$ is $f$-local if and only if any monoid homomorphism $\mathbb{N} \to M$ automatically extends to a homomorphism $\mathbb{Z} \to M$, which is the same as asking that every element in $M$ has an inverse. Therefore, $f$-local monoids are precisely *groups*. The natural transformation $M \to M^{gp}$, from a monoid to its group completion, is an $f$-localization.

*Example* 3.11. Consider the map $f: F_2 \to \mathbb{Z}^2$, from a free group on two generators $x$ and $y$ to its abelianization. A group $G$ is $f$-local if and only if *every* homomorphism $F_2 \to G$, equivalent to choosing a pair of elements $x$ and $y$ of $G$, can be factored through $\mathbb{Z}^2$, which happens exactly when the commutator $[x, y]$ is sent to the trivial element. Therefore, $f$-local groups are precisely *abelian* groups. The natural transformation $G \to G_{ab}$, from a group to its abelianization, is an $f$-localization.

These two localizations are left adjoints to the inclusion of a subcategory, and this phenomenon is completely general.

**Proposition 3.12**. *Let $S$ be a class of morphisms in $\mathcal{A}$, and suppose that $\mathcal{A}$ has $S$-localizations. Then the inclusion $L^S\mathcal{A} \to \mathcal{A}$ is part of an adjoint pair*

$$\mathcal{A} \overset{L}{\rightleftarrows} L^S\mathcal{A}.$$

*As a result, $L$ is a reflective localization onto the subcategory $L^S\mathcal{A}$.*

*Proof.* In this situation, the functor $Loc^S(\mathcal{A}) \to \mathcal{A}$ is fully faithful and surjective on objects. Therefore, it is an equivalence of categories and we can choose an inverse,[3] functorially sending $X$ to a pair $(X \to LX)$ in $Loc^S(\mathcal{A})$. The composite functor sending $X$ to $LX$ is the desired left adjoint. □

*Remark* 3.13. Embedding the category $\mathcal{A}$ as a full subcategory of a larger category can change localization drastically. Consider a set $S$ of maps in $\mathcal{A} \subset \mathcal{B}$. Then the $S$-local objects of $\mathcal{A}$ are simply the $S$-local objects of $\mathcal{B}$ that happen to be in $\mathcal{A}$, but because there may be more local objects in $\mathcal{B}$ there may be fewer $S$-equivalences in $\mathcal{B}$ than in $\mathcal{A}$. Localization in $\mathcal{B}$ may not preserve objects of $\mathcal{A}$; a localization map in $\mathcal{A}$ might not be an equivalence in $\mathcal{B}$; there might, in general, be no comparison map between the two localizations.

For example, consider the set $S$ of multiplication-by-$p$ maps $\mathbb{Z} \to \mathbb{Z}$ (as $p$ ranges over primes) in the category of finitely generated abelian groups, considered as a full subcategory of all abelian groups. An abelian group is $S$-local if and only if it is a rational vector space, and the only finitely generated group of this form is trivial. A map $A \to B$ of finitely generated abelian groups is an $S$-equivalence in the larger category of all abelian groups if and only if it induces an isomorphism $A \otimes \mathbb{Q} \to B \otimes \mathbb{Q}$, whereas it is *always* an equivalence within the smaller category of finitely generated abelian groups. Within all abelian groups, $S$-localization is rationalization, whereas within finitely generated abelian groups, $S$-localization takes all groups to zero.

# 4 Localization using mapping spaces

We now consider the case where $\mathcal{C}$ is a category enriched in spaces. The previous definitions and results apply perfectly well to the homotopy category $h\mathcal{C}$. The following illustrates that the homotopy category may be an inappropriate place to carry out such localizations.

*Example* 4.1. Let us start with the homotopy category of spaces $h\mathcal{S}$, and fix an $n \geq 0$. Suppose that we want to invert the inclusion $S^n \to D^{n+1}$. We fairly readily find that any space $X$ has a map $X \to X'$ such that $[D^{n+1}, X'] \to [S^n, X']$ is an isomorphism: construct $X'$ by attaching $(n+1)$-dimensional cells to $X$ until the $n$'th homotopy group $\pi_n(X', x) = 0$ is trivial at any basepoint.

[3] If the category $\mathcal{A}$ is large then we need to be a little bit more honest here, and worry about whether a fully faithful and essentially surjective map between large categories has an inverse equivalence. This depends on our model for set theory: it is asking for us to make a distinguished choice of objects for our inverse functor, which may require an axiom of choice for proper classes. It is an awkward situation, because choosing these inverses isn't categorically interesting unless we can't do it.

However, this construction lacks ***universality***. If $Y$ is any other space whose $n$'th homotopy groups are trivial, then any map $X \to Y$ can be extended to a map $X' \to Y$ because the attaching maps for the cells of $X'$ are trivial. However, this extension is ***not unique*** up to homotopy: any two extensions $D^{n+1} \to X' \to Y$ of a cell $S^n \to X \to Y$ glue together to an obstruction class in $[S^{n+1}, Y]$. As a result, if we construct two spaces $X'$ and $X''$ as attempted localizations of $X$, we can find maps $X' \to X''$ and $X'' \to X'$ but cannot establish that they are mutually inverse in the homotopy category.

In short, in order for $Y$ to have ***uniqueness*** for filling maps from $n$-spheres, we have to have ***existence*** for filling maps from $(n+1)$-spheres. Thus, to make this localization work canonically we would need to enlarge our class $S$ to contain $S^{n+1} \to D^{n+2}$. The same argument then repeats, showing that a canonical localization for $S$ requires that $S$ also contain $S^m \to D^{m+1}$ for $m \geq n$.

The example in the previous section leads to the following principle. In our definitions, we must replace isomorphism on the path components of mapping spaces with homotopy equivalence.

**Definition 4.2**. Let $S$ be a class of morphisms in the category $\mathcal{C}$. An object $Y \in \mathcal{C}$ is ***S-local*** if, for all $f\colon A \to B$ in $S$, the map

$$\mathrm{Map}_{\mathcal{C}}(B, Y) \xrightarrow{f^*} \mathrm{Map}_{\mathcal{C}}(A, Y)$$

is a weak equivalence.[4] We write $L^S(\mathcal{C})$ for the full subcategory of $S$-local objects.

If $S = \{f\colon A \to B\}$ consists of just one map, we simply refer to this property as being ***f-local*** and write $L^f(\mathcal{C})$ for the category of $f$-local objects.

**Definition 4.3**. A map $A \to B$ in $\mathcal{C}$ is an ***S-equivalence*** if, for all $S$-local objects $Y$, the map

$$\mathrm{Map}_{\mathcal{C}}(B, Y) \to \mathrm{Map}_{\mathcal{C}}(A, Y)$$

is a weak equivalence.

**Definition 4.4**. A map $X \to Y$ is an ***S-localization*** if it is an $S$-equivalence and $Y$ is $S$-local, and under these conditions we say that *$X$ has an $S$-localization*. If all objects in $\mathcal{C}$ have $S$-localizations, we say that *$\mathcal{C}$ has $S$-localizations*.

By applying $\pi_0$ to mapping spaces, we find that some of this passes to the homotopy category.

**Proposition 4.5**. *Let $\bar{S}$ be the image of $S$ in the homotopy category $h\mathcal{C}$. If $Y$ is $S$-local in $\mathcal{C}$, then its image in the homotopy category $h\mathcal{C}$ is $\bar{S}$-local.*

*Remark* 4.6. An $S$-equivalence in $\mathcal{C}$ does not necessarily becomes an $\bar{S}$-equivalence in $h\mathcal{C}$ because there is potentially a larger supply of $\bar{S}$-local objects.

[4]Note that the homotopy class of the map $\mathrm{Map}_{\mathcal{C}}(B, Y) \to \mathrm{Map}_{\mathcal{C}}(A, Y)$ only depends on the image of $f\colon A \to B$ in the homotopy category $h\mathcal{C}$, and so we may simply view $S$ as a collection of representatives for a class of maps $\bar{S}$ in $h\mathcal{C}$.

**Proposition 4.7.** *Any two $S$-localizations $f_1\colon X \to Y_1$ and $f_2\colon X \to Y_2$ become isomorphic under $X$ in the homotopy category $h\mathcal{C}$.*

*Proof.* This proceeds exactly as in the proof of Proposition 3.7. Applying $\mathrm{Map}_{\mathcal{C}}(-, Y_i)$ to the $S$-equivalence $X \to Y_j$, we find that the maps $X \to Y_i$ extend to maps $Y_j \to Y_i$ which are unique up to homotopy. By first taking $i \neq j$ we construct maps between the $Y_i$ whose restrictions to $X$ are homotopic to the originals, and taking $i = j$ shows that the double composites are homotopic under $X$. □

*Remark* 4.8. At this point it would be very useful to show that, if they exist, localizations can be made functorial in the spirit of Proposition 3.8. There is typically no easy way to produce a functorial localization because many choices are made up to homotopy equivalence, and this leads to coherence issues: for example, if we have a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ LX \arrow[r, dotted] & LX' \end{tikzcd}$$

where the vertical maps are $S$-localization, then we can construct at best the dotted map together with a *homotopy* between the two double composites. Larger diagrams do get more extensive families of homotopies, but these take work to describe. This is a *rectification problem* and in general it is not solvable without asking for more structure on $\mathcal{C}$. The small object argument, which we will discuss in §6, can often be done carefully enough to give some form of functorial construction of the localization.

**Proposition 4.9.** *The following properties hold for a class $S$ of morphisms in $\mathcal{C}$.*

1. *The collection of $S$-local objects is closed under equivalence in the homotopy category.*
2. *The collection of $S$-equivalences is closed under equivalence in the homotopy category.*
3. *The collection of $S$-local objects is closed under homotopy limits.*
4. *The collection of $S$-equivalences is closed under homotopy colimits.*
5. *The homotopy pushout of an $S$-equivalence is an $S$-equivalence.*
6. *The $S$-equivalences satisfy the two-out-of-three axiom: given maps $A \xrightarrow{f} B \xrightarrow{g} C$, if any two of $f$, $g$, and $gf$ are $S$-equivalences then so is the third.*

*Proof.* If $X \to Y$ becomes an isomorphism in the homotopy category, then one can choose an inverse map and homotopies between the double composites. Composing with these makes $\mathrm{Map}_{\mathcal{C}}(-, X) \to \mathrm{Map}_{\mathcal{C}}(-, Y)$ a homotopy equivalence of functors on $\mathcal{C}$, and so $X$ is $S$-local if and only if $Y$ is.

Similarly, if two maps $f\colon A \to B$ and $f'\colon A' \to B'$ become isomorphic in the homotopy category, there exist homotopy equivalences $A' \to A$ and $B \to B'$ such that the composite $A' \to A \to B \to B'$ is homotopic to $f'$, and applying $\mathrm{Map}_{\mathcal{C}}(-, Y)$ we obtain the desired result.

If $f\colon A \to B$ is in $S$ and $\{Y_j\}$ is a diagram of $S$-local objects, then

$$\mathrm{Map}_{\mathcal{C}}(B, Y_j) \to \mathrm{Map}_{\mathcal{C}}(A, Y_j)$$

is a diagram of weak equivalences of spaces, and taking homotopy limits we find that we have an equivalence

$$\mathrm{Map}_{\mathcal{C}}(B, \operatorname*{holim}_J Y_j) \to \mathrm{Map}_{\mathcal{C}}(A, \operatorname*{holim}_J Y_j).$$

Since $A \to B$ was an arbitrary map in $S$, this shows that $\mathrm{holim}_J Y_j$ is $S$-local.

Similarly, if $\{A_i \to B_i\}$ is a diagram of $S$-equivalences and $Y$ is $S$-local, then

$$\mathrm{Map}_{\mathcal{C}}(B_i, Y) \to \mathrm{Map}_{\mathcal{C}}(A_i, Y)$$

is a diagram of weak equivalences of spaces, and so

$$\mathrm{Map}_{\mathcal{C}}(\operatorname*{hocolim}_I B_i, Y) \to \mathrm{Map}_{\mathcal{C}}(\operatorname*{hocolim}_I A_i, Y)$$

is also a weak equivalence. Since $Y$ was an arbitrary $S$-local object, this shows that the map $\mathrm{hocolim}_I A_i \to \mathrm{hocolim}_I B_i$ is an $S$-equivalence.

Suppose that we have a homotopy pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

where $f\colon A \to B$ is an $S$-equivalence. Given any $S$-local object $Y$, we get a homotopy pullback diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(A, Y) & \mathrm{Map}_{\mathcal{C}}(B, Y) \arrow[l] \\ \mathrm{Map}_{\mathcal{C}}(A', Y) \arrow[u] & \mathrm{Map}_{\mathcal{C}}(B', Y) \arrow[l] \arrow[u] \end{tikzcd}$$

The top arrow is an equivalence by the assumption that $f$ is an $S$-equivalence, and hence the bottom arrow is an equivalence. Since $Y$ was an arbitrary $S$-local object, we find that $f'$ is an $S$-equivalence.

The 2-out-of-3 property is obtained by first applying $\mathrm{Map}_{\mathcal{C}}(-, Y)$ to the diagram $A \to B \to C$ and then using the 2-out-of-3 axiom for weak equivalences. $\square$

If we expand a class $S$ to a larger class $T$ of equivalences, our work so far gives us an automatic relation between $S$-localization and $T$-localization.

**Proposition 4.10**. *Suppose that $S$ and $T$ are classes of morphisms such that every map in $S$ is a $T$-equivalence. Then the following properties hold.*

1. *Every $T$-local object is also $S$-local.*

2. *Every $S$-equivalence is also a $T$-equivalence.*

3. *Suppose $X \to L_S X$ is an $S$-localization and $X \to L_T X$ is a $T$-localization. Then there exists an essentially unique factorization $X \to L_S X \to L_T X$, and the map $L_S X \to L_T X$ is a $T$-localization.*

*Proof.* 1. By assumption, every map $f\colon A \to B$ in $S$ is a $T$-equivalence, and so for any $T$-local object $Y$ we get an equivalence $\mathrm{Map}_{\mathcal{C}}(B, Y) \to \mathrm{Map}_{\mathcal{C}}(A, Y)$. Thus by definition $Y$ is $S$-local.

2. If $f\colon A \to B$ is an $S$-equivalence, and $Y$ is any $T$-local object, then by the previous point $Y$ is also $S$-local, and so we get an equivalence $\mathrm{Map}_{\mathcal{C}}(B, Y) \to \mathrm{Map}_{\mathcal{C}}(A, Y)$. Since $Y$ was an arbitrary $T$-local object, $f$ is therefore a $T$-equivalence.

3. Since $X \to L_S X$ is an $S$-equivalence, the previous point shows that it is a $T$-equivalence and so we have an equivalence

$$\mathrm{Map}_{\mathcal{C}}(L_S X, L_T X) \to \mathrm{Map}_{\mathcal{C}}(X, L_T X).$$

As a result, the chosen map $X \to L_T X$ has a contractible space of homotopy commuting factorizations $X \to L_S X \to L_T X$. As the maps $X \to L_S X$ and $X \to L_T X$ are both $T$-equivalences, the 2-out-of-3 property implies that $L_S X \to L_T X$ is also a $T$-equivalence whose target is $T$-local. By definition, this makes $L_T X$ into a $T$-localization of $L_S X$. □

# 5 Lifting criteria for localizations

In this section we will observe that, if $\mathcal{C}$ has homotopy pushouts, we can characterize local objects in terms of a lifting criterion. To do so, we will need to establish a few preliminaries. Fix a collection $S$ of maps in $\mathcal{C}$.

**Proposition 5.1.** *Suppose that $f\colon A \to B$ is an $S$-equivalence, and that $\mathcal{C}$ has homotopy pushouts. Then the map*

$$\mathrm{hocolim}(B \leftarrow A \to B) \to B$$

*is an $S$-equivalence.*

*Proof.* The map in question is equivalent to the map of homotopy pushouts induced by the diagram

$$\begin{tikzcd} B \arrow[d, equal] & A \arrow[l, "f"'] \arrow[r, "f"] \arrow[d, "f"] & B \arrow[d, equal] \\ B & B \arrow[l] \arrow[r] & B. \end{tikzcd}$$

However, the vertical maps are $S$-equivalences, and so by Proposition 4.9 the map $\mathrm{hocolim}(B \leftarrow A \to B) \to B$ is an $S$-equivalence. □

The lifting criterion we are about to describe rests on the following useful characterization of connectivity of a map.

**Lemma 5.2.** *Suppose that $f\colon X \to Y$ is a map of spaces and $N \geq 0$. Then $f$ is $N$-connected if and only if the following two criteria are satisfied:*

1. *the map $\pi_0(X) \to \pi_0(Y)$ is surjective, and*
2. *the diagonal map $X \to \mathrm{holim}(X \to Y \leftarrow X)$ is $(N-1)$-connected.*

*Proof.* The map $f$ is $N$-connected if and only if it is surjective on $\pi_0$ and, for all basepoints $x \in X$, the homotopy fiber $Ff$ over $f(x)$ is $(N-1)$-connective.

However, $Ff$ is equivalent to the homotopy fiber of $\mathrm{holim}(X \to Y \leftarrow X) \to X$ over $x$, and so this second condition is equivalent to $\mathrm{holim}(X \to Y \leftarrow X) \to X$ being $N$-connected. The composite $X \to \mathrm{holim}(X \to Y \leftarrow X) \to X$ is the identity, and the map $\mathrm{holim}(X \to Y \leftarrow X) \to X$ is $N$-connected if and only if the map $X \to \mathrm{holim}(X \to Y \leftarrow X)$ is $(N-1)$-connected. □

**Corollary 5.3.** *Suppose that $\mathcal{C}$ has homotopy pushouts and that we have a map $f_0\colon A_0 \to B$ in $\mathcal{C}$. Inductively define the $n$-fold double mapping cylinder $f_n$ as the map*

$$A_n = \mathrm{hocolim}(B \leftarrow A_{n-1} \to B) \to B.$$

*Then an object $Y$ is $f_0$-local if and only if the maps*

$$\mathrm{Hom}_{h\mathcal{C}}(B, Y) \to \mathrm{Hom}_{h\mathcal{C}}(A_n, Y)$$

*are surjective; equivalently, for any map $A_n \to Y$ there is a map $B \to Y$ such that the diagram*

$$\begin{tikzcd} A_n \arrow[r] \arrow[d, "f_n"'] & Y \\ B \arrow[ur, dotted] & \end{tikzcd}$$

*is homotopy commutative.*

*Proof.* We note that the definition of $A_n$ gives an identification

$$\mathrm{Map}_{\mathcal{C}}(A_n, Y) \simeq \mathrm{holim}\left[\mathrm{Map}_{\mathcal{C}}(B, Y) \to \mathrm{Map}_{\mathcal{C}}(A_{n-1}, Y) \leftarrow \mathrm{Map}_{\mathcal{C}}(B, Y)\right].$$

Inductive application of Lemma 5.2 shows that the map

$$\mathrm{Map}_{\mathcal{C}}(B, Y) \to \mathrm{Map}_{\mathcal{C}}(A_0, Y)$$

is $N$-connected if and only if the maps

$$\mathrm{Hom}_{h\mathcal{C}}(B, Y) \to \mathrm{Hom}_{h\mathcal{C}}(A_n, Y)$$

are surjective for $0 \leq n \leq N$. Letting $N$ grow arbitrarily large, we find that $Y$ is $f_0$-local if and only of the maps

$$\mathrm{Hom}_{h\mathcal{C}}(B, Y) \to \mathrm{Hom}_{h\mathcal{C}}(A_n, Y)$$

are surjective for all $n \geq 0$. □

*Example* 5.4. Suppose that $\mathcal{C}$ has homotopy pushouts and that $f\colon W \to *$ is a map to a homotopy terminal object of $\mathcal{C}$. Then the iterated double mapping cylinders are the maps $\Sigma^t W \to *$, and an object of $\mathcal{C}$ is $f$-local if and only if every map $\Sigma^t W \to Y$ factors, up to homotopy, through $*$.

*Example* 5.5. In the category of spaces $\mathcal{S}$, the iterated double mapping cylinders $f_n$ of a cofibration $f_0\colon A \to B$ have a more familiar description as the ***pushout-product*** maps

$$(S^{n-1} \times B) \underset{S^{n-1}\times A}{\cup} (D^n \times A) \to D^n \times B \to B.$$

# 6 The small object argument

We now sketch how, when we have some form of colimits in our category, Bousfield localizations can often be constructed using the small object argument.

From the previous section we know that we can replace the mapping space criterion for local objects with a lifting criterion when $\mathcal{C}$ has homotopy colimits, as follows. Given a map $f_0\colon A_0 \to B$, we construct iterated double mapping cylinders $f_n\colon A_n \to B$, and we find that an object is $Y$ is $f_0$-local if and only if every map $g\colon A_n \to Y$ can be extended to a map $\tilde{g}\colon B \to Y$ up to homotopy. More generally we can enlarge a collection of maps $S$ to a collection $T$ closed under double mapping cylinders, and ask whether $Y$ satisfies an extension property with respect to $T$.

This leads to an inductive method.

1. Start with $Y_0 = Y$.

2. Given $Y_\alpha$, either $Y_\alpha$ is local (in which case we are done) or there exists some set of maps $A_i \to B_i$ in $T$ and maps $g_i\colon A_i \to Y_\alpha$ which do not extend to $B_i$ up to homotopy. Form the homotopy pushout of the diagram

$$\coprod_i B_i \leftarrow \coprod_i A_i \to Y_\alpha$$

   and call it $Y_{\alpha+1}$. The map $Y_\alpha \to Y_{\alpha+1}$ is an $S$-equivalence because it is a homotopy pushout along an $S$-equivalence, and all the extension problems that $Y_\alpha$ had now have solutions in $Y_{\alpha+1}$.

3. Once we have constructed $Y_0, Y_1, Y_2, \ldots$, define $Y_\omega = \operatorname{hocolim} Y_n$. More generally, once we have constructed $Y_\alpha$ for all ordinals $\alpha$ less than some limit ordinal $\beta$, we define $Y_\beta = \operatorname{hocolim} Y_\alpha$. The map $Y \to Y_\beta$ is a homotopy colimit of $S$-equivalences and hence an $S$-equivalence.

The critical thing that we need is that *this procedure can be stopped at some point*, and for this we typically need to know that there will be some ordinal $\beta$ which is so big that any map $A_i \to Y_\beta$ automatically factors, up to homotopy, through some object $Y_\alpha$ with $\alpha < \beta$. This is a *compactness* property of the objects $A_i$, and this argument is called the *small object argument.* If we work on the point-set level this can be addressed using Smith's theory of combinatorial model categories; if we work on the homotopical level this can be addressed using Lurie's theory of presentable $\infty$-categories. We will discuss these approaches in §10 and §11.

Another important aspect of the small object argument is that it can prove additional properties about localization maps. If $S$ is a collection of maps all satisfying some property $P$ of maps in the homotopy category, and property $P$ is preserved under homotopy pushouts and transfinite homotopy colimits, then this process constructs a localization $Y \to LY$ that also has property $P$. Since localizations are essentially unique, any localization automatically has property $P$ as well.

*Remark* 6.1. If our category $\mathcal{C}$ does not have enough colimits, the small object argument may not apply. However, Bousfield localizations may still exist even if this particular construction cannot be applied.

# 7 Unstable settings

The classical examples of Bousfield localization are localizations of spaces. It is worthwhile first relating the localization condition to based mapping spaces.

**Proposition 7.1.** *Suppose $f\colon A \to B$ is a map of well-pointed spaces with basepoint. Then a space $Y$ is $f$-local in the category of* unbased *spaces if and only if, for all basepoints $y \in Y$, the restriction*

$$f^*\colon \mathrm{Map}_*(B,Y) \to \mathrm{Map}_*(A,Y)$$

*of based mapping spaces is a weak equivalence.*

*Proof.* Evaluation at the basepoint gives a map of fibration sequences

$$\begin{tikzcd}
\mathrm{Map}_*(B,Y) \arrow[r] \arrow[d] & \mathrm{Map}(B,Y) \arrow[r] \arrow[d] & Y \arrow[d, equal] \\
\mathrm{Map}_*(A,Y) \arrow[r] & \mathrm{Map}(A,Y) \arrow[r] & Y.
\end{tikzcd}$$

The center vertical map is an isomorphism on $\pi_*$ at any basepoint if and only if the left-hand map is. ☐

*Remark* 7.2. As $S$-equivalences are preserved under homotopy pushouts and the 2-out-of-3 axiom, we find that any space $Y$ local with respect to $f\colon A \to B$ is also local with respect to the map $B/A \to *$ from the homotopy cofiber to a point, and thus that every path component of $Y$ has a contractible space of based maps $B/A \to Y$. However, we will see shortly that the converse does not hold in general.

*Example* 7.3. Let $\mathcal{S}$ be the category of spaces, and take $f$ to be the map $S^n \to *$. Then a space $X$ is $f$-local if and only if, for any basepoint $x \in X$, the iterated loop space $\Omega^n X$ at $x$ is weakly contractible. Equivalently, for $n \geq 1$ the space $X$ is $f$-local if and only if it is $(n-1)$-*truncated*: $\pi_k(X,x)$ is trivial for all $k \geq n$ and all $x \in X$.

A map $A \to B$ of CW-complexes, by obstruction theory, is an $f$-equivalence if and only if it is $(n-1)$-connected. Therefore, for $n > 0$ a map $A \to B$ of CW-complexes is an $f$-localization if and only if $\pi_k(A) \to \pi_k(B)$ is an isomorphism for $0 \le k < n$ and all basepoints, but $\pi_k B$ vanishes for all $k \ge n$ and all basepoints.[5] This characterizes a stage $P_{n-1}(X)$ in the Postnikov tower of $X$.

*Example* 7.4. Let $f$ be the inclusion $S^n \vee S^m \to S^n \times S^m$ of spaces. The Cartesian product is formed by attaching an $(n+m)$-cell to $S^n \vee S^m$ along an attaching map given by a Whitehead product $[\iota_n, \iota_m] \in \pi_{n+m-1}(S^n \vee S^m)$. Any map $S^n \vee S^m \to X$, classifying a pair of elements $\alpha \in \pi_n(X)$ and $\beta \in \pi_m(X)$ at some basepoint $x$, sends this attaching map to $[\alpha, \beta]$. The fiber of $\mathrm{Map}(S^n \times S^m, X) \to \mathrm{Map}(S^n \vee S^m, X)$ over the corresponding point is either empty (if $[\alpha, \beta]$ is nontrivial) or equivalent to the iterated loop space $\Omega^{n+m} X$ at $x$ (if $[\alpha, \beta]$ is trivial). A space $X$ is therefore local with respect to $f$ if and only if, at any basepoint, the homotopy groups $\pi_k(X)$ are zero for all $k \ge n+m$ and the Whitehead products

$$\pi_n(X, x) \times \pi_m(X, x) \to \pi_{n+m-1}(X, x)$$

vanish at any basepoint $x$.

Consider the case $n = m = 1$. For a path-connected CW-complex $X$ with fundamental group $G$, the map $X \to K(G_{ab}, 1)$ is an $f$-localization.

*Example* 7.5. If $A$ is nonempty, then a space $Y$ is local with respect to $f \colon \emptyset \to A$ if and only if $Y$ is weakly contractible. All maps are $f$-equivalences, and $X \to *$ is always an $f$-localization.

*Example* 7.6. Consider a degree-$p$ map $f \colon S^1 \to S^1$. A space $Y$ is $f$-local if and only if it is local for degree-$p$ maps $S^n \to S^n$, and this occurs if and only if the $p$'th power maps $\pi_n(Y) \to \pi_n(Y)$ are all isomorphisms.

By contrast, let $M(\mathbb{Z}/p, 1)$ be the Moore space constructed as the cofiber of $f$, and consider the map $g \colon M(\mathbb{Z}/p, 1) \to *$. A space $Y$ is $g$-local if and only if it satisfies the extension condition for the maps $M(\mathbb{Z}/p, n) \to *$ for all $n \ge 1$, or equivalently if the mod-$p$ homotopy sets $\pi_n(Y; \mathbb{Z}/p)$ vanish for all $n \ge 2$. This is equivalent to the $p$'th-power maps being isomorphisms on $\pi_n(Y)$ for all $n > 1$ and injective on $\pi_1(Y)$.

*Example* 7.7 ([Nei10]). Let $S$ be the set of maps $\{K(\mathbb{Z}/p, 1) \to *\}$ as $p$ ranges over the prime numbers. Then the Sullivan conjecture, as proven by Miller [Mil84], is equivalent to the statement that any finite CW-complex $X$ is $S$-local. Since $S$-equivalences are closed under products and homotopy colimits, the expression of $K(\mathbb{Z}/p, n+1)$ as the geometric realization of the bar construction $\{K(\mathbb{Z}/p, n)^q\}$ shows inductively that the maps $K(\mathbb{Z}/p, n) \to *$ are all $S$-equivalences. However, if $Y$ is any nontrivial 1-connected space with finitely generated homotopy groups and a finite Postnikov tower, then $Y$ accepts a nontrivial map from some $K(\mathbb{Z}/p, n)$ and hence cannot be $S$-local. This argument shows that a simply-connected finite CW-complex

[5] We should be careful about edge cases. When $n = 0$, $X$ is $(-1)$-truncated if and only if it is either empty or weakly contractible. By convention, $S^{-1} = \emptyset$, and $X$ is $(-2)$-truncated if and only if it is weakly contractible.

When $n = 0$ a map $A \to B$ is an $f$-equivalence if and only if either both $A$ and $B$ are empty or neither of them is, and a map $A \to X$ is an $f$-localization if and only if either $A$ is nonempty and $X$ is weakly contractible, or $A$ and $X$ are both empty. When $n = -1$ any map is an $f$-equivalence, and a map $A \to X$ is an $f$-localization if and only if $X$ is weakly contractible.

with nonzero mod-$p$ homology has $p$-torsion in infinitely many nonzero homotopy groups, which was conjectured by Serre in the early 1950's and proven by McGibbon and Neisendorfer [MN84].

Localization still applies to other categories closely related to topological spaces.

*Example* 7.8. Let $\mathcal{C}$ be the category of based spaces. A based space $Y$ is local with respect to the based map $* \to S^1$ if and only if the loop space $\Omega Y$ is weakly contractible, or equivalently if and only if the path component $Y_0$ of the basepoint is weakly contractible. A model for the Bousfield localization is given by the mapping cone of the map $Y_0 \to Y$.

*Example* 7.9. Fix a discrete group $G$, and consider the category of $G$-spaces: spaces with a continuous action of a group $G$, with maps being continuous maps. For example, the empty space has a unique $G$-action, while the orbit spaces $G/H$ have continuous actions under the discrete topology. Every $G$-space has fixed-point subspaces $X^H \cong \mathrm{Map}_G(G/H, X)$ for subgroups $H$ of $G$. In this context, there is an abundance of examples of localizations.

A $G$-space $Y$ is local with respect to $\emptyset \to *$ if and only if the fixed-point subspace $Y^G$ is contractible. A model for Bousfield localization is given by the mapping cone of the map $Y^G \to Y$.

Fix a model for the universal contractible $G$-space $EG$. A $G$-space $Y$ is local with respect to $EG \to *$ if and only if the map from the fixed point space $Y^G$ to the homotopy fixed point space $\mathrm{Map}_G(EG, Y) = Y^{hG}$ is a weak equivalence. Since there is a $G$-equivariant homotopy equivalence $EG \times EG \to EG$, a model for the Bousfield localization is the space of nonequivariant maps $\mathrm{Map}(EG, Y)$, with $G$ acting by conjugation.

A $G$-space $Y$ is local with respect to $\emptyset \to G$ if and only if the underlying space $Y$ is contractible. A model for the Bousfield localization is given by the mapping cone of the map $EG \times Y \to Y$, sometimes called $\widetilde{EG} \wedge Y_+$.

*Example* 7.10. Fix a collection $S$ of maps and a space $Z$, letting $\mathcal{C}$ be the category of spaces over $Z$. We say that a map $X \to Y$ of spaces over $Z$ is a ***fiberwise $S$-equivalence*** if the map of homotopy fibers over any point $z \in Z$ is an $S$-equivalence, and refer to the corresponding localizations as ***fiberwise $S$-localizations***.

A map $X \to Y$ over $Z$ which is a weak equivalence on underlying spaces is in particular a fiberwise $S$-equivalence. Applying this to the lifting characterization of fibrations, we can find that for an object $Y \to Z$ of $\mathcal{C}$ to be fiberwise $S$-local the map $Y \to Z$ must be a fibration. Moreover, for fibrations $Y \to Z$ we can recharacterize being local. Given any map $f \colon A \to B$ in $S$ and any point $z \in Z$, there is a map in $\mathcal{C}$ of the form $f_z \colon A \to B \to \{z\} \subset Z$ concentrated entirely over the point $z$; let $S_Z$ be the set of all such maps. A fibration $Y \to Z$ in $\mathcal{C}$ fiberwise $S$-local if and only if it is $S_Z$-local in $\mathcal{C}$.

Fiberwise localizations were constructed by Farjoun in [Far96, 1.F.3]; they are also constructed in [Hir03, §7] and characterized from several perspectives.

*Example* 7.11. The category of topological monoids and continuous homomorphisms has its own homotopy theory. Consider the inclusion $f \colon \mathbb{N} \to \mathbb{Z}$ of discrete monoids. Then $\mathrm{Map}^{mon}(\mathbb{Z}, M) \to \mathrm{Map}^{mon}(\mathbb{N}, M)$ is isomorphic to the map $M^\times \to M$ from the space of invertible elements of $M$ to the space $M$.[6] An $f$-local object is a topological group, and localization is a topologized version of group-completion.

We note, however, that the map $\mathbb{N} \to \mathbb{Z}$ does not participate well with *weak* equivalences of topological monoids: weakly equivalent topological monoids do not have weakly equivalent spaces of invertible elements because homomorphisms out of $\mathbb{Z}$ are not homotopical. We can get a version that respects weak equivalences in two ways. With model categories, we can factor the map $\mathbb{N} \to \mathbb{Z}$ as $\mathbb{N} \hookrightarrow \mathbb{Z}_c \xrightarrow{\simeq} \mathbb{Z}$ in the category of topological monoids, where $\mathbb{Z}_c$ is a cofibrant topological monoid, and there are explicit models for such. We could instead use coherent multiplications, where a map $\mathbb{Z} \to M$ is no longer required to strictly be a homomorphism but instead be a coherently multiplicative map.

Using either correction, the space $M^\times$ of strict units becomes replaced, up to equivalence, by the pullback

$$\begin{tikzcd} M^{inv} \arrow[r] \arrow[d] & M \arrow[d] \\ \pi_0(M)^\times \arrow[r] & \pi_0 M, \end{tikzcd}$$

the union of the components of $M$ whose image in $\pi_0(M)$ has an inverse. A local object is then a *grouplike* topological monoid, and localization is homotopy-theoretic group completion. These play a key role the study of iterated loop spaces and algebraic $K$-theory [May72, Seg74, MS76].

# 8 Stable settings

One of the great benefits of the stable homotopy category, and stable settings in general, is that a map $f\colon X \to Y$ becoming an equivalence is roughly the same as the cofiber $Y/X$ becoming trivial.

We recall the definition of stability from [Lur17, §1.1.1].

**Definition 8.1.** The category $\mathcal{C}$ is *stable* if it satisfies the following properties:

1. $\mathcal{C}$ is (homotopically) *pointed*: there is an object $*$ such that, for all $X \in \mathcal{C}$, the spaces $\mathrm{Map}_{\mathcal{C}}(X, *)$ and $\mathrm{Map}_{\mathcal{C}}(*, X)$ are contractible.
2. $\mathcal{C}$ has homotopy pushouts of diagrams $* \leftarrow X \to Y$ and homotopy pullbacks of diagrams $* \to Y \leftarrow X$.

   As a special case, we have suspension and loop objects:

$$\Sigma X = \mathrm{hocolim}(* \leftarrow X \to *)$$
$$\Omega X = \mathrm{holim}(* \to X \leftarrow *)$$

[6] As a point-set digression the reader should, as usual, be warned that the source may not have the subspace topology. The space of invertible elements is, instead, homeomorphic to the subspace of $M \times M$ of pairs of elements $(x, y)$ such that $xy = yx = 1$.

3. Suppose that we have a homotopy coherent diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ * \arrow[r] & Z, \end{tikzcd}$$

meaning maps as given and a homotopy between the double composites. Then the induced map

$$\operatorname{hocolim}(* \leftarrow X \to Y) \to Z$$

is a homotopy equivalence if and only if the map

$$X \to \operatorname{holim}(* \to Z \leftarrow Y)$$

is a homotopy equivalence.

Taking $Y = *$, we find that a map $X \to \Omega Z$ is an equivalence if and only if the homotopical adjoint $\Sigma X \to Z$ is an equivalence.

*Example* 8.2. The category of (cofibrant–fibrant) spectra is the canonical example of a stable category.

*Example* 8.3. For any ring $R$, there is a category $\mathcal{K}_R$ of chain complexes of $R$-modules. Any two complexes $C$ and $D$ have a Hom-complex $\operatorname{Hom}_R(C,D)$, and the Dold–Kan correspondence produces a simplicial set $\operatorname{Map}_{\mathcal{K}_R}(C,D)$ whose homotopy groups satisfy

$$\pi_n \operatorname{Map}_{\mathcal{K}_R}(C,D) \cong H_n \operatorname{Hom}_R(C,D)$$

for $n \geq 0$.[7] This gives the category $\mathcal{K}_R$ of complexes an enrichment in simplicial sets, and these mapping spaces make the category $\mathcal{K}_R$ stable. Within this category there are many stable subcategories: categories of complexes which are bounded above or below or both, with homology groups bounded above or below or both, which are made up of projectives or injectives, and so on.

We will write $\mathcal{C}_R$ be the category of cofibrant objects in the projective model structure on $R$, whose homotopy category is the derived category $D(R)$.

**Theorem 8.4** (see [Lur17, Theorem 1.1.2.14]). *If $\mathcal{C}$ is stable, then the homotopy category $h\mathcal{C}$ has the structure of a triangulated category.*

In a stable category, every object $Y$ has an equivalence $Y \to \Omega\Sigma Y$. However, there is a natural weak equivalence

$$\begin{aligned} \operatorname{Map}_{\mathcal{C}}(X, \Omega Z) &\simeq \operatorname{holim}\left[\operatorname{Map}_{\mathcal{C}}(X,*) \to \operatorname{Map}_{\mathcal{C}}(X,Z) \leftarrow \operatorname{Map}_{\mathcal{C}}(X,*)\right] \\ &\simeq \operatorname{holim}(* \to \operatorname{Map}_{\mathcal{C}}(X,Z) \leftarrow *) \\ &\simeq \Omega \operatorname{Map}_{\mathcal{C}}(X,Z), \end{aligned}$$

[7] More generally, if $R[m]$ is the complex equal to $R$ in degree $n$ and zero elsewhere, then for all complexes $C$ we have $[R[m], C]_{h\mathcal{K}_R} \cong H_m(C)$.

and hence the mapping spaces

$$\mathrm{Map}_{\mathcal{C}}(X,Y) \simeq \Omega^n \mathrm{Map}_{\mathcal{C}}(X, \Sigma^n Y)$$

can be extended to be valued in $\Omega$-spectra. This makes it much easier to detect equivalences: we only need to check the homotopy groups of $\Omega^t \mathrm{Map}_{\mathcal{C}}(X,Y)$ at the basepoint.

**Definition 8.5**. Suppose that $\mathcal{C}$ is stable and $S$ is a class of maps in $\mathcal{C}$. We say that $S$ is *shift-stable* if the image $\bar{S}$ in $h\mathcal{C}$ is closed under suspension and desuspension, up to isomorphism.

**Proposition 8.6**. *Suppose that $\mathcal{C}$ is stable and $S$ is a shift-stable class of maps $\{f_\alpha : A_\alpha \to B_\alpha\}$. Then an object $Y$ in $\mathcal{C}$ is $S$-local if and only if the homotopy classes of maps $[B_\alpha/A_\alpha, X]_{h\mathcal{C}}$ are trivial.*

*Proof.* The individual fiber sequences

$$\Omega^t \mathrm{Map}_{\mathcal{C}}(B_\alpha/A_\alpha, Y) \to \Omega^t \mathrm{Map}_{\mathcal{C}}(B_\alpha, Y) \to \Omega^t \mathrm{Map}_{\mathcal{C}}(A_\alpha, Y),$$

on homotopy classes classes of maps, are part of a long exact sequence

$$\cdots \to [\Sigma^t B_\alpha/A_\alpha, Y]_{h\mathcal{C}} \to \pi_t \mathrm{Map}_{\mathcal{C}}(B_\alpha, Y) \to \pi_t \mathrm{Map}_{\mathcal{C}}(A_\alpha, Y) \to [\Sigma^{t-1} B_\alpha/A_\alpha, Y]_{h\mathcal{C}} \to \dots$$

from the triangulated structure. We get an isomorphism on homotopy groups if and only if the terms $[\Sigma^t B_\alpha/A_\alpha, Y]_{h\mathcal{C}}$ vanish for all values of $t$. $\square$

By contrast with the unstable case where basepoints are a continual issue, these shift-stable localizations in a stable category are always nullifications, and they are *equivalent* to nullifications of the triangulated homotopy category by a class $S$ that is closed under shift operations.

**Definition 8.7**. Suppose that $\mathcal{D}$ is a triangulated category. A full subcategory $\mathcal{T}$ is called a *thick subcategory* if its objects are closed under closed under isomorphism, shifts, cofibers, and retracts. If $\mathcal{D}$ has coproducts, a thick subcategory $\mathcal{T}$ is *localizing* if it is also closed under coproducts.

**Proposition 8.8**. *Suppose that $\mathcal{D}$ is a triangulated category and that $\mathcal{T} \subset \mathcal{D}$ is a thick subcategory. Then there exists a triangulated category $\mathcal{D}/\mathcal{T}$ called the* Verdier quotient *of $\mathcal{D}$ by $\mathcal{T}$, with a functor $\mathcal{D} \to \mathcal{D}/\mathcal{T}$. The Verdier quotient is universal among triangulated categories under $\mathcal{D}$ such that the objects of $\mathcal{T}$ map to trivial objects.*

This universal characterization allows us to strongly relate Bousfield localization of stable categories to localization of the homotopy category.

**Proposition 8.9**. *Suppose that $\mathcal{C}$ is stable, and that $S$ is a shift-stable collection of maps in $\mathcal{C}$.*

1. *An object in $\mathcal{C}$ is $S$-local if and only if its image in the homotopy category $h\mathcal{C}$ is $S$-local.*

2. *A map in $\mathcal{C}$ is an $S$-equivalence if and only if its image in the homotopy category is an $S$-equivalence.*

3. *The subcategories $L^S\mathcal{C}$ of $S$-local objects and $\mathcal{T}$ of $S$-trivial objects are thick subcategories of $\mathcal{C}$.*

4. *The subcategory $\mathcal{T}$ of $S$-trivial objects is closed under all coproducts that exist in $\mathcal{C}$. If $\mathcal{C}$ has small coproducts then it is a localizing subcategory.*

5. *If all objects in $\mathcal{C}$ have $S$-localizations, then the left adjoint to the inclusion $hL^S\mathcal{C} \to h\mathcal{C}$ has a factorization*

$$h\mathcal{C} \to h\mathcal{C}/h\mathcal{T} \to hL^S\mathcal{C}.$$

*The latter functor is an equivalence of categories.*

*Remark* 8.10. The fact that Bousfield localization of $\mathcal{C}$ is determined by a construction purely in terms of $h\mathcal{C}$ is special to the stable setting.

*Remark* 8.11. This relates Verdier quotients in a stable category to Bousfield localization, but only quotients by a *localizing* subcategory. For a homotopical interpretation of more general Verdier quotients, see [NS17, §I.3].

*Example* 8.12. Let $S$ be the collection of multiplication-by-$m$ maps $S^n \to S^n$ for $n \in \mathbb{Z}$, $m > 0$. A spectrum $Y$ is $S$-local if and only if multiplication by $m$ is an isomorphism on the homotopy groups $\pi_* Y$ for all positive $m$, or equivalently if the maps $\pi_* Y \to \mathbb{Q} \otimes \pi_* Y$ are isomorphisms. Such spectra are called *rational.*

If $Y$ is such a spectrum, we can calculate that the natural map

$$[X, Y] \to \prod_n \mathrm{Hom}(\pi_n X, \pi_n Y)$$

is an isomorphism for any spectrum $X$: because $\pi_n Y$ is a graded vector space, $\mathrm{Hom}(-, \pi_n Y)$ is exact and so both sides are cohomology theories in $X$ that satisfy the wedge axiom and agree on spheres. Therefore, $A \to B$ is an $S$-equivalence if and only if $\mathbb{Q} \otimes \pi_n(A) \to \mathbb{Q} \otimes \pi_n(B)$ is an isomorphism for all $n$, and such maps are called *rational equivalences.* In this case, this is the same as the map $H_*(A; \mathbb{Q}) \to H_*(B; \mathbb{Q})$ being an isomorphism.

This analysis allows us to conclude that $X \to H\mathbb{Q} \wedge X = X_{\mathbb{Q}}$ is a rationalization for all $X$.

*Example* 8.13. In the above, we can make $S$ smaller. If $S$ is the set of multiplication-by-$p$ maps $S^n \to S^n$, we similarly find that $S$-local spectra are those whose homotopy groups are $\mathbb{Z}[1/p]$-modules, and that equivalences are those maps which induce isomorphisms on homotopy groups after inverting $p$. The localization of $\mathbb{S}$ is the homotopy colimit

$$\mathbb{S}[1/p] = \mathrm{hocolim}(\mathbb{S} \xrightarrow{p} \mathbb{S} \xrightarrow{p} \mathbb{S} \xrightarrow{p} \dots),$$

which is also a Moore spectrum for $\mathbb{Z}[1/p]$. We similarly find that $X \to \mathbb{S}[1/p] \wedge X$ is an $S$-localization for all $X$.

We could also let $S$ be the set of multiplication-by-$m$ maps for $m$ relatively prime to $p$, which replaces the ring $\mathbb{Z}[1/p]$ with the local ring $\mathbb{Z}_{(p)}$ in the above.

*Example* 8.14. Fix a commutative ring $R$ and a multiplicatively closed subset $W \subset R$, recalling that localization with respect to $W$ is exact. If we define $S$ to be the set of maps of the form $R[n] \xrightarrow{w} R[n]$ for $w \in W$, then a complex $C$ of $R$-modules is $S$-local if and only if the multiplication-by-$w$ maps $H_*(C) \to H_*(C)$ are isomorphisms, or equivalently if and only if $H_*(C) \to W^{-1}H_*(C) \cong H_*(W^{-1}C)$ is an isomorphism. A map $A \to B$ of complexes is an $S$-equivalence if and only if the map $W^{-1}A \to W^{-1}B$ is an equivalence.

The natural map $C \to W^{-1}C \cong W^{-1}R \otimes_R C$ is an $S$-localization.

These examples have such nice properties that it is convenient to axiomatize them.

**Definition 8.15**. A stable Bousfield localization on spectra[8] is a *smashing localization* if either of the following equivalent conditions hold.

1. There is a map of spectra $\mathbb{S} \to L\mathbb{S}$ such that, for any $X$, the map $X \to L\mathbb{S} \wedge X$ is a localization.

2. Local objects are closed under arbitrary homotopy colimits.

The equivalence between these two characterizations is not immediately obvious. The first implies the second, because

$$L\mathbb{S} \wedge \operatorname{hocolim} X_i \to \operatorname{hocolim}(L\mathbb{S} \wedge X_i)$$

is always an equivalence and the former is always local. The converse follows because the only homotopy-colimit preserving functors on spectra are all equivalent to functors of the form $X \mapsto A \wedge X$ for some $A$, and the resulting localization map $\mathbb{S} \to A$ is of the desired form.

*Example* 8.16. A spectrum $Y$ is local for the maps $\mathbb{S}[1/p] \wedge S^n \to *$ if and only if the homotopy limit

$$\operatorname{holim}(\cdots \to Y \xrightarrow{p} Y \xrightarrow{p} Y) \simeq F(\mathbb{S}[1/p], Y)$$

of function spectra is weakly contractible. However, taking homotopy limits of the natural fiber sequences

$$\begin{tikzcd}
\cdots \arrow[r] & Y \arrow[r, "p"] \arrow[d, "p^2"'] & Y \arrow[r, "p"] \arrow[d, "p"'] & Y \arrow[d, "1"'] \\
\cdots \arrow[r] & Y \arrow[r, "1"] \arrow[d] & Y \arrow[r, "1"] \arrow[d] & Y \arrow[d] \\
\cdots \arrow[r] & Y/p^2 \arrow[r] & Y/p \arrow[r] & *
\end{tikzcd}$$

shows that $Y$ is local if and only if the map $Y \to Y^\wedge_p = \operatorname{holim} Y/p^k$ is an equivalence. Therefore, we refer to a spectrum local for these maps as *$p$-complete*; a Bousfield

[8] This definition extends if we have a stable category $\mathcal{C}$ with a symmetric monoidal structure appropriately compatible with the stable structure.

localization of $Y$ will be called the *$p$-completion*; a trivial object is called *$p$-adically trivial*; an equivalence is called a *$p$-adic equivalence*. The above presents $Y_p^\wedge$ as a candidate for the $p$-completion of $Y$.

If we construct the fiber sequence

$$\Sigma^{-1}\mathbb{S}/p^\infty \to \mathbb{S} \to \mathbb{S}[1/p],$$

we find that we can identify $Y_p^\wedge$ with the function spectrum $F(\Sigma^{-1}\mathbb{S}/p^\infty, Y)$. Moreover, the map $Y_p^\wedge \to (Y_p^\wedge)_p^\wedge$ is always an equivalence. Therefore, $Y_p^\wedge$ is always $p$-complete.

If multiplication-by-$p$ is an equivalence on $Z$, then $Z \simeq Z \wedge \mathbb{S}[1/p]$, and so maps $Z \to Y$ are equivalent to maps $Z \to F(\mathbb{S}[1/p], Y)$. For any $Y$ which is $p$-adically complete, this is trivial, so such objects $Z$ are $p$-adically trivial. In particular, the fiber of $Y \to Y_p^\wedge$ is always trivial and so $Y \to Y_p^\wedge$ is a $p$-adic equivalence. Therefore, this is a $p$-adic completion.

If each homotopy group of $Y$ has a bound on the order of $p$-power torsion, we can further identify the homotopy groups of $Y_p^\wedge$ as the ordinary $p$-adic completions of the homotopy groups of $Y$; if the homotopy groups of $Y$ are finitely generated, then $\pi_*(Y_p^\wedge) \to \pi_*(Y) \otimes \mathbb{Z}_p$.[9]

*Remark* 8.17. Note that the previous example is not a smashing localization. For any connective spectrum $X$, the map $\mathbb{S}_p^\wedge \wedge X \to X_p^\wedge$ induces the map $\pi_*(X) \otimes \mathbb{Z}_p \to \pi_*(X)_p^\wedge$ on homotopy groups; this is typically only an isomorphism if the homotopy groups $\pi_*(X)$ are finitely generated.

*Example* 8.18. For an element $x$ in a commutative ring $R$, let $K_x$ be the complex

$$\cdots \to 0 \to R \to x^{-1}R \to 0 \to \dots$$

concentrated in degrees 0 and $-1$, with a map $K_X \to R$. For a sequence of elements $(x_1, \dots, x_n)$, let $K_{(x_1,\dots,x_n)} = \bigotimes_R K_{x_i}$ be the ***stable Koszul complex***. If $y$ is in the ideal generated by $(x_1, \dots, x_n)$, then the inclusion $K_{(x_1,\dots,x_n)} \to K_{(x_1,\dots,x_n,y)}$ is a quasi-isomorphism, and so up to quasi-isomorphism the Koszul complex only depends on the ideal. Let $K_I$ be a cofibrant replacement.

We say that a complex $C$ is *$I$-complete* if and only if it is local with respect to the shifts of the map $K_I \to R$. This is true if and only if the homology groups of $C$ are $I$-complete in the derived sense. If $R$ is Noetherian and the homology groups of $C$ are finitely generated, this is true if and only if the homology groups of $C$ are $I$-adically complete in the ordinary sense.

These frameworks for the study of localization and completion, and many generalizations of it, were developed by Greenlees and May [GM95].

*Example* 8.19. Fix a ring $R$, and let $\mathcal{C}$ be the category of unbounded complexes of finitely generated projective left $R$-modules that only have nonzero homology groups in finitely many degrees. Consider the set $S$ of maps $R[n] \to 0$. An object $C$ is $S$-local if and only if its homology groups are trivial.

[9] In general, the homotopy groups of the $p$-adic completion are somewhat sensitive and one needs to be careful about derived functors of completion.

We can inductively take mapping cones of maps $R[n] \to C$ to construct a localization $C \to LC$, embedding $C$ into an unbounded complex of finitely generated projective modules with trivial homology groups. Therefore, localizations exist in this category.

For two such complexes $C$ and $D$ with trivial homology, we have

$$\mathrm{Hom}_{h\mathcal{C}}(C,D) \cong \lim_n \mathrm{Hom}_R(Z_n C, Z_n D)/\mathrm{Hom}_R(Z_n C, D_{n+1})$$

where $D_{n+1} \to Z_n(D)$ is the boundary map—a surjective map from a projective module.

This can be interpreted in terms of the stable module category of $R$. Defining $W_n(C) = Z_{-n}(C)$, the short exact sequences $0 \to Z_{-n}(C) \to C_{-n} \to Z_{-n-1}(C) \to 0$ determine isomorphisms $W_n(C) \cong \Omega W_{n+1}(C)$ in the stable module category, assembling the $W_n$ into an "$\Omega$-spectrum". Maps $C \to D$ are then equivalent to maps of $\Omega$-spectra in the stable module category.[10]

# 9 Homology localizations

## 9.1 Homology localization of spaces

**Definition 9.1.** Suppose $E_*$ is a homology theory on spaces. Then we say that a map $f\colon A \to B$ of spaces is an ***$E_*$-equivalence*** if it induces an isomorphism $f_*\colon E_*A \to E_*B$. A space is ***$E_*$-local*** if it is local with respect to the class of $E_*$-equivalences.

*Example* 9.2. Suppose that $E_*$ is integral homology $H_*$. Any Eilenberg–Mac Lane space $K(A,n)$ is $H_*$-local by the universal coefficient theorem for cohomology. Moreover, any simply-connected space $X$ is the homotopy limit of a Postnikov tower built from fibration sequences $P_n X \to P_{n-1} X \to K(\pi_n X, n+1)$. Since local objects are closed under homotopy limits, we find that simply-connected spaces are $H_*$-local.[11]

*Remark* 9.3. This example illustrates a very different approach to the construction of localizations. Because homology isomorphisms are detected by the $K(A,n)$, these spaces are automatically local; therefore, any object built from these using homotopy limits is automatically local. Such objects are often called ***nilpotent***. Thus gives us a dual approach to building the Bousfield localization of $X$: construct a natural diagram of nilpotent objects that receive maps from $X$, and try to verify that the homotopy limit is a localization of $X$.

*Example* 9.4. Serre's rational Hurewicz theorem implies that a map of simply-connected spaces is an isomorphism on rational homology groups if and only if it is an isomorphism on rational homotopy groups. A simply-connected space is local for rational homology if and only if it its homotopy groups are rational vector spaces.

[10] In certain cases, such as for Frobenius algebras, $\Omega$ is an autoequivalence. This definition then simply recovers the stable module category of $R$ by itself. If $R$ has finite projective dimension, $\Omega$-spectrum objects are necessarily trivial.

[11] This argument can be refined to show that ***nilpotent spaces*** (where $\pi_1(X)$ is nilpotent, and acts nilpotently on the higher homotopy groups) are $H_*$-local.

The same is not true for general spaces. The map $\mathbb{RP}^2 \to *$ is a rational homology isomorphism, and the covering map $S^2 \to \mathbb{RP}^2$ is an isomorphism on rational homotopy groups, but the composite $S^2 \to *$ is neither. The problem here is the failure of a simple Postnikov tower for $\mathbb{RP}^2$ due to the action of $\pi_1$ on the higher homotopy groups.

*Example* 9.5. If $X$ is a connected space with perfect fundamental group, then Quillen's plus-construction gives a map $X \to X^+$ that induces an $H_*$-isomorphism such that $X^+$ is simply-connected. This makes $X^+$ into an $H_*$-localization of $X$.

Classically, Quillen's plus-construction can be applied to groups with a perfect subgroup. In order to properly identify the universal property, we need to work in a relative situation.

*Example* 9.6. Fix a group $G$, and let $\mathcal{C}$ be the category of spaces over $BG$. Given an abelian group $A$ with $G$-action, there is an associated local coefficient system $\underline{A}$ on $BG$, and so given any object $X \to BG$ of $\mathcal{C}$ we can define the homology groups $H_*(X;\underline{A})$. We say that a map $X \to Y$ over $BG$ is a relative homology equivalence if it induces isomorphisms on homology with coefficients in any $\underline{A}$. Taking $A$ to be the group algebra $\mathbb{Z}[G]$, we find that this is equivalent to the map of homotopy fibers $F_X \to F_Y$ being a homology isomorphism, so this is the same as a ***fiberwise* $H_*$*-equivalence***. If an object $Y$ over $BG$ has simply-connected homotopy fiber it is automatically local.

Suppose that $X$ is any connected space such that $\pi_1(X)$ contains a perfect normal subgroup $P$ with quotient group $G$. The homomorphism $\pi_1(X) \to G$ lifts to a map $X \to BG$. The plus-construction with respect to $P$ is a fiber homology equivalence $X \to X^+$ where $X^+ \to BG$ has simply-connected homotopy fiber, and thus is a localization in $\mathcal{C}$.

Localization with respect to homology is very difficult to analyze in the case when a space is not simply-connected, especially if the space is not ***simple*** (either the fundamental group is not nilpotent or it does not act nilpotently on the higher homotopy groups). Many natural spaces are not local. Here are some basic tools to prove this.

**Lemma 9.7**. *Suppose that $F_n$ is a free group on $n$ generators and $\alpha\colon F_n \to F_n$ is a homomorphism, with induced map $\alpha_{ab}\colon \mathbb{Z}^n \to \mathbb{Z}^n$. Under the identification $\mathrm{Hom}(F_n, G) \cong G^n$ for any group $G$, write $\alpha^*$ for the natural map of sets $G^n \to G^n$.*

*Suppose the map $\alpha_{ab}$ becomes an isomorphism after tensoring with a ring $R$. Then, for any space $X$, a necessary condition for $X$ to be $H_*(-;R)$-local is that $\alpha^*\colon \pi_1(X,x)^n \to \pi_1(X,x)^n$ must be a bijection at any basepoint.*

*Proof.* The map $\alpha_{ab}$, after tensoring with $R$, can be identified with the map $H_1(F_n;R) \to H_1(F_n;R)$ on homology induced by $\alpha$. If $\alpha_{ab}$ becomes an isomorphism after tensoring with $R$, then $\alpha\colon K(F_n,1) \to K(F_n,1)$ is an $H_*(-;R)$-equivalence.

For a space $X$ to be $H_*(-;R)$-local, the induced map

$$\mathrm{Map}_*(K(F_n,1),X) \to \mathrm{Map}_*(K(F_n,1),X)$$

must be a weak equivalence. Taking a wedge of circles as our model, we find that the induced map

$$(\Omega X)^n \to (\Omega X)^n$$

must be a weak equivalence. On $\pi_0$, this is the map $\alpha^*$ on $\pi_1(X)^n$. $\square$

*Example* 9.8. For $n \neq 0$, the multiplication-by-$n$ map $\mathbb{Z} \to \mathbb{Z}$ is a rational isomorphism. Therefore, for $X$ to be rationally local, the $n$'th power map $\pi_1(X) \to \pi_1(X)$ should be a bijection: every element $g \in \pi_1(X)$ has a unique $n$'th root $g^{1/n}$. Such groups are called uniquely divisible, or sometimes $\mathbb{Q}$-groups. The structure of free $\mathbb{Q}$-groups was studied in [Bau60].

*Example* 9.9. Let $F_2$ be free on the generators $x$ and $y$, and define $\alpha\colon F_2 \to F_2$ by

$$\alpha(x) = x^{-9}y^{-20}(y^2x)^{10}$$
$$\alpha(y) = x^{-9}y^{10}(yx^{-1})^{-9}.$$

The map $\alpha_{ab}$ is the identity map. Therefore, for a space with fundamental group $G$ to be local with respect to integral homology, any pair of elements $(z,w) \in G$ has to be uniquely of the form $(z,w) = (x^{-9}y^{-20}(y^2x)^{10}, x^{-9}y^{-10}(yx^{-1})^{-9})$ for some $x$ and $y$ in $G$. Most groups do not satisfy this property.

We can use this to show that any space whose fundamental group $G$ has a surjective homomorphism $\phi\colon G \to A_5$ cannot be local with respect to integral homology—in particular, this applies to a free group $F_2$. Choose elements $x$ and $y$ in $G$ with $\phi(x) = (123)$ and $\phi(y) = (12345)$. Then $\phi(y^2x) = (14)(25)$ and $\phi(yx^{-1}) = (145)$, and $\phi \circ \alpha$ is the trivial homomorphism while $\phi$ is surjective.[12]

Several other, more easily defined, maps $\alpha$ can be shown to not be bijective. For example, the map $(x,y) \mapsto (x[x,y], y[x,y])$ can be shown to not be a bijection, e.g. by using Fox's free differential calculus [Fox53].

**Lemma 9.10**. *Let $G$ be a group, $R$ a ring, and $\beta \in \mathbb{Z}[G]$ an element such that the composite ring homomorphism $\mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to R$ sends $\beta$ to zero.*

*Then, for any based space $X$ with fundamental group $G$, a necessary condition for $X$ to be $H_*(-;R)$-local is that $\pi_k(X)$ must be complete in the topology defined by $\beta$.*[13]

*Proof.* Fix the space $X$ and basepoint and consider the space $Y = X \vee S^k$. The group $\pi_k(Y)$ is isomorphic to $\pi_k(X) \oplus \mathbb{Z}[G]$, and so the element $\beta \in \mathbb{Z}[G]$ lifts to a map $\beta\colon Y \to Y$ given by the identity on $X$ together with the map $S^k \to Y$ corresponding to the element $(0,\beta) \in \pi_k(X) \oplus \mathbb{Z}[G]$. The induced self-map of

$$H_*(Y;R) \cong H_*(X;R) \oplus \widetilde{H}_*(S^k;R)$$

is given by the identity on $H_*(X;R)$ together with the map $\epsilon(\beta)$ tensored with $R$ on the second factor. If $\epsilon(\beta)$ becomes zero after tensoring with $R$, then this map is zero on the second factor.

[12] In order to use this *particular* technique to show that $\phi$ was not a bijection, we needed to have a homomorphism $\phi$ whose image was a perfect group—the image of $\alpha_{ab}$ is contained in the kernel of $\phi_{ab}$. This particular map $\alpha$ is complicated because it was reverse-engineered from $\phi$.

[13] This refers to being *derived* complete in the sense of Example 8.18.

Define

$$X' = \mathrm{hocolim}(Y \xrightarrow{\beta} Y \xrightarrow{\beta} \cdots).$$

By construction, the map

$$H_*(X;R) \to H_*(X';R) = \mathrm{colim}\, H_*(Y;R)$$

is an isomorphism. Therefore, $X \to X'$ is an $H_*(-;R)$-equivalence.

For $X$ to be $H_*(-;R)$-local, the induced map

$$\mathrm{Map}(X',X) \to \mathrm{Map}(X,X)$$

must be a weak equivalence. Taking the fiber over the identity map of $X$, we find that there is an induced equivalence

$$\mathrm{holim}(\cdots \xrightarrow{\beta} \Omega^k X \xrightarrow{\beta} \Omega^k X) \xrightarrow{\sim} *.$$

Using the Milnor $\lim^1$-sequence, we find that all of the homotopy groups of $X$ must be derived-complete with respect to $\beta$. $\square$

*Remark* 9.11. If $R = \mathbb{Z}$, then this implies that any element $s \in \mathbb{Z}[G]$ with $\epsilon(s) = \pm 1$ must act invertibly on the higher homotopy groups of $X$, and so the action must factor through a large localization $S^{-1}\mathbb{Z}[G]$.

*Example* 9.12. Consider $X = S^1 \vee S^2$, whose fundamental group is isomorphic to $\mathbb{Z}$ with generator $t$. The second homotopy group satisfies

$$\pi_2(S^1 \vee S^2) \cong \mathbb{Z}[t^{\pm 1}]$$

as a module over $\mathbb{Z}[t^{\pm 1}]$. This is not complete with respect to the ideal generated by $\beta = (t-1)$ even though $\epsilon(\beta) = 0$. Therefore, $S^1 \vee S^2$ is not local with respect to integral homology.

*Example* 9.13. The space $\mathbb{RP}^2$ has fundamental group $\mathbb{Z}/2$ generated by an element $\sigma$, and the second homotopy group $\mathbb{Z}$ satisfies $\sigma(y) = -y$. The element $(1-\sigma)$ has $\epsilon(1-\sigma) = 0$ and acts as multiplication by 2. Since $\mathbb{Z}$ is not complete in the 2-adic topology we find that $\mathbb{RP}^2$ is not local with respect to integral homology.[14]

*Example* 9.14. If $R = \mathbb{Q}$, then any element $S \in \mathbb{Z}[G]$ with $\epsilon(s) \neq 0$ must act invertibly on the higher homotopy groups of $X$ for $X$ to be local with respect to rational homology. The homotopy groups of $K(\mathbb{Q},1) \vee (S^3)_{\mathbb{Q}}$ are $\mathbb{Q}$ in degree 1 and the rational group algebra $\mathbb{Q}[\mathbb{Q}]$ in degree 3. If $t$ is the generator of $\mathbb{Z} \subset \mathbb{Q}$, the element $2t-1$ has $\epsilon(2t-1) = 1$ and does not act invertibly on this group algebra. Therefore, this space is not local with respect to rational homology even though its homotopy groups are rational.

*Remark* 9.15. Bousfield localization with respect to $E_*$-equivalences leads us to some uncomfortable pressure with our previous notation. At first glance, it is not clear whether being an equivalence on $E_*$-homology is the same as having the same

[14]The homology localization of $\mathbb{RP}^2$ has, in fact, a fiber sequence $(S^2)^\wedge_2 \to L\mathbb{RP}^2 \to K(\mathbb{Z}/2,1)$.

mapping spaces into any $E_*$-local object.[15] To prove this, one needs to prove that there is a sufficient supply of $E_*$-local objects: for any $X$, we need to be able to construct an $E_*$-homology isomorphism $X \to L_E X$ such that $L_E X$ is $E_*$-local. Here is how Bousfield addressed this in [Bou75, Theorem 11.1]. It is essentially a cardinality argument, whose general form is called the Bousfield–Smith cardinality argument in [Hir03, §2.3].

Let $E_*$ be a homology theory on spaces. We then have a class $S$ of $E_*$-equivalences, which are those maps which induce equivalences on $E_*$-homology. Unfortunately, this is a proper class of morphisms, and so we cannot immediately apply the small object argument to construct localizations. Moreover, because we do not know anything about local objects we cannot assert that an $S$-equivalence $X \to Y$ is the same as a map inducing an isomorphism $E_* X \to E_* Y$.

Bousfield addresses this by showing the following. Suppose $K \to L$ is an inclusion of simplicial sets such that $E_* K \to E_* L$ is an isomorphism, and that we choose any simplex $\sigma$ of $L$. Then there exists a subcomplex $L' \subset L$ with the following properties:

1. The simplex $\sigma$ is contained in $L'$.
2. The map $E_*(K \cap L') \to E_*(L')$ is an isomorphism on $E_*$.
3. The complex $L'$ has size bounded by a cardinal $\kappa$, which *depends only on* $E$.

Because of the cardinality bound on $L'$, we can find a *set* $T$ of $E_*$-equivalences $A \to B$ so that any such map $K \cap L' \to L'$ must be isomorphic to one of them; an arbitrary $E_*$-equivalence $K \to L$ can then be factored as a (possibly transfinite) sequence of pushouts along the maps in the set $T$ followed by an equivalence. The maps in $T$ are $E_*$-isomorphisms, and an object is $S$-local if and only if it is $T$-local. The small object argument then applies to $T$, allowing us to construct $T$-localizations $Y \to LY$ which are also $E_*$-isomorphisms.

We will see in § 10 and § 11, in general constructions of Bousfield localization, that this verification is the key step.

## 9.2 Homology localization of spectra

**Definition 9.16.** For a spectrum $E$, a map $f : X \to Y$ is an $E$-*homology equivalence* (or simply an $E$-equivalence) if the corresponding map $E_* X \to E_* Y$ is an isomorphism, and we say that $Z$ is $E$-*trivial* if $E_* Z = 0$. A map $f$ is an $E$-equivalence if and only if the cofiber of $f$ is $E$-trivial.[16]

This is most often employed when $E$ is a ring spectrum.

**Proposition 9.17.** *If $E$ has a multiplication $m : E \wedge E \to E$ with a left unit $\eta : \mathbb{S} \to E$ in the homotopy category, then any spectrum $Y$ with a unital map $E \wedge Y \to Y$ is $E$-local.*

[15] One could, but should not, say it this way: it is not clear that an ($E_*$-equivalence)-equivalence is automatically an $E_*$-equivalence.

[16] Again, the definitions of this section can be applied to a stable category $\mathcal{C}$ with a compatible symmetric monoidal structure.

*Remark* 9.18. Such spectra $Y$ are sometimes called ***homotopy** $E$**-modules***. Any spectrum of the form $E \wedge W$ is a homotopy $E$-module.

*Proof.* Any map $f\colon Z \to Y$ has the following factorization in the homotopy category:

$$Z \xrightarrow{\eta\wedge 1} E \wedge Z \xrightarrow{1\wedge f} E \wedge Y \xrightarrow{m} Y$$

If $Z$ has trivial $E$-homology, then $E \wedge Z$ is trivial and so the composite $Z \to Y$ is nullhomotopic. Therefore, $[Z, Y] = 0$ for all $E$-trivial $Z$, as desired. $\square$

**Corollary 9.19**. *If $E$ has a multiplication $m\colon E \wedge E \to E$ with a left unit $\eta\colon \mathbb{S} \to E$ in the homotopy category, then any homotopy limit of spectra that admit homotopy $E$-module structures is $E$-local.*

*Example* 9.20. A particular case of interest is when $E = H\mathbb{Z}$. Any Eilenberg–Mac Lane spectrum $HA$ is $H\mathbb{Z}$-local, being of the form $H\mathbb{Z} \wedge MA$ for a Moore spectrum for $A$.

Then any connective spectrum $Y$ is $H\mathbb{Z}$-local, as follows. As $H\mathbb{Z}$-local objects form a thick subcategory, any spectrum with finitely many nonzero homotopy groups is therefore $H\mathbb{Z}$-local. If $Y$ is connective then $P_n Y$ is $H\mathbb{Z}$-local due to having a finite Postnikov tower. Therefore, $Y = \operatorname{holim} P_n Y$ is the homotopy limit of $H\mathbb{Z}$-local spectra, and is thus $H\mathbb{Z}$-local.

Similarly, any product of Eilenberg–Mac Lane spectra $\prod \Sigma^n HA_n$ is also $H\mathbb{Z}$-local. Any rational spectrum is of this form.

However, not all spectra are $H\mathbb{Z}$-local. For any prime $p$ and integer $n > 0$, there are $p$-primary Morava $K$-theories $K(n)$ such that $H\mathbb{Z} \wedge K(n)$ is trivial; these are $H\mathbb{Z}$-acyclic. The complex $K$-theory spectrum $KU$ satisfies the property that $H_*(KU;\mathbb{Z}) \to H_*(KU;\mathbb{Q})$ is an isomorphism: from this we can find that $KU \to KU_{\mathbb{Q}}$ is an $H\mathbb{Z}$-equivalence. The target is also $H\mathbb{Z}$-local because it is rational, and so $KU_{\mathbb{Q}}$ is the $H\mathbb{Z}$-localization of $KU$.

*Example* 9.21. We can consider the case where $E = H\mathbb{Z}/p$. By a similar argument, we find that any connective spectrum which is $p$-adically complete in the sense of Example 8.16 is also $H\mathbb{Z}/p$-complete. Again, in connective cases there is not a difference between being $p$-adically complete and being $H\mathbb{Z}/p$-local.

For nonconnective spectra, these are quite different. The Morava $K$-theories $K(n)$ are $p$-adically complete but $H\mathbb{Z}/p$-trivial. The periodic complex $K$-theory spectrum $KU$ has $\pi_*(KU_p^\wedge) \cong (\pi_* KU)_p^\wedge$, but $KU$ is also $H\mathbb{Z}/p$-trivial.

These localizations have the flavor of completion with respect to an ideal. In some cases we can express them as such.

**Definition 9.22**. Suppose that $E$ has a binary multiplication $m$ with a left unit $\eta\colon \mathbb{S} \to E$, and let $j\colon I \to \mathbb{S}$ be the fiber of $\eta\colon \mathbb{S} \to E$. Assemble these into the inverse system

$$\cdots \to I^{\wedge 3} \xrightarrow{j\wedge 1\wedge 1} I \wedge I \xrightarrow{j\wedge 1} I \xrightarrow{j} \mathbb{S}$$

The *$E$-**nilpotent completion*** $X_E^\wedge$ is the homotopy limit

$$\operatorname*{holim}_n (\mathbb{S}/I^{\wedge n}) \wedge X,$$

with map $X \to X_E^\wedge$ induced by the maps $\mathbb{S} \to \mathbb{S}/I^{\wedge n}$.

**Proposition 9.23**. *The $E$-nilpotent completion is always $E$-local.*

*If $E$ is a finite complex, or $X$ and $I$ are connective and $E$ is of finite type, then the map $X \to X_E^\wedge$ is an $E$-localization.*

*Proof.* The cofiber sequence $I \to \mathbb{S} \to E$, after smashing with $I^{\wedge(n-1)}$, becomes a cofiber sequence $I^{\wedge n} \to I^{\wedge(n-1)} \to E \wedge I^{\wedge(n-1)}$, and so there are cofiber sequences

$$\mathbb{S}/I^{\wedge n} \wedge X \to \mathbb{S}/I^{\wedge(n-1)} \wedge X \to E \wedge I^{\wedge(n-1)} \wedge X.$$

By induction on $n$ we find that $\mathbb{S}/I^{\wedge n} \wedge X$ is $E$-local, and so the homotopy limit $X_E^\wedge$ is $E$-local.

After smashing with $E$, the cofiber sequence

$$E \wedge I^{\wedge n} \wedge X \to E \wedge I^{\wedge(n-1)} \wedge X \to E \wedge E \wedge I^{\wedge(n-1)} \wedge X$$

has a retraction of the second map via the (opposite) multiplication of $E$, and so the first map is nullhomotopic. Therefore, the homotopy limit $\operatorname{holim} E \wedge (I^{\wedge n} \wedge X)$ is trivial, and from the cofiber sequences

$$E \wedge (I^{\wedge n} \wedge X) \to E \wedge X \to E \wedge (\mathbb{S}/I^{\wedge n} \wedge X)$$

we find that $E \wedge X \to \operatorname{holim}(E \wedge (\mathbb{S}/I^{\wedge n} \wedge X)$ is an equivalence.

This reduces us to proving that the map

$$E \wedge \operatorname{holim}(\mathbb{S}/I^{\wedge n} \wedge X) \to \operatorname{holim}(E \wedge \mathbb{S}/I^{\wedge n} \wedge X)$$

is an equivalence: we can move the smash product with $E$ inside the homotopy limit. This is always true if $E$ is finite or if $E$ is of finite type and the homotopy limit is of connective objects. □

*Remark* 9.24. The spectral sequence arising from the inverse system defining $X_E^\wedge$ is the *generalized Adams–Novikov spectral sequence based on $E$-homology.* It often abuts to the homotopy groups of the Bousfield localization with respect to $E$.

We can generalize our construction by allowing more general towers with a nilpotence property, after Bousfield in [Bou79], or by extending these methods to the category of modules over a ring spectrum, as Baker–Lazarev did in [BL01] or Carlsson did in [Car08].

*Example* 9.25. For any prime $p$ and any $n > 0$, we have the Johnson–Wilson homology theories $E(n)_*$ and the Morava $K$-theories $K(n)_*$. Associated to these we have $E(n)$-localization functors and $K(n)$-localization functors, as well as categories of $E(n)$-local and $K(n)$-local spectra, which play an essential role in chomatic homotopy theory. Ravenel conjectured, and Devinatz–Hopkins–Smith proved, that the localization $L_{E(n)}$ is a smashing localization [Rav84, DHS88, Rav92]. These localizations also have *chromatic fractures* which are built using the following result.

**Proposition 9.26.** *Suppose that $E$ and $K$ are spectra such that $L_K L_E X$ is always trivial. Then, for all $X$, there is a homotopy pullback diagram*

$$\begin{tikzcd} L_{E\vee K}X \arrow[r] \arrow[d] & L_E X \arrow[d] \\ L_K X \arrow[r] & L_E L_K X. \end{tikzcd}$$

*Proof.* The objects in the diagram

$$L_E X \to L_E L_K X \leftarrow L_K X$$

are either $E$-local or $K$-local, and hence automatically $E \vee K$-local; therefore, the homotopy pullback $P$ is $E \vee K$-local. It then suffices to show that the fiber of the map $X \to P$ is $E \vee K$-trivial, which is equivalent to showing that

$$\begin{tikzcd} X \arrow[r] \arrow[d] & L_E X \arrow[d] \\ L_K X \arrow[r] & L_E L_K X. \end{tikzcd}$$

becomes a homotopy pullback after smashing with $E \vee K$. After smashing with $E$, the horizontal maps become equivalences, and so the diagram is a pullback. After smashing with $K$, the left-hand vertical map is an equivalence and the right-hand vertical map is between trivial objects, so the diagram is also a pullback. Therefore, the diagram becomes a pullback after smashing with $E \vee K$. □

# 10 Model categories

The lifting characterization of local objects from §5 falls very naturally into the framework of Quillen's model categories. The groundwork for this is in [Bou75, §10].

**Definition 10.1.** Suppose that $\mathcal{M}$ is a category with a model structure. We say that a second model structure $\mathcal{M}'$ with the same underlying category is a ***left Bousfield localization*** of $\mathcal{M}$ if $\mathcal{M}'$ has the same family of cofibrations but a larger family of weak equivalences than $\mathcal{M}$.

As a first consequence, note that the identity functor (which is its own right and left adjoint) preserves cofibrations and takes the weak equivalences in $\mathcal{M}$ to weak equivalences in $\mathcal{M}'$. This makes it part of a Quillen adjunction

$$\mathcal{M} \rightleftarrows \mathcal{M}'.$$

This has the immediate consequence that the induced adjunction on homotopy categories is a reflective localization.

**Proposition 10.2.** *Suppose that $L\colon \mathcal{M} \rightleftarrows \mathcal{M}'\colon R$ is the adjunction associated to a left Bousfield localization. Then the right adjoint $R$ identifies the homotopy category $h\mathcal{M}'$ with a full subcategory of $h\mathcal{M}$.*

*Proof.* It is necessary and sufficient to show that the counit $\epsilon\colon LRx \to x$ of the adjunction on homotopy categories is always an isomorphism, for this is the same as asking that, in the factorization

$$\mathrm{Hom}_{h\mathcal{M}}(Rx, Ry) \cong \mathrm{Hom}_{h\mathcal{M}'}(LRx, y) \to \mathrm{Hom}_{h\mathcal{M}'}(x, y),$$

the second map is an isomorphism.

For an object of $y$, the composite functor $LR$ on homotopy categories is calculated as follows: find a fibrant replacement $y \xrightarrow{\simeq'} y_{f'}$ in $\mathcal{M}'$, apply the identity functor to get to $\mathcal{M}$, find a cofibrant replacement $(y_{f'})_c \xrightarrow{\simeq} y_{f'}$ in $\mathcal{M}$, and apply the identity functor to get to $\mathcal{M}'$. The counit of the adunction is represented in the homotopy category of $\mathcal{M}'$ by the composite

$$(y_{f'})_c \xrightarrow{\simeq} y_{f'} \xleftarrow{\simeq'} y.$$

However, equivalences in $\mathcal{M}$ are automatically equivalences in $\mathcal{M}'$, and so the counit is an isomorphism in the homotopy category of $\mathcal{M}'$. □

Because fibrations and acyclic fibrations are determined by having the right lifting property against acyclic cofibrations and fibrations, the new model structure has the same acyclic fibrations but fewer fibrations. For example, a fibrant object in the left Bousfield localization has to have a lifting property against the cofibrations which are weak equivalences in $\mathcal{M}'$.

The next proposition establishes the connection between left Bousfield localization and ordinary Bousfield localization when both are defined and compatible: the case of a simplicial model category.

**Proposition 10.3.** *Suppose that $\mathcal{M}$ is a simplicially enriched category with two model structures, making $\mathcal{M} \to \mathcal{M}'$ is a left Bousfield localization of simplicial model categories. Let $S$ be the collection of weak equivalences between cofibrant objects in $\mathcal{M}'$. Then, in the category of cofibrant-fibrant objects of $\mathcal{M}$, the objects which are fibrant in $\mathcal{M}'$ are precisely the $S$-local fibrant objects.*

*Proof.* Fix an object $Y$ of $\mathcal{M}'$. For it to be fibrant in $\mathcal{M}'$, it must also be fibrant in $\mathcal{M}$. Suppose $Y$ is a fibrant object in $\mathcal{M}'$. Given any acyclic cofibration $A \to B$ in $\mathcal{M}'$, the map of simplicial sets $\mathrm{Map}_{\mathcal{M}'}(A, Y) \to \mathrm{Map}_{\mathcal{M}'}(B, Y)$ is an acyclic fibration by the SM7 axiom of simplicial model categories. Thus, the functor $\mathrm{Map}_{\mathcal{M}'}(-, Y)$ from $\mathcal{M}'$ to the homotopy category of spaces takes acyclic cofibrations to isomorphisms. Thus, Ken Brown's lemma implies that it also takes weak equivalences between cofibrant objects in $\mathcal{M}'$ to isomorphisms in the homotopy category of spaces.

Suppose that we have a map $f\colon A \to B$ in $S$ between cofibrant objects of $\mathcal{M}$ that is also a weak equivalence in $\mathcal{M}'$. Then $f$ is also a weak equivalence between cofibrant objects of $\mathcal{M}'$. The induced map $\mathrm{Map}_{\mathcal{M}}(B, Y) \to \mathrm{Map}_{\mathcal{M}}(A, Y)$ is a weak equivalence because the mapping spaces in $\mathcal{M}$ and $\mathcal{M}'$ are the same. Thus, $Y$ is $S$-local. □

We would now like to establish results in the other direction. Namely, given a model category $\mathcal{M}$ and a collection $S$ of maps $A_i \to B_i$ in $\mathcal{M}$, we would like to establish the existence of a Bousfield localization $\mathcal{M}'$ of $\mathcal{M}$. Because we want to work within the already-established homotopy theory of $\mathcal{M}$, we want to use derived mapping spaces out of $A$ and $B$ and replace homotopy lifting properties with strict lifting properties. We assume without loss of generality that our set $S$ is made up of cofibrations $A_i \to B_i$ between cofibrant objects.

**Definition 10.4**. Suppose that $\mathcal{M}$ is a simplicial model category, and that $f\colon A \to B$ is a map. Then the *iterated double mapping cylinders* are the maps

$$(B \otimes \partial\Delta^n) \coprod_{A \otimes \partial\Delta^n} (A \otimes \Delta^n) \to B \otimes \Delta^n.$$

This definition is rigged so that an object $Y$ has the right lifting property with respect to the iterated double mapping cylinders if and only if the map $\mathrm{Map}_{\mathcal{M}}(B, Y) \to \mathrm{Map}_{\mathcal{M}}(A, Y)$ is an acyclic fibration of simplicial sets. One of the equivalent formulations of the SM7 axioms for a simplicial model category is that double mapping cylinders are always cofibrations, as follows.

**Proposition 10.5**. *Suppose that $f\colon A \to B$ is a map. If $f$ is a cofibration, then the iterated double mapping cylinders are cofibrations. If $A$ is also cofibrant, then the iterated double mapping cylinders have cofibrant source.*

*Remark* 10.6. If $\mathcal{M}$ does not have a simplicial model structure, we can obtain replacements for these objects by iteratively replacing the maps $B \coprod_A B \to B$ with equivalent cofibrations.

**Definition 10.7**. Suppose that $\mathcal{M}$ is a simplicial model category, that $S$ is a collection of maps, and that $T$ is the collection of iterated double mapping cylinders of maps in $S$. We say that a map in $\mathcal{M}$ is an *$S$-cofibration* if it is a cofibration in $\mathcal{M}$, and that it is an *$S$-fibration* if it has the right lifting property with respect to the maps in $T$. If these determine a new model structure $\mathcal{M}'$, we call this the *left Bousfield localization with respect to $S$*.

This gives us two fundamentally different approaches to the process of constructing a left Bousfield localization. In the first, we may try to expand our family of weak equivalences to some new family $\mathcal{W}$; we must then prove that we can construct enough fibrations and fibrant objects to make the model structure work. In the second, we may try to start with some collection of maps $S$ which serve as new "cells" to build acyclic cofibrations, and use them to contract our family of fibrations; we then lose control over the weak equivalences, and typically must work to prove that cofibrations which are weak equivalences can be built out of our new cells.

The most advanced technology available for Bousfield localization is Jeff Smith's theory of combinatorial model categories.

**Definition 10.8**. A model category $\mathcal{M}$ is *cofibrantly generated* if there are sets $I$ and $J$ of maps satisfying the following properties:

1. the fibrations in $\mathcal{M}$ are the maps that have the right lifting property with respect to $J$;
2. the acyclic fibrations in $\mathcal{M}$ are the maps that have the right lifting property with respect to $I$;
3. $I$ permits the small object argument, so that from any object $X$ we can construct a map $X \to X'$, as a transfinite composition of pushouts along coproducts of maps in $I$, that has the right lifting property with respect to $I$;
4. $J$ also permits the small object argument.

We refer to $I$ as the set of *generating cofibrations* and to $J$ as the set of *generating acyclic cofibrations* respectively.

The cofibrantly generated model category is also *combinatorial* if it is also locally presentable, meaning there exists a regular cardinal $\kappa$ and a set $\mathcal{M}_0$ of objects satisfying the following properties:

1. any small diagram in $\mathcal{M}$ has a colimit;
2. for any object $x$ in $\mathcal{M}_0$, the functor $\mathrm{Hom}_{\mathcal{M}}(x,-)$ commutes with $\kappa$-filtered colimits;
3. every object in $\mathcal{M}$ is a $\kappa$-filtered colimit of objects in $\mathcal{M}_0$.

**Theorem 10.9** (Dugger's theorem [Dug01]). *Any combinatorial model category is Quillen equivalent to a left proper simplicial model category.*

*Remark* 10.10. The axioms of a cofibrantly generated model category and a locally presentable category have nontrivial overlap. In one direction, the model category axioms already ask that $\mathcal{M}$ has all colimits. In the other direction, being locally presentable means that *every* set of maps admits the small object argument.

*Example* 10.11. Simplicial sets are the motivating example of a combinatorial model category. Fibrations and acyclic fibrations are defined as having the right lifting property with respect to the generating acyclic cofibrations $\Lambda^n_i \to \Delta^n$ and the generating cofibrations $\partial\Delta^n \to \Delta^n$. The category is also locally presentable because it is generated by finite simplicial sets. Every simplicial set is the filtered colimit of its finite subobjects; there are only countably many isomorphism classes of finite simplicial sets; for any finite simplicial set $X$, $\mathrm{Hom}(X,-)$ commutes with filtered colimits.

**Theorem 10.12** (Smith's theorem [Bek00, Bar10, Lur09]). *Suppose that $\mathcal{M}$ is a locally presentable category with a family $\mathcal{W}$ of* weak equivalences *and a set $I$ of* generating cofibrations. *Call those maps which have the right lifting property with respect to $I$ the* acyclic fibrations, *and those maps which have the left lifting property with respect to acyclic fibrations the* cofibrations. *Suppose that we have the following:*

1. *$\mathcal{W}$ satisfies the 2-out-of-3 axiom;*
2. *acyclic fibrations are in $\mathcal{W}$;*

3. *the class of cofibrations which are in $\mathcal{W}$ is closed under pushout and transfinite composition; and*

4. *maps in $\mathcal{W}$ are closed under $\kappa$-filtered colimits for some regular cardinal $\kappa$, and generated under $\kappa$-filtered colimits by some set of maps in $\mathcal{W}$.*

*Then there exists a combinatorial model structure on $\mathcal{M}$ with set $I$ of generating cofibrations and set $\mathcal{W}$ of weak equivalences. This model structure on $\mathcal{M}$ has cofibrant and fibrant replacement functors. Moreover, any combinatorial model structure arises in this fashion.*

**Corollary 10.13.** *Suppose that $\mathcal{M}$ is a combinatorial model category with set $I$ of generating cofibrations and class $\mathcal{W}$ of weak equivalences. Given a functor $E\colon \mathcal{M} \to \mathcal{D}$ factoring through the homotopy category $h\mathcal{M}$, define a map to be an $E$-*equivalence *if its image under $E$ is an isomorphism. Then there exists a left Bousfield localization $\mathcal{M}_E$, whose equivalences are the $E$-equivalences, if the following conditions hold:*

1. *$E$-equivalence is preserved by transfinite composition along cofibrations;*

2. *pushouts of $E$-acyclic cofibrations are $E$-equivalences; and*

3. *there exists a set of $E$-acyclic cofibrations that generate all $E$-acyclic cofibrations under $\kappa$-filtered colimits.*

*Proof.* The 2-out-of-3 axiom is automatic: if two of $E(g)$, $E(f)$ and $E(gf) = E(g)E(f)$ are isomorphisms, then so is the third. The fact that $E$ factors through the homotopy category automatically implies that acyclic fibrations are taken by $E$ to isomorphisms. □

*Example* 10.14. Let $E_*$ be a homology theory on the category of simplicial sets. The excision and direct limit axioms for homology imply that $E$-equivalences are preserved by homotopy pushouts and transfinite compositions. Therefore, the verification that we have a model structure is immediately reduced to the core of the Bousfield–Smith cardinality argument of Example 9.15: that there is a set of $E$-acyclic cofibrations generating all others under filtered colimits.

The great utility of combinatorial model structures is that they allow us to *build* new model categories: categories of diagrams and Bousfield localizations.

**Theorem 10.15** ([Lur09, A.2.8.2, A.3.3.2]). *Suppose that $\mathcal{M}$ is a combinatorial model category and that $I$ is a small category. Then there exists a* projective *(resp.* injective*) model structure on the functor category $\mathcal{M}^I$, where a natural transformation of diagrams is an equivalence or fibration (resp. cofibration) if and only if it is an objectwise equivalence or fibration (resp. cofibration).*

*If $\mathcal{M}$ is a simplicial model category, then the natural simplicial enrichment on $\mathcal{M}^I$ makes the injective and projective model structures into simplicial model categories.*

**Theorem 10.16** ([Lur09, A.3.7.3]). *Suppose that $\mathcal{M}$ is a left proper combinatorial simplicial model category and that $S$ is a set of cofibrations in $\mathcal{M}$. Let $S^{-1}\mathcal{M}$ have the same underlying category as $\mathcal{M}$ and the same cofibrations, but with weak equivalences the $S$-equivalences.*

*Then $S^{-1}\mathcal{M}$ has the structure of a left proper combinatorial model category, whose fibrant objects are precisely the $S$-local fibrant objects of $\mathcal{M}$.*

## 11 Presentable $\infty$-categories

Bousfield localization for model categories has the useful property that it *keeps the category in place* and merely changes the equivalences. One cost is that making localization canonical or extending monoidal structures to localized objects takes hard work. By contrast, localization for $\infty$-categories has the useful property that it is genuinely *defined by a universal property*, automatically making localization canonical and making it much easier to extend a monoidal structure to local objects without rectifying structure. Of course, this comes at the cost of coming to grips with coherent category theory itself.

The homotopy theory of presentable $\infty$-categories is equivalent, in a precise sense, to the homotopy theory of combinatorial model categories [Lur09, A.3.7.6]. However, by contrast with our techniques for Bousfield localization using model categories and fibrant replacement functors, it allows us to rephrase some of our localization techniques in a way that connects more directly with the homotopical techniques that we originally used in §5.

In this section, we will let $\mathcal{C}$ be an $\infty$-category in the sense of [Lur09]. It is outside our scope to give a technically correct discussion of these. However, the study of $\infty$-categories is equivalent to the study of categories with morphism spaces, and where possible we will attempt to make connection with classical techniques. With this in mind, if $\mathcal{C}$ is an enriched category we will say that a *coherent diagram* $I \to \mathcal{C}$ is a coherent functor in the sense of Vogt [Vog73]. This is equivalent to either the notion of a functor $\mathfrak{C}[I] \to \mathcal{C}$ from a certain simplicially enriched category or to the notion of a functor $I \to N\mathcal{C}$ of simplicial sets to the coherent nerve in the sense of [Lur09]. As before a *homotopy colimit* for such a diagram is based on classical homotopy limits and colimits in spaces, and is characterized by having natural weak equivalences

$$\mathrm{Map}_{\mathcal{C}}(\operatorname*{hocolim}_{I} F(i), Y) \simeq \operatorname*{holim}_{I} \mathrm{Map}_{\mathcal{C}}(F(i), Y).$$

**Definition 11.1** ([Lur09, 5.5.1.1]). An $\infty$-category $\mathcal{C}$ is *presentable* if there there exists a regular cardinal $\kappa$ and a set $\mathcal{C}_0$ of objects satisfying the following properties:

1. any small diagram in $\mathcal{C}$ has a homotopy colimit;
2. for any object $x$ in $\mathcal{C}_0$, the functor $\mathrm{Hom}_{\mathcal{C}}(x, -)$ commutes with $\kappa$-filtered homotopy colimits;
3. every object in $\mathcal{C}$ is a $\kappa$-filtered homotopy colimit of objects in $\mathcal{C}_0$.

This definition is precisely parallel to the definition of local presentability in an ordinary category (see Definition 10.8). In essence, $\mathcal{C}$ is a large category that is formally generated under colimits by a small category.

Given such an $\infty$-category $\mathcal{C}$ and a collection $S$ of morphisms in $\mathcal{C}$, it makes sense to define the $S$-local objects and $S$-equivalences just as in §4: an object $Y$ is $S$-local if and only if the mapping spaces $\mathrm{Map}_{\mathcal{C}}(-, Y)$ take maps in $S$ to equivalences of spaces.

**Definition 11.2** ([Lur09, 5.5.4.5]). Suppose that $\mathcal{C}$ is an $\infty$-category with small colimits and that $\mathcal{W}$ is a collection of maps in $\mathcal{C}$. We say that $\mathcal{W}$ is *strongly saturated* if it satisfies the following conditions:

1. given a homotopy pushout diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d] & D \arrow[d] \\ C' \arrow[r, "f'"'] & D', \end{tikzcd}$$

   if $f$ is in $\mathcal{W}$ then so is $f'$;

2. the class $\mathcal{W}$ is closed under homotopy colimits;

3. the class $\mathcal{W}$ is closed under equivalence, and its image in the homotopy category satisfies the 2-out-of-3 axiom.

**Proposition 11.3** ([Lur09, 5.5.4.7]). *Given a set $S$ of morphisms in $\mathcal{C}$, there is a smallest saturated class of morphisms containing $S$. We denote this as $\bar{S}$. If $\mathcal{W} = \bar{S}$ for some set $S$, then we say that $\mathcal{W}$ is* of small generation.

*Example* 11.4. Suppose that $E\colon \mathcal{C} \to \mathcal{C}'$ is a functor of $\infty$-categories that preserves homotopy colimits. Then the set $\mathcal{W}^E$ of maps in $\mathcal{C}$ that map to equivalences is strongly saturated.

The presentability axioms for an $\infty$-category provide a homotopical version of what we needed to construct localizations by ensuring that the small object argument goes through. As a result, we obtain a result on the existence of Bousfield localizations for presentable $\infty$-categories.

**Theorem 11.5** ([Lur09, 5.5.4.15]). *Let $\mathcal{C}$ be a presentable $\infty$-category and $S$ a set of morphisms in $\mathcal{C}$, generating the saturated class $\bar{S}$. Let $L^S\mathcal{C}$ be the full subcategory of $S$-local objects. Then the following hold:*

1. *for every object $C \in \mathcal{C}$, there is a map $C \to C'$ in $\bar{S}$ such that $C'$ is $S$-local;*

2. *the $\infty$-category $L^S\mathcal{C}$ is presentable;*

3. *the inclusion $L^S\mathcal{C} \to \mathcal{C}$ has a (homotopical) left adjoint $L$;*

4. *the class of $S$-equivalences coincides with both the saturated class $\bar{S}$ and the set of maps taken to equivalences by $L$.*

*Remark* 11.6. The homotopical left adjoint can be rephrased as follows. If we write $Loc^S(\mathcal{C})$ for the category of $S$-localizations $C \to C'$, then the forgetful functor

$$Loc^S(\mathcal{C}) \to \mathcal{C},$$

sending $(C \to C')$ to $C$, is an equivalence of categories (in fact, a trivial fibration of quasicategories). By choosing a section, given by $C \mapsto (C \to LC)$, we obtain a localization functor $L$.

As in the case of Bousfield localization of combinatorial model categories, this connects the two approaches to Bousfield localization. We can start with a set $S$ of generating equivalences and construct localizations from those, so for a given class $\mathcal{W}$ of weak equivalences we are reduced to showing that $\mathcal{W}$ is generated by a set $S$ of maps. Moreover, if the maps in $S$ all happen to be in a particular saturated class, then so are the maps in $\mathcal{W}$.

# 12 Multiplicative properties

Many of the categories where we carry out Bousfield localization have monoidal structures, and under good circumstances localization is compatible with them. In this section we will briefly discuss the circumstances under which this is true.

## 12.1 Enriched monoidal structures

In order to begin to work with these definitions, we need a monoidal or symmetric monoidal structure on $\mathcal{C}$ that respects morphism spaces.

**Definition 12.1**. Suppose $\mathcal{C}$ is a category enriched in spaces. The structure of an *enriched monoidal category* on $\mathcal{C}$ consists of a functor $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ of enriched categories, a unit object $\mathbb{I}$ of $\mathcal{C}$, and natural associativity and commutativity isomorphisms that satisfy the axioms for a monoidal category.

A compatible symmetric monoidal structure on $\mathcal{C}$ is defined similarly.

Throughout this section we will fix such an enriched monoidal category $\mathcal{C}$.

**Definition 12.2**. Suppose that $S$ is a class of morphisms in $\mathcal{C}$. We say that $S$-equivalences are *compatible with the monoidal structure* (or simply that $S$ is compatible) if, for any $S$-equivalence $f\colon Y\to Y'$ and any object $X\in\mathcal{C}$, the maps $id_X\otimes f$ and $f\otimes id_X$ are $S$-equivalences.

**Proposition 12.3**. *Suppose that $S$ is compatible with the monoidal structure. Then localization respects the monoidal structure: any choices of localization give an equivalence*

$$L(X_1\otimes\cdots\otimes X_n)\to L(LX_1\otimes\cdots\otimes LX_n).$$

*Proof.* By induction, the map $X_1\otimes\cdots\otimes X_n\to LX_1\otimes\cdots\otimes LX_n$ is an $S$-equivalence, and therefore any $S$-localization of the latter is equivalent to any $S$-localization of the former. □

**Corollary 12.4**. *The monoidal structure on the homotopy category of $\mathcal{C}$ induces a monoidal structure on the homotopy catogory of the localization $L^S\mathcal{C}$, making any localization functor into a monoidal functor. If $\mathcal{C}$ was symmetric monoidal, then so is the localization.*

*Remark* 12.5. The inclusion $L^S\mathcal{C}\to\mathcal{C}$ is almost never monoidal. For example, it usually does not preserve the unit.

*Example* 12.6. Let $\mathcal{C}$ be the category of spaces with cartesian product, and let $E_*$ be a homology theory. Then any map $X \to X'$ which induces an isomorphism on $E_*$-homology also induces isomorphisms $E_*(X \times Y) \to E_*(X' \times Y)$ for any CW-complex $Y$: one can prove this inductively on the cells of $Y$. Therefore, $E$-homology equivalences are compatible with the Cartesian product monoidal structure.

Similarly, $E$-homology equivalences are compatible with the smash product on based spaces (using that based spaces are built from $S^0$) or the smash product on spectra (using that all spectra are built from spheres $S^n$).

*Example* 12.7. Let $\mathcal{C}$ be the category of spectra, and $f$ be the map $S^n \to *$. Then $f$-equivalences are maps inducing isomorphisms in degree strictly less than $n$. This is not compatible with the smash product on spectra: for example, smashing with $\Sigma^{-1}\mathbb{S}$ does not preserve $f$-equivalences. If one restricts to the subcategory of ***connective*** spectra, however, one finds that $f$-equivalences are compatible with the smash product.

*Example* 12.8. Consider the map $f\colon S^n \to *$ of spaces, so that $S$-equivalences are maps inducing an isomorphism on all homotopy groups in degrees less than $n$. This map is compatible with several symmetric monoidal structures, such as:

1. spaces with Cartesian product;
2. spaces with disjoint union;
3. based spaces with wedge product; and
4. based spaces with smash product.

Despite the usefulness of these results, the existence of a (symmetric) monoidal localization functor on the homotopy category does not, by itself, allow us to extend very structured multiplication from an object $X$ to its localization $LX$. To counter this we typically require the theory of operads.

**Definition 12.9**. Suppose that $\mathcal{C}$ is (symmetric) monoidal, and that $X$ is an object of $\mathcal{C}$. The endomorphism operad $\mathrm{End}_{\mathcal{C}}(X)$ is the (symmetric) sequence of spaces $\mathrm{Map}_{\mathcal{C}}(X \otimes \cdots \otimes X, X)$, with (symmetric) operad structure given by composition.

Given a map $f\colon X \to Y$, the endomorphism operad $\mathrm{End}_{\mathcal{C}}(f)$ is the (symmetric) sequence which in degree $n$ is the pullback diagram

$$\begin{tikzcd}
\mathrm{End}_{\mathcal{C}}(f)_n \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(X \otimes \cdots \otimes X, X) \arrow[d] \\
\mathrm{Map}_{\mathcal{C}}(Y \otimes \cdots \otimes Y, Y) \arrow[r] & \mathrm{Map}_{\mathcal{C}}(X \otimes \cdots \otimes X, Y).
\end{tikzcd}$$

The space $\mathrm{End}_{\mathcal{C}}(f)_n$ is the space of strictly commutative diagrams

$$\begin{tikzcd}
X^{\otimes n} \arrow[r] \arrow[d, "f^{\otimes n}"'] & X \arrow[d, "f"] \\
Y^{\otimes n} \arrow[r] & Y,
\end{tikzcd}$$

and as such the operad structure is given by composition.

The operad $\mathrm{End}_{\mathcal{C}}(f)$ has forgetful maps to $\mathrm{End}_{\mathcal{C}}(X)$ and $\mathrm{End}_{\mathcal{C}}(Y)$.

**Proposition 12.10**. *Suppose that the (symmetric) monoidal structure on $\mathcal{C}$ is compatible with $S$ and that $f\colon X \to LX$ is an $S$-localization. If the maps $\mathrm{Map}_{\mathcal{C}}(LX^{\otimes n}, LX) \to \mathrm{Map}_{\mathcal{C}}(X^{\otimes n}, LX)$ are fibrations for all $n \geq 0$, then in the diagram of operads*

$$\mathrm{End}_{\mathcal{C}}(X) \leftarrow \mathrm{End}_{\mathcal{C}}(f) \rightarrow \mathrm{End}_{\mathcal{C}}(LX),$$

*the left-hand arrow is an equivalence on the level of underlying spaces.*

*Proof.* This is merely the observation that $\mathrm{End}_{\mathcal{C}}(f) \to \mathrm{End}_{\mathcal{C}}(X)$ is, level by level, a homotopy pullback of the equivalences $\mathrm{Map}_{\mathcal{C}}(LX^{\otimes n}, LX) \to \mathrm{Map}_{\mathcal{C}}(X^{\otimes n}, LX)$. □

This condition then allows us to lift structured multiplication.

**Corollary 12.11**. *Suppose that a (symmetric) operad $\mathcal{O}$ acts on $X$ via a map $\mathcal{C} \to \mathrm{End}_{\mathcal{C}}(X)$. Then there exists a weak equivalence $\mathcal{O}' \to \mathcal{O}$ of operads and an action of $\mathcal{O}'$ on $LX$ such that $f$ is a map of $\mathcal{O}'$-algebras.*

*Proof.* We define $\mathcal{O}'$ to be the fiber product of the diagram $\mathcal{O} \to \mathrm{End}_{\mathcal{C}}(X) \leftarrow \mathrm{End}_{\mathcal{C}}(f)$. The map $\mathcal{O}' \to \mathcal{O}$ is an equivalence by the fibration condition, and the map $\mathcal{O}' \to \mathrm{End}_{\mathcal{C}}(f)$ of operads precisely states that $f$ is a map of $\mathcal{O}'$-algebras. [17] □

This means that $A_\infty$ and $E_\infty$ multiplications on $X$ extend automatically to $A_\infty$ and $E_\infty$ multiplications on $LX$. However, this is the best we can do in general: lifting more refined multiplicative structures requires stronger assumptions.

In cases where the category $\mathcal{C}$ has more structure, it is typically easier to verify that $S$ is compatible with the monoidal structure.

**Proposition 12.12**. *Suppose that the monoidal structure on $\mathcal{C}$ has internal function objects $F^L(X,Y)$ and $F^R(X,Y)$ that are adjoint to the monoidal structure: there are isomorphisms*

$$\mathrm{Map}_{\mathcal{C}}(X, F^L(Y,Z)) \cong \mathrm{Map}(X \otimes Y, Z) \cong \mathrm{Map}_{\mathcal{C}}(Y, F^R(X,Z))$$

*that are natural in $X$, $Y$, and $Z$. Then $S$ is compatible with the monoidal structure on $\mathcal{C}$ if and only if, for any $f\colon A \to B$ in $S$ and any object $X \in \mathcal{C}$, the maps $id_X \otimes f$ and $f \otimes id_X$ are $S$-equivalences.*

*Proof.* Suppose that for any $f\colon A \to B$ in $S$ and any object $X \in \mathcal{C}$, the maps $id_X \otimes f$ are $S$-equivalences. Using the unit isomorphisms, we find that if $Z$ is $S$-local the maps in the diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(X \otimes B, Z) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(X \otimes A, Z) \arrow[d] \\
\mathrm{Map}_{\mathcal{C}}(B, F^R(X,Z)) \arrow[r] & Map_{\mathcal{C}}(A, F^R(X,Z))
\end{tikzcd}$$

[17] If $\mathcal{O}$ happens to be a cofibrant (symmetric) operad $\mathcal{O}$ in Berger–Moerdijk's model structure [BM13] we can do better. Any map $\mathcal{O} \to \mathrm{End}_{\mathcal{C}}(X)$ lifts, up to homotopy, to a map $\mathcal{O} \to \mathrm{End}_{\mathcal{C}}(f) \to \mathrm{End}_{\mathcal{C}}(LX)$.

are equivalences. Therefore, $F^R(X, Z)$ is $S$-local, and so for any $S$-equivalence $f\colon Y \to Y'$ the maps in the diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(X \otimes Y', Z) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(X \otimes Y, Z) \arrow[d] \\
\mathrm{Map}_{\mathcal{C}}(Y', F^R(X, Z)) \arrow[r] & Map_{\mathcal{C}}(Y, F^R(X, Z))
\end{tikzcd}$$

are all equivalences. Similar considerations apply to $F^L$. $\square$

## 12.2 Monoidal model categories

The necessary conditions for compatibility between model structures and monoidal structures were determined by Schwede–Shipley [SS00] and Hovey [Hov99, §4.2], in the symmetric and nonsymmetric cases respectively. This structure allows us, after [SS00], to construct model structures on categories of algebras and modules in $\mathcal{M}'$ such that the localization functor $\mathcal{M} \to \mathcal{M}'$ preserves this structure.

**Definition 12.13.** A *(symmetric) monoidal model category* $\mathcal{M}$ is a model category with a (symmetric) monoidal closed structure[18] satisfying the following axioms.

1. (Pushout-product) Given cofibrations $i\colon A \to A$ and $j\colon B \to B'$ in $\mathcal{M}$, the induced pushout-product map

$$i \boxtimes j\colon (A \otimes B') \coprod_{A \otimes B} (A' \otimes B) \to A' \otimes B'$$

is a cofibration, which is acyclic if either $i$ or $j$ is.

2. (Unit) Let $Q\mathbb{I} \to \mathbb{I}$ be a cofibrant replacement of the unit. Then the natural maps $Q\mathbb{I} \otimes X \to X \leftarrow X \otimes Q\mathbb{I}$ are isomorphisms for all cofibrant $X$.

**Proposition 12.14.** *Suppose that $\mathcal{M}$ is a monoidal model category. Then, for cofibrant objects $X$, the functors $X \otimes (-)$ and $(-) \otimes X$ preserve cofibrations, acyclic cofibrations, and weak equivalences between cofibrant objects.*

*Proof.* Since $\otimes$ has adjoints, it preserves colimits in each variable. In particular, any object tensored with an initial object of $\mathcal{M}$ is an initial object of $\mathcal{M}$. Applying the pushout-product axiom to the map $\emptyset \to X$ in either variable, we find that the two functors in question preserve cofibrations and acyclic cofibrations. By Ken Brown's lemma, they also automatically take weak equivalences between cofibrant objects to weak equivalences. $\square$

This connects with our work in the the previous section, which only asked that the tensor product preserved equivalences in each variable. The pushout-product axiom for monoidal model categories looks stronger, in principle, but Proposition 12.14 has a partial converse.

[18] Analogously to the previous section, this means that the symmetric monoidal structure must have left and right function objects which are adjoints in each variable.

**Proposition 12.15.** *Suppose that $j\colon B \to B'$ is a map such that $(-) \otimes B$ preserves acyclic cofibrations and that $(-) \otimes B'$ preserves weak equivalences between cofibrant objects. If $i$ is an acyclic cofibration with cofibrant source, then the pushout-product map $i \boxtimes j$ is an equivalence.*

*Proof.* Without loss of generality, let $i\colon A \to A'$ be an acyclic cofibration and $j\colon B \to B'$ a cofibration, with all four objects cofibrant. Then the pushout-product $i \boxtimes j$ is part of the following diagram:

$$\begin{tikzcd}
 & A' \otimes B \arrow[d] \arrow[dr] & \\
A \otimes B \arrow[ur, "\sim"] \arrow[dr] & P \arrow[r, "i \boxtimes j"] & A' \otimes B' \\
 & A \otimes B' \arrow[u, "\sim"'] \arrow[ur, "\sim"'] &
\end{tikzcd}$$

The upper-left and lower-right maps are equivalences because they are obtained by tensoring an acyclic cofibration with the cofibrant objects $B$ and $B'$. The map $A \otimes B' \to P$ is the pushout of an acyclic cofibration, and so it is an acyclic cofibration. Therefore, by the 2-out-of-3 axiom the map $i \boxtimes j$ is an equivalence. □

The adunction isomorphism $\mathrm{Hom}_{\mathcal{M}}(X \otimes Y, Z) \cong \mathrm{Hom}_{\mathcal{M}}(X, F^R(Y, Z))$, and similarly for the left, allows us to rephrase the pushout-product axiom in multiple ways.

**Proposition 12.16** ([Hov99, 4.2.2]). *The following are equivalent for a model category $\mathcal{M}$ with a closed monoidal structure.*

1. *The model category $\mathcal{M}$ satisfies the pushout-product axiom.*
2. *For a cofibration $i\colon A \to B$ and a fibration $p\colon X \to Y$ in $\mathcal{M}$, the induced map*
$$F^R(B, X) \to F^R(B, Y) \times_{F^R(A,Y)} F^R(A, X)$$
*is a fibration, which is acyclic if either $i$ or $p$ are.*
3. *For a cofibration $i\colon A \to B$ and a fibration $p\colon X \to Y$ in $\mathcal{M}$, the induced map*
$$F^L(B, X) \to F^L(B, Y) \times_{F^L(A,Y)} F^L(A, X)$$
*is a fibration, which is acyclic if either $i$ or $p$ are.*

**Corollary 12.17** ([Hov99, 4.2.5]). *Suppose that $\mathcal{M}$ is a cofibrantly generated model category with a closed monoidal structure, a set $I$ of generating cofibrations and $J$ of generating acyclic cofibrations. Then the pushout-product axiom for $\mathcal{M}$ holds if and only if the pushout-product takes $I \times I$ to cofibrations in $\mathcal{M}$ and takes both $I \times J$ and $J \times I$ to acyclic cofibrations.*

Because left Bousfield localization doesn't change the cofibrations in a model structure, one is reduced to a few key verifications.

**Proposition 12.18.** *Suppose that $\mathcal{M}$ is a (symmetric) monoidal closed model category with left Bousfield localization $\mathcal{M}'$. Then $\mathcal{M}'$ is compatibly a (symmetric) monoidal model category if and only if, for cofibrations $i$ and $j$ such that one is acyclic, the pushout-product map $i \boxtimes j$ is acyclic.*

*If $\mathcal{M}'$ is cofibrantly generated, then it suffices to check that the pushout-product of a generating acyclic cofibration with a generating cofibration, in either order, is a weak equivalence.*

*Remark* 12.19. If the generating cofibrations and generating acyclic cofibrations of $\mathcal{M}'$ have cofibrant source, then by Proposition 12.15 we only need to show that tensoring with the sources or target of any map in $I$ or $J$ takes generating cofibrations in $\mathcal{M}'$ to weak equivalences.

*Remark* 12.20. Bousfield localization of stable model categories has been more extensively studied by Barnes and Roitzheim [BR14, BR15]. To have homotopical control over *commutative* algebra objects in a symmetric monoidal model category, one needs to obtain control over the extended power constructions; see [Whi].

## 12.3 Monoidal $\infty$-categories

We will begin by giving a brief background on monoidal structures on $\infty$-categories which is light on technical details.

Recall that a *multicategory* $\mathcal{O}$ is equivalent to the following data:

1. a collection of objects of $\mathcal{O}$;
2. for any object $Y$ and indexed set of objects $\{X_s\}_{s\in S}$ of $\mathcal{O}$, a space $\mathrm{Map}_{\mathcal{O}}(\{X_s\}_{s\in S}; Y)$ of multimaps; and
3. for a surjection $p\colon S \to T$ of finite sets, natural composition maps

$$\mathrm{Map}_{\mathcal{O}}(\{Y_t\}_{t\in T}; Z) \times \prod_{t\in T} \mathrm{Map}_{\mathcal{O}}(\{X_s\}_{s\in p^{-1}(t)}; Y_t) \to \mathrm{Map}_{\mathcal{O}}(\{X_s\}_{s\in S}; Z)$$

   that are compatible with composing surjections $S \to T \to U$.

*Remark* 12.21. As a special case, for $\sigma$ a permutation of $S$ there is an isomorphism $\mathrm{Map}_{\mathcal{O}}(\{X_s\}_{s\in S}; Y) \to \mathrm{Map}_{\mathcal{O}}(\{X_{\sigma(s)}\}_{s\in S}; Y)$, and the composition operations are appropriately equivariant with respect to these isomorphisms.

For such a multicategory, we could give a prototype definition of an $\mathcal{O}$-monoidal $\infty$-category $\mathcal{C}$ as an enriched functor from $\mathcal{O}$ to $\infty$-categories. This data specifies, for each object $X$ of $\mathcal{O}$, a category $\mathcal{C}_X$. For each object $Y$ and indexed set $\{X_s\}_{s\in S}$ of objects, there is a specified continuous map from $\mathrm{Map}_{\mathcal{O}}(\{X_s\}_{s\in S}; Y)$ to the space of functors $\prod_{s\in S} \mathcal{C}_{X_s} \to \mathcal{C}_Y$. Moreover, these maps must be compatible with composition on both sides.

The definition of an $\infty$-operad $\mathcal{O}$ and an $\mathcal{O}$-monoidal $\infty$-category $\mathcal{C}$ is slightly different from this [Lur17, §2.1]. Roughly, it is an *unstraightened* definition where the spaces of multimaps in $\mathcal{O}$ and the product functors on $\mathcal{C}$ are only specified up to a contractible space of choices; the technical details are related in spirit to Segal's work [Seg74]. Even though the functors induced from $\mathcal{O}$ are specified only up to contractible indeterminacy, it still makes sense to ask about compatibility of the monoidal structure with localization.

The following result very general result encodes the situations under which homotopical localization is compatible with monoidal structures.

**Theorem 12.22** ([Lur17, 2.2.1.9]). *Let $\mathcal{O}^{\otimes}$ be an $\infty$-operad and let $\mathcal{C}$ be an $\mathcal{O}$-monoidal $\infty$-category. Suppose that for all objects $X$ of $\mathcal{O}$ we have a localization functor $L_X\colon \mathcal{C}_X \to \mathcal{C}_X$, and that for any map $\alpha\colon \{X_s\}_{s\in S} \to Y$ in $\mathcal{O}^{\otimes}$ the induced functor $\prod_{s\in S} \mathcal{C}_{X_s} \to \mathcal{C}_Y$ preserves $L$-equivalences in each variable. Then there exists an $\mathcal{O}$-monoidal structure on the category $L\mathcal{C}$ of local objects making the localization $L\colon \mathcal{C} \to L\mathcal{C}$ into an $\mathcal{O}$-monoidal functor.*

**Corollary 12.23**. *Suppose that $\mathcal{C}$ is a (symmetric) monoidal $\infty$-category and that $L$ is a localization functor on $\mathcal{C}$ such that $L(X\otimes Y) \to L(LX \otimes LY)$ is always an equivalence. Then the subcategory $L\mathcal{C}$ of local objects has the structure of a (symmetric) monoidal $\infty$-category and any localization functor $L$ has the structure of a (symmetric) monoidal functor.*

*Example* 12.24. In the category of spaces, we can use the mapping space adjunctions and find that for any $S$-local object $Z$, we have

$$\begin{aligned}\operatorname{Map}(X\times Y, Z) &\simeq \operatorname{Map}(X, \operatorname{Map}(Y,Z))\\ &\simeq \operatorname{Map}(X, \operatorname{Map}(LY, Z)\\ &\simeq \operatorname{Map}(X\times LY, Z)\end{aligned}$$

and similarly on the other side, showing that $LX \times LY$ is a localization of $X\times Y$. This gives the cartesian product on spaces the special property that it is compatible with *all* localization functors.

*Example* 12.25. Fix an $E_n$-operad $\mathcal{O}$ and an $\mathcal{O}$-algebra $B$ in spaces representing an $n$-fold loop space. Consider the category $\mathcal{C}$ of functors $B \to \mathcal{S}$, viewed as local systems of spaces over $B$. Then the category $\mathcal{C}$ has a *Day convolution*, developed by Glasman [Gla16] in the $E_\infty$-case and by Lurie [Lur17, §2.2.6] in general, making $\mathcal{C}$ into an $\mathcal{O}$-monoidal category. The category $\mathcal{C}$ is equivalent (via unstraightening) to the category of spaces over $B$. In these terms the $\mathcal{O}$-monoidal structure is given by maps

$$\begin{aligned}\mathcal{O}(n) &\to \operatorname{Map}(B^n, B)\\ &\to \operatorname{Fun}((\mathcal{S}/B)^n, \mathcal{S}/B)\end{aligned}$$

that respect composition. Here $f \in \mathcal{O}(n)$ first goes to $f\colon B^n \to B$, then to the functor sending $\{X_i \to B\}$ to the map $\prod X_i \to B^n \xrightarrow{f} B$. An $\mathcal{O}$-algebra in $\mathcal{C}$ is equivalent to an $E_n$-space $X$ with a map $X \to B$ of $E_n$-spaces.

Suppose $L$ is a Bousfield localization on spaces, and consider the associated pointwise localization on the functor category $\mathcal{C}$ (which corresponds to the fiberwise localization on spaces over $B$). All operations in $\mathcal{O}$ are, up to homotopy, composites of the binary multiplication operation, and so it suffices to show that this preserves localization. However, if the maps $X_i \to B$ have homotopy fibers $F_i$, then the homotopy fiber of the map $X_1 \times X_2 \to B \times B \to B$ is, up to equivalence, the geometric realization of the bar construction

$$B(F_1, \Omega B, F_2).$$

Since any localization preserves homotopy colimits and products of spaces, this bar construction preserves it also. Therefore, fiberwise localization is an $E_n$-monoidal functor on the category of spaces over $B$.[19]

# References


[Bar10] Clark Barwick, *On left and right model categories and left and right Bousfield localizations*, Homology Homotopy Appl. **12** (2010), no. 2, 245–320. MR 2771591

[Bau60] Gilbert Baumslag, *Some aspects of groups with unique roots*, Acta Math. **104** (1960), 217–303. MR 0122859

[Bek00] Tibor Beke, *Sheafifiable homotopy model categories*, Math. Proc. Cambridge Philos. Soc. **129** (2000), no. 3, 447–475. MR 1780498

[BK72] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, Vol. 304, Springer-Verlag, Berlin, 1972. MR 0365573 (51 #1825)

[BL01] Andrew Baker and Andrej Lazarev, *On the Adams spectral sequence for $R$-modules*, Algebr. Geom. Topol. **1** (2001), 173–199. MR 1823498 (2002a:55011)

[BM13] Clemens Berger and Ieke Moerdijk, *On the homotopy theory of enriched categories*, Q. J. Math. **64** (2013), no. 3, 805–846. MR 3094501

[Bou75] A. K. Bousfield, *The localization of spaces with respect to homology*, Topology **14** (1975), 133–150. MR 0380779

[Bou79] ______, *The localization of spectra with respect to homology*, Topology **18** (1979), no. 4, 257–281. MR 551009 (80m:55006)

[BR14] David Barnes and Constanze Roitzheim, *Stable left and right Bousfield localisations*, Glasg. Math. J. **56** (2014), no. 1, 13–42. MR 3137847

[BR15] ______, *Homological localisation of model categories*, Appl. Categ. Structures **23** (2015), no. 3, 487–505. MR 3351091


[19] For *grouplike* $E_n$-spaces over a grouplike $B$, this is roughly the statement that we can take $n$-fold classifying spaces, apply the fiberwise localization, and then take $n$-fold loop spaces.

[Car08] Gunnar Carlsson, *Derived completions in stable homotopy theory*, J. Pure Appl. Algebra **212** (2008), no. 3, 550–577. MR 2365333 (2008m:55018)

[DHS88] Ethan S. Devinatz, Michael J. Hopkins, and Jeffrey H. Smith, *Nilpotence and stable homotopy theory. I*, Ann. of Math. (2) **128** (1988), no. 2, 207–241.

[Dug01] Daniel Dugger, *Universal homotopy theories*, Adv. Math. **164** (2001), no. 1, 144–176. MR 1870515

[Far96] Emmanuel Dror Farjoun, *Cellular spaces, null spaces and homotopy localization*, Lecture Notes in Mathematics, vol. 1622, Springer-Verlag, Berlin, 1996. MR 1392221

[Fox53] Ralph H. Fox, *Free differential calculus. I. Derivation in the free group ring*, Ann. of Math. (2) **57** (1953), 547–560. MR 0053938

[Gla16] Saul Glasman, *Day convolution for $\infty$-categories*, Math. Res. Lett. **23** (2016), no. 5, 1369–1385. MR 3601070

[GM95] J. P. C. Greenlees and J. P. May, *Completions in algebra and topology*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 255–276.

[Hir03] Philip S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003.

[Hov99] Mark Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999. MR 1650134 (99h:55031)

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659 (2010j:18001)

[Lur17] ———, *Higher Algebra*, Draft version available at: http://www.math.harvard.edu/~lurie/papers/higheralgebra.pdf, 2017.

[May72] J. Peter May, *The geometry of iterated loop spaces*, Springer-Verlag, Berlin, 1972, Lecture Notes in Mathematics, Vol. 271.

[Mil84] Haynes Miller, *The Sullivan conjecture on maps from classifying spaces*, Ann. of Math. (2) **120** (1984), no. 1, 39–87. MR 750716

[MN84] C. A. McGibbon and J. A. Neisendorfer, *On the homotopy groups of a finite-dimensional space*, Comment. Math. Helv. **59** (1984), no. 2, 253–257. MR 749108

[Mor85] Jack Morava, *Noetherian localisations of categories of cobordism comodules*, Ann. of Math. (2) **121** (1985), no. 1, 1–39. MR 782555 (86g:55004)

[MS76] D. McDuff and G. Segal, *Homology fibrations and the "group-completion" theorem*, Invent. Math. **31** (1975/76), no. 3, 279–284. MR 0402733

[Nei10] Joseph Neisendorfer, *Algebraic methods in unstable homotopy theory*, New Mathematical Monographs, vol. 12, Cambridge University Press, Cambridge, 2010. MR 2604913

[NS17] T. Nikolaus and P. Scholze, *On topological cyclic homology*, ArXiv e-prints (2017).

[Qui69a] Daniel Quillen, *On the formal group laws of unoriented and complex cobordism theory*, Bull. Amer. Math. Soc. **75** (1969), 1293–1298.

[Qui69b] ———, *Rational homotopy theory*, Ann. of Math. (2) **90** (1969), 205–295. MR 0258031

[Rav84] Douglas C. Ravenel, *Localization with respect to certain periodic homology theories*, Amer. J. Math. **106** (1984), no. 2, 351–414.

[Rav92] ———, *Nilpotence and periodicity in stable homotopy theory*, Annals of Mathematics Studies, vol. 128, Princeton University Press, Princeton, NJ, 1992, Appendix C by Jeff Smith. MR 1192553 (94b:55015)

[Seg74] Graeme Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312. MR 0353298 (50 #5782)

[Ser53] Jean-Pierre Serre, *Groupes d'homotopie et classes de groupes abéliens*, Ann. of Math. (2) **58** (1953), 258–294. MR 0059548

[SS00] Stefan Schwede and Brooke E. Shipley, *Algebras and modules in monoidal model categories*, Proc. London Math. Soc. (3) **80** (2000), no. 2, 491–511. MR 1734325 (2001c:18006)

[Sul74] Dennis Sullivan, *Genetics of homotopy theory and the Adams conjecture*, Ann. of Math. (2) **100** (1974), 1–79. MR 0442930

[Sul05] Dennis P. Sullivan, *Geometric topology: localization, periodicity and Galois symmetry*, $K$-Monographs in Mathematics, vol. 8, Springer, Dordrecht, 2005, The 1970 MIT notes, Edited and with a preface by Andrew Ranicki. MR 2162361

[Vog73] Rainer M. Vogt, *Homotopy limits and colimits*, Math. Z. **134** (1973), 11–52. MR 0331376

[Whi] David White, *Model structures on commutative monoids in general model categories*, arXiv:1403.6759.