THE CUBE AND THE BURNSIDE CATEGORY

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Abstract. In this note we present a combinatorial link invariant that underlies some recent stable homotopy refinements of Khovanov homology of links. The invariant takes the form of a functor between two combinatorial 2-categories, modulo a notion of stable equivalence. We also develop some general properties of such functors.

1. Introduction

In the last five years, two stable homotopy refinements of the Khovanov homology of links were introduced [LS14a, HKK]. In a recent paper, we showed that these two refinements agree, and in the process gave a simplified construction of these invariants [LLS]. That paper still involved a certain amount of topology. In this note, we present a combinatorial link invariant from which one can extract the Khovanov homotopy types, with the goal of making aspects of our earlier work more broadly accessible.

We will study functors from the cube category $\underline{2}^n$, the small category associated to the partially ordered set $\{0,1\}^n$ (Section 2), to the Burnside category $\mathscr{B}$, which is the 2-category whose objects are finite sets and morphisms are finite correspondences (Section 3). The original construction of Khovanov homology [Kho00] using Kauffman's n-dimensional cube of resolutions [Kau87] of an n-crossing link diagram K can be generalized to construct a 2-functor

$$F_{Kh}(K)\colon \underline{2}^n \to \mathscr{B}.$$

See Section 6 for the exact definition, and for some additional grading shifts that are involved. (Such functors also arise in other contexts, such as from a simplicial complex with n vertices; see Example 4.2.)

To any such functor $F\colon \underline{2}^n \to \mathscr{B}$, one can associate a chain complex $\mathsf{Tot}(F) \in \mathsf{Kom}$, the *totalization* of F. Indeed, this construction can be thought of a functor

$$\mathsf{Tot}\colon \mathscr{B}^{\underline{2}^n} \to \mathsf{Kom}.$$

The totalization of the functor F_{Kh} recovers the dual of the Khovanov complex of K:

$$(\mathcal{C}_{Kh}(K))^* = \mathsf{Tot}(F_{Kh}).$$

(The dual only appears to maintain consistency with earlier conventions [LS14a].)

Given a functor $F\colon \underline{2}^n \to \mathscr{B}$ and a sufficiently large $\ell \in \mathbb{Z}$, one can construct a based CW complex $\|F\|_\ell \in \mathsf{CW}$. The dependence on ℓ is simple: $\|F\|_{\ell+1}$ is merely the reduced suspension $\Sigma\|F\|_\ell$. The reduced cellular chain complex of $\|F\|_\ell$ is the earlier chain complex up to a shift:

$$\widetilde{C}^{cell}_\bullet(\|F\|_\ell) = \Sigma^\ell \mathsf{Tot}(F).$$

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Similarly, to any natural transformation $\eta\colon F \to F'$ between two such 2-functors $F, F'\colon \underline{2}^n \to \mathscr{B}$, one can associate a pointed cellular map $\|F\|_\ell \to \|F'\|_\ell$ that induces the map $\Sigma^\ell \mathsf{Tot}(\eta)$ on the reduced cellular chain complexes. Unfortunately, the definitions of these spaces and these maps depend on certain choices, and therefore the construction is not strictly functorial. We may eliminate these choices if we are willing to work with spectra. Indeed, if $\mathscr{S}$ is any reasonable category of spectra, then there is a canonical functor

$$|\cdot|\colon \mathscr{B}^{\underline{2}^n} \to \mathscr{S}$$

so that for any $F\colon \underline{2}^n \to \mathscr{B}$, $|F|$ is isomorphic to $\Sigma^{-\ell}(\Sigma^\infty \|F\|_\ell)$, the ℓ^{th} desuspension of the suspension spectrum of $\|F\|_\ell$. Up to a grading shift, the Khovanov homotopy type associated to the n-crossing link diagram K is

$$\mathcal{X}_{Kh}(K) = |F_{Kh}|,$$

and therefore, its reduced cohomology $\widetilde{H}^\bullet(\mathcal{X}_{Kh}(K))$ is isomorphic to the Khovanov homology $Kh(K) = H_\bullet(\mathcal{C}_{Kh}(K))$.

In this note, we only sketch the construction of $\|F\|_\ell$ and only hint at the construction of $|F|$ (Section 7). We will focus on functors from the cube to the Burnside category and define a (stable) equivalence relation on such functors, generated by the following two relations:

(1) If $\eta\colon F \to F'$ is a natural transformation between two functors $F, F'\colon \underline{2}^n \to \mathscr{B}$ and the induced map $\mathsf{Tot}(\eta)\colon \mathsf{Tot}(F) \to \mathsf{Tot}(F')$ is a chain homotopy equivalence, then F is stably equivalent to F'.
(2) If $\iota\colon \underline{2}^n \hookrightarrow \underline{2}^N$ is a face inclusion, and $F\colon \underline{2}^n \to \mathscr{B}$, then there is an induced functor $F_\iota\colon \underline{2}^N \to \mathscr{B}$, defined by declaring that $F_\iota \circ \iota = F$ and that for any $v \in \underline{2}^N \setminus \iota(\underline{2}^n)$, $F_\iota(v) = \varnothing$. The functor F_ι is stably equivalent to F (modulo some grading shift).

See Definition 5.9 for the precise version. This notion of stable equivalence ensures that if F, F' are stably equivalent functors, then $|F|$ and $|F'|$ are homotopy equivalent spectra; or equivalently, $\|F\|_\ell$ and $\|F'\|_\ell$ are stably homotopy equivalent CW complexes.

The main result of this paper is that the Khovanov functor F_{Kh} is a link invariant. Namely, if K and K' are two isotopic link diagrams, then the Khovanov functors $F_{Kh}(K)$ and $F_{Kh}(K')$ are stably equivalent in the above sense (Theorem 1, Section 6).

2. The cube

The one-dimensional cube is $\underline{2} = \{0, 1\}$. It can be viewed as a partially ordered set by declaring $1 > 0$. It can also be viewed as a category with a single non-identity morphism from 1 to 0. There is a notion of grading, where we declare the grading of $v \in \underline{2}$ to be simply $|v| = v$.

The n-dimensional cube is the Cartesian product $\underline{2}^n = \{0, 1\}^n$. It has an induced partial order, where

$$(u_1, \dots, u_n) \geq (v_1, \dots, v_n) \text{ if and only if } \forall i (u_i \geq v_i),$$

and an induced categorical structure: the morphism set $\mathrm{Hom}_{\underline{2}^n}(u, v)$ has a single element if $u \geq v$, and is empty otherwise. For $u \geq v$, we will write $\varphi_{u,v}$ to denote the unique morphism in $\mathrm{Hom}_{\underline{2}^n}(u, v)$. Finally, there is an induced grading, which is simply the L^1-norm:

$$|v| = \sum_i v_i.$$

For convenience, we will write $u \geq_k v$ if $u \geq v$ and $|u| - |v| = k$; and we will sometimes write $u \multimap v$ if $u \geq_1 v$.

We will need the following *sign assignment* function.

Definition 2.1. Given $u = (u_1, \dots, u_n) \geq_1 v = (v_1, \dots, v_n)$, let k be the unique element in $\{1, \dots, n\}$ satisfying $u_k > v_k$, and define

$$s_{u,v} = \sum_{i=1}^{k-1} u_i \pmod 2.$$

3. The Burnside category

For us, the Burnside category $\mathscr{B}$ is the 2-category of finite sets, finite correspondences, and bijections of correspondences. The objects $\mathrm{Ob}(\mathscr{B})$ are finite sets; for any two objects A, B, the morphisms $\mathrm{Hom}_{\mathscr{B}}(A, B)$ are the finite correspondences (or spans) from A to B, that is, triples (X, s, t) where X is a finite set, and $s\colon X \to A$ and $t\colon X \to B$ are set maps, called the *source map* and the *target map*, respectively. We usually denote such correspondences by diagrams $A \xleftarrow{s} X \xrightarrow{t} B$; and we often drop s and t from the notation if they are irrelevant to the discussion. The identity morphism $\mathrm{Id}_A \in \mathrm{Hom}_{\mathscr{B}}(A, A)$ is the correspondence $A \xleftarrow{\mathrm{Id}} A \xrightarrow{\mathrm{Id}} A$. Composition is given by fiber product. That is, for X in $\mathrm{Hom}_{\mathscr{B}}(A, B)$ and Y in $\mathrm{Hom}_{\mathscr{B}}(B, C)$, the composition $Y \circ X$ is defined to be the fiber product $Y \times_B X = \{(y, x) \in Y \times X \mid s(y) = t(x)\}$ in $\mathrm{Hom}_{\mathscr{B}}(A, C)$:

$$\begin{array}{ccccccc} & & & Y \times_B X & & & \\ & & \swarrow & & \searrow & & \\ & X & & & & Y & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ A & & & B & & & C. \end{array}$$

For morphisms X, Y in $\mathrm{Hom}_{\mathscr{B}}(A, B)$, define the 2-morphisms from X to Y to be the bijections $X \xrightarrow{\cong} Y$ so that the following diagram commutes:

$$\begin{array}{ccccc} & & X & & \\ & \swarrow & \downarrow & \searrow & \\ & & Y & & \\ & \swarrow & & \searrow & \\ A & & & & B. \end{array}$$

The Burnside category $\mathscr{B}$ is self-dual, in the sense of being isomorphic to its own opposite $\mathscr{B}^{\mathrm{op}}$. The isomorphism preserves objects, and sends $(X, s, t) \in \mathrm{Hom}_{\mathscr{B}}(A, B)$ to $(X, t, s) \in \mathrm{Hom}_{\mathscr{B}^{\mathrm{op}}}(A, B)$. The category of finite sets is a subcategory of $\mathscr{B}$ since we can view any set map $f\colon A \to B$ as the correspondence $A \xleftarrow{\mathrm{Id}} A \xrightarrow{f} B$.

We will need the following functor from $\mathscr{B}$ to the category of Abelian groups Ab.

Definition 3.1. Define the functor $\mathcal{A}\colon \mathscr{B} \to \mathsf{Ab}$ as follows. For $A \in \mathrm{Ob}(\mathscr{B})$, define $\mathcal{A}(A) = \mathbb{Z}\langle A \rangle$, the free Abelian group with basis A. For a correspondence $(X, s, t) \in \mathrm{Hom}_{\mathscr{B}}(A, B)$, define the map $\mathcal{A}(X)\colon \mathcal{A}(A) \to \mathcal{A}(B)$ by defining it on the basis elements $a \in A$ as

$$\mathcal{A}(X)(a) = \sum_{b \in B} \left| s^{-1}(a) \cap t^{-1}(b) \right| b.$$

If, in $\mathscr{B}$, we replace correspondences with isomorphism classes of correspondences, we obtain an ordinary category. The above functor identifies this category with a subcategory of Ab, whose objects are the finitely generated free Abelian groups with a chosen basis and whose morphisms are represented by matrices of nonnegative integers.

4. Functors from the cube to $\mathscr{B}$

The central objects that we will study are pairs (F, r), where F is a functor $\underline{2}^n \to \mathscr{B}$ and $r \in \mathbb{Z}$ is an integer; we will use the notation $\Sigma^r F$ to denote such a pair, and call it a *stable functor*. The grading shifts are usually inconsequential up to some signs, and consequently we will often suppress Σ^r from the notation.

The cube category $\underline{2}^n$ is a strict category, while the Burnside category $\mathscr{B}$ is a weak 2-category, so we owe some explanation regarding what we mean by a functor $F\colon \underline{2}^n \to \mathscr{B}$. We will treat $\underline{2}^n$ as a 2-category with no non-identity 2-morphisms, and the functor F will be a strictly unitary lax 2-functor. (For general definitions, see [Bén67, Definition 4.1 and Remark 4.2] which calls such functors *strictly unitary homomorphisms*.)

Definition 4.1. A *strictly unitary lax* 2*-functor* F from the cube category $\underline{2}^n$ to the Burnside category $\mathscr{B}$ consists of the following data:

(1) a finite set $F(v) \in \mathrm{Ob}(\mathscr{B})$ for every $v \in \{0,1\}^n$;
(2) a finite correspondence $F(\varphi_{u,v}) \in \mathrm{Hom}_{\mathscr{B}}(F(u), F(v))$ for every $u > v$;
(3) a 2-isomorphism $F_{u,v,w}\colon F(\varphi_{v,w}) \times_{F(v)} F(\varphi_{u,v}) \to F(\varphi_{u,w})$ for every $u > v > w$;

so that for all $u > v > w > z$, the following diagram commutes:

$$\begin{array}{ccc} F(\varphi_{w,z}) \times_{F(w)} F(\varphi_{v,w}) \times_{F(v)} F(\varphi_{u,v}) & \xrightarrow{\mathrm{Id} \times F_{u,v,w}} & F(\varphi_{w,z}) \times_{F(w)} F(\varphi_{u,w}) \\ \Big\downarrow {\scriptstyle F_{v,w,z} \times \mathrm{Id}} & & \Big\downarrow {\scriptstyle F_{u,w,z}} \\ F(\varphi_{v,z}) \times_{F(v)} F(\varphi_{u,v}) & \xrightarrow[F_{u,v,z}]{} & F(\varphi_{u,z}). \end{array} \tag{4.1}$$

(Here, $F(\varphi_{w,z}) \times_{F(w)} F(\varphi_{v,w}) \times_{F(v)} F(\varphi_{u,v})$ denotes either $\big(F(\varphi_{w,z}) \times_{F(w)} F(\varphi_{v,w})\big) \times_{F(v)} F(\varphi_{u,v})$ or $F(\varphi_{w,z}) \times_{F(w)} \big(F(\varphi_{v,w}) \times_{F(v)} F(\varphi_{u,v})\big)$, which are not the same but are canonically identified.)

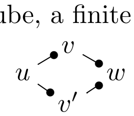
We will typically refer to strictly unitary lax 2-functors simply as 2*-functors* or even just *functors*.

Example 4.2. Let X be a finite Δ-complex (as in [Hat02, Section 2.1]) such that every k-simplex contains $(k+1)$ distinct vertices. (For example, X could be a finite simplicial complex.) For each k, let $X(k+1)$ denote the set of k-simplices of X, and let $X(0) = \varnothing$. Let $\{p_1, \dots, p_n\} = X(1)$ be the vertices of X. Then we get a functor $F_\Delta\colon \Sigma^{-1}\underline{2}^n \to \mathscr{B}$ with

$$F_\Delta(v) = \{\Delta \in X(|v|) \mid \forall i \big((v_i = 1) \iff (p_i \in \Delta)\big)\} \qquad \text{for every } v \in \{0,1\}^n,$$
$$F_\Delta(\varphi_{u,v}) = \{(\Delta_v, \Delta_u) \in F_\Delta(v) \times F_\Delta(u) \mid \Delta_v \subset \Delta_u\} \qquad \text{for every } u > v \text{ in } \{0,1\}^n.$$

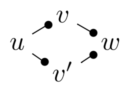
The source and target maps for the correspondences are the two projection maps. For every $u > v > w$, the 2-isomorphisms $F_{u,v,w}$ are uniquely determined, and the uniqueness forces Diagram (4.1) to commute.

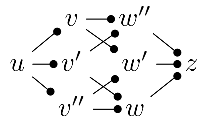
Like the name, the data for a strictly unitary lax 2-functor $\underline{2}^n \to \mathscr{B}$ might seem unwieldy, but fortunately there is a smaller formulation. Consider the following three pieces of data:

(D-1) for every vertex $v \in \{0,1\}^n$ of the cube, a finite set $F(v) \in \mathrm{Ob}(\mathscr{B})$,
(D-2) for every edge $u \to v$ of the cube, a finite correspondence $F(\varphi_{u,v}) \in \mathrm{Hom}_{\mathscr{B}}(F(u), F(v))$,
(D-3) for every two-dimensional face  of the cube, a 2-morphism

$$F_{u,v,v',w}\colon F(\varphi_{v,w}) \times_{F(v)} F(\varphi_{u,v}) \to F(\varphi_{v',w}) \times_{F(v')} F(\varphi_{u,v'}),$$

satisfying the following two conditions:

(C-1) for every two-dimensional face  , $F_{u,v',v,w} = F^{-1}_{u,v,v',w}$,

(C-2) for every three-dimensional face  , the following commutes:

$$\begin{array}{ccc}
F(\varphi_{w'',z}) \times_{F(w'')} F(\varphi_{v,w''}) \times_{F(v)} F(\varphi_{u,v}) & \xrightarrow{F_{v,w'',w',z}\times \mathrm{Id}} & F(\varphi_{w',z}) \times_{F(w')} F(\varphi_{v,w'}) \times_{F(v)} F(\varphi_{u,v}) \\
\big\downarrow {\scriptstyle \mathrm{Id}\times F_{u,v,v',w''}} & & \big\downarrow {\scriptstyle \mathrm{Id}\times F_{u,v,v'',w'}} \\
F(\varphi_{w'',z}) \times_{F(w'')} F(\varphi_{v',w''}) \times_{F(v')} F(\varphi_{u,v'}) & & F(\varphi_{w',z}) \times_{F(w')} F(\varphi_{v'',w'}) \times_{F(v'')} F(\varphi_{u,v''}) \\
\big\downarrow {\scriptstyle F_{v',w'',w,z}\times \mathrm{Id}} & & \big\downarrow {\scriptstyle F_{v'',w',w,z}\times \mathrm{Id}} \\
F(\varphi_{w,z}) \times_{F(w)} F(\varphi_{v',w}) \times_{F(v')} F(\varphi_{u,v'}) & \xrightarrow{\mathrm{Id}\times F_{u,v',v'',w}} & F(\varphi_{w,z}) \times_{F(w)} F(\varphi_{v'',w}) \times_{F(v'')} F(\varphi_{u,v''}).
\end{array}$$

A strictly unitary lax 2-functor F produces such data (D-1)–(D-3) satisfying conditions (C-1)–(C-2), by simply declaring that $F_{u,v,v',w} = F^{-1}_{u,v',w} \circ F_{u,v,w}$. Conversely:

Proposition 4.3. *Assume we are given data (D-1)–(D-3) satisfying conditions (C-1)–(C-2). Then up to natural isomorphism, there is exactly one strictly unitary 2-functor $F\colon \underline{2}^n \to \mathscr{B}$ that produces it.*

Proof. This is [LLS, Lemma 2.12], but for completeness, we give a proof. For both existence and uniqueness, we need the following facts about maximal chains on the cube $\underline{2}^n$. Fix $u \geq_k v$, and consider maximal chains $u = z_0 \to \cdots \to z_i \to \cdots \to z_k = v$. Then:

(m-1) Any two such maximal chains $\mathfrak{m}_1$ and $\mathfrak{m}_2$ can be connected by a sequence of swaps across two-dimensional faces, that is, by a sequence of replacements of chains $\big(\cdots \to z_{i-1} \to z_i \to z_{i+1} \to \cdots\big)$ by $\big(\cdots \to z_{i-1} \to z'_i \to z_{i+1} \to \cdots\big)$.

(m-2) Any two such sequences $\mathfrak{s}_1$ and $\mathfrak{s}_2$ connecting any two such maximal chains $\mathfrak{m}_1$ and $\mathfrak{m}_2$ can be related by a sequence of moves of the following three types:
  (a) Replacing a sequence of the form $\{\dots, \mathfrak{m}_\ell, \mathfrak{m}'_\ell, \mathfrak{m}_\ell, \dots\}$ with the sequence $\{\dots, \mathfrak{m}_\ell, \dots\}$.
  (b) Exchanging a sequence of one of the following two forms for the other:

$$\begin{array}{c}
\vdots \\
\cdots \to z_{i-1} \to z_i \to z_{i+1} \to \cdots \to z_{j-1} \to z_j \to z_{j+1} \to \cdot \\
\cdots \to z_{i-1} \to z'_i \to z_{i+1} \to \cdots \to z_{j-1} \to z_j \to z_{j+1} \to \cdot \\
\cdots \to z_{i-1} \to z'_i \to z_{i+1} \to \cdots \to z_{j-1} \to z'_j \to z_{j+1} \to \cdot \\
\vdots
\end{array}
\longleftrightarrow
\begin{array}{c}
\vdots \\
\cdot \to z_{i-1} \to z_i \to z_{i+1} \to \cdots \to z_{j-1} \to z_j \to z_{j+1} \to \cdots \\
\cdot \to z_{i-1} \to z_i \to z_{i+1} \to \cdots \to z_{j-1} \to z'_j \to z_{j+1} \to \cdots \\
\cdot \to z_{i-1} \to z'_i \to z_{i+1} \to \cdots \to z_{j-1} \to z'_j \to z_{j+1} \to \cdots \\
\vdots
\end{array},$$

where $j - i \geq 2$. (This corresponds to exchanging the order of swaps across two faces which share no edges.)

(c) Exchanging a sequence of one of the following two forms for the other:

$$\begin{array}{c} \vdots \\ \cdots \to z_i \to z_{i+1} \to z''_{i+2} \to z_{i+3} \to \cdots \\ \cdots \to z_i \to z'_{i+1} \to z''_{i+2} \to z_{i+3} \to \cdots \\ \cdots \to z_i \to z'_{i+1} \to z_{i+2} \to z_{i+3} \to \cdots \\ \cdots \to z_i \to z''_{i+1} \to z_{i+2} \to z_{i+3} \to \cdots \\ \vdots \end{array} \longleftrightarrow \begin{array}{c} \vdots \\ \cdots \to z_i \to z_{i+1} \to z''_{i+2} \to z_{i+3} \to \cdots \\ \cdots \to z_i \to z_{i+1} \to z'_{i+2} \to z_{i+3} \to \cdots \\ \cdots \to z_i \to z''_{i+1} \to z'_{i+2} \to z_{i+3} \to \cdots \\ \cdots \to z_i \to z''_{i+1} \to z_{i+2} \to z_{i+3} \to \cdots \\ \vdots \end{array}.$$

(This corresponds to the six faces of the cube $\cdots \to z_i \to \{z_{i+1}, z'_{i+1}, z''_{i+1}\} \to \{z''_{i+2}, z'_{i+2}, z_{i+2}\} \to z_{i+3} \to \cdots$.)

These facts are easy to check directly. Alternatively, they are obvious from a geometric reformulation. The maximal chains in the cube correspond to the vertices of the permutohedron Π_{k-1}. The edges of Π_{k-1} correspond to swaps across two-dimensional faces, as described in (m-1), and the two-dimensional faces of Π_{k-1} are either squares or hexagons, corresponding to the second and third moves of (m-2). (For more details about the permutohedron, see, for instance, [Zie95].) Therefore, (m-1) can be restated by saying that any two vertices of Π_{k-1} can be connected by a path along the edges, and (m-2) can be restated by saying that any two such paths can be connected by homotoping across two-dimensional faces.

Now consider the given data (D-1)–(D-3) that satisfies conditions (C-1)–(C-2). For each $u \geq_k v$, choose a maximal chain $\mathfrak{m}^{u,v} = \{\, u = z_0^{u,v} \to \cdots \to z_i^{u,v} \to \cdots \to z_k^{u,v} = v \,\}$ and define

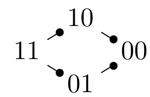
$$F(\varphi_{u,v}) = F(\varphi_{z_{k-1}^{u,v}, z_k^{u,v}}) \times_{F(z_{k-1}^{u,v})} \cdots \times_{F(z_1^{u,v})} F(\varphi_{z_0^{u,v}, z_1^{u,v}}).$$

For $u \geq_k v \geq_\ell w$, define the 2-isomorphism $F_{u,v,w} \colon F(\varphi_{v,w}) \times_{F(v)} F(\varphi_{u,v}) \to F(\varphi_{u,w})$ by choosing a sequence of maximal chains connecting $\mathfrak{m}^{v,w} \cup \mathfrak{m}^{u,v}$ to $\mathfrak{m}^{u,w}$ of the type described in (m-1), and using the maps provided by the data (D-3). Since the data satisfies conditions (C-1)–(C-2), (m-2) implies that the 2-isomorphism is independent of the choice of the sequence of maximal chains. The same argument shows that these 2-isomorphisms satisfy Diagram (4.1). Therefore, this defines a strictly unitary lax 2-functor $F \colon \underline{2}^n \to \mathscr{B}$. The construction is clearly unique up to natural isomorphism. □

Notice that the existence of the $F_{u,v,v',w}$ implies the following:

(C-0) for every two-dimensional face $u \to v, v' \to w$, $\left|F(\varphi_{v,w}) \times_{F(v)} F(\varphi_{u,v})\right| = \left|F(\varphi_{v',w}) \times_{F(v')} F(\varphi_{u,v'})\right|$.

The converse is not true: given data (D-1)–(D-2) satisfying condition (C-0), there might be no way, one way, or more than one way of constructing data (D-3) satisfying conditions (C-1)–(C-2), as the following two examples illustrate.

Example 4.4. Assume we have the following data (D-1)–(D-2) on the 2-dimensional cube 

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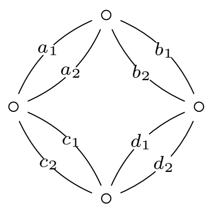
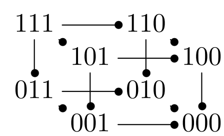


(We have not labeled the elements of the one-element sets $F(v)$, only the elements of the two-element correspondences $F(\varphi_{u,v})$.) This does not uniquely specify data (D-3). Up to natural isomorphism, we may assume

$$F_{11,10,01,00}\colon \{a_1b_1, a_1b_2, a_2b_1, a_2b_2\} \to \{c_1d_1, c_1d_2, c_2d_1, c_2d_2\}$$

sends a_1b_1 to c_1d_1; but there are still six ways to define the bijection $F_{11,10,01,00}$ that are not naturally isomorphic. (Indeed, some of these six functors are not even equivalent in the sense of Definition 5.9; see Remark 5.13.)

Example 4.5. Assume we have the following data (D-1)–(D-2) on the 3-dimensional cube 

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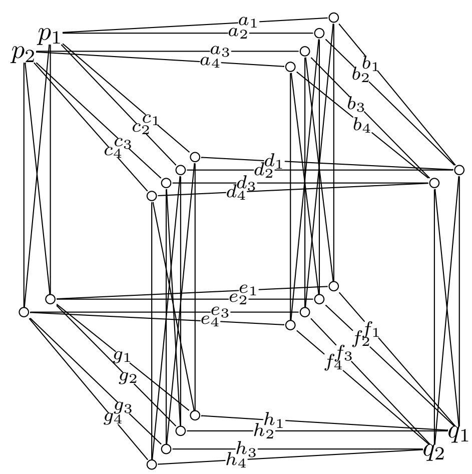


(We have only labeled the elements of the two-element sets $F(111)$ and $F(000)$, and the elements of some of the correspondences $F(\varphi_{u,v})$. In case the picture is not clear, the correspondence $F(\varphi_{110,010})$ sends $t(a_1)$ to $t(e_1)$ and $t(e_3)$, $t(a_2)$ to $t(e_2)$ and $t(e_4)$, $t(a_3)$ to $t(e_1)$ and $t(e_3)$, and $t(a_4)$ to $t(e_2)$ and $t(e_4)$; and the correspondence $F(\varphi_{101,001})$ sends $t(c_1)$ to $t(g_1)$ and $t(g_3)$, $t(c_2)$ to $t(g_2)$ and $t(g_4)$, $t(c_3)$ to $t(g_2)$ and $t(g_3)$, and $t(c_4)$ to $t(g_1)$ and $t(g_4)$.) Let us attempt to construct data (D-3). Up to natural isomorphism, we may assume the bijection $F_{111,110,101,100}$ sends a_1b_1 to c_1d_1. Looking at the correspondence from p_1 to q_1, conditions (C-1)–(C-2) force $F_{011,010,001,000}(e_1f_1) = g_1h_1$. Similarly, using the correspondence from p_1 to q_2, we get $F_{011,010,001,000}(e_3f_3) = g_3h_3$. Using the correspondence from p_2 to q_1, the former implies $F_{111,110,101,100}(a_3b_3) = c_4d_4$; but using the correspondence from p_2 to q_2, the latter implies

$F_{111,110,101,100}(a_3b_3) = c_3d_3$, which is a contradiction. So even though the given data satisfies condition (C-0), it is impossible to construct data (D-3) satisfying conditions (C-1)–(C-2).

5. Properties of such functors

In this section, we will discuss a few constructions that can be carried out with such functors from the cube $\underline{2}^n$ to the Burnside category $\mathscr{B}$.

Definition 5.1. If $F\colon \underline{2}^n \to \mathscr{B}$, then $\mathsf{Tot}(F) \in \mathsf{Kom}$ is defined to be the following chain complex. The chain group is defined to be

$$\mathsf{Tot}(F) = \bigoplus_{v \in \underline{2}^n} \mathcal{A}(F(v)),$$

the homological grading of the summand $\mathcal{A}(F(v))$ is defined to be $|v|$, and the differential $\partial\colon \mathsf{Tot}(F) \to \mathsf{Tot}(F)$ is defined by declaring the component $\partial_{u,v}$ of ∂ that maps from $\mathcal{A}(F(u))$ to $\mathcal{A}(F(v))$ to be

$$\partial_{u,v} = \begin{cases} (-1)^{s_{u,v}} \mathcal{A}(F(\varphi_{u,v})) & \text{if } u \geq_1 v, \\ 0 & \text{otherwise.} \end{cases}$$

(Here $s_{u,v}$ is the sign assignment from Definition 2.1 and $\mathcal{A}\colon \mathscr{B} \to \mathsf{Ab}$ is the functor defined in Definition 3.1.) For all $r \in \mathbb{Z}$, define $\mathsf{Tot}(\Sigma^r F) = \Sigma^r \mathsf{Tot}(F)$, the chain complex with gradings shifted up by r.

Remark 5.2. In terms of the reformulation from Proposition 4.3, the functor $F\colon \underline{2}^n \to \mathscr{B}$ is equivalent to data (D-1)–(D-3) satisfying conditions (C-1)–(C-2). In order to define the chain complex $\mathsf{Tot}(F)$, it is enough to have data (D-1)–(D-2) satisfying condition (C-0).

Definition 5.3. If $F, F'\colon \underline{2}^n \to \mathscr{B}$ are two 2-functors, then the *coproduct* $F \amalg F'\colon \underline{2}^n \to \mathscr{B}$ is defined as follows.

(1) For all $v \in \underline{2}^n$, $(F \amalg F')(v) = F(v) \amalg F'(v)$.
(2) For all $u > v$, $(F \amalg F')(\varphi_{u,v}) = F(\varphi_{u,v}) \amalg F'(\varphi_{u,v})$ with the source and target maps defined component-wise.
(3) For all $u > v > w$, the map $(F \amalg F')_{u,v,w}$ from $(F \amalg F')(\varphi_{v,w}) \times_{(F \amalg F')(v)} (F \amalg F')(\varphi_{u,v}) \cong \big(F(\varphi_{v,w}) \times_{F(v)} F(\varphi_{u,v})\big) \amalg \big(F'(\varphi_{v,w}) \times_{F'(v)} F'(\varphi_{u,v})\big)$ to $(F \amalg F')(\varphi_{u,w}) = F(\varphi_{u,w}) \amalg F'(\varphi_{u,w})$ is defined to be $F_{u,v,w}$ on the first component and $F'_{u,v,w}$ on the second component.

It is straightforward to check that this defines a strictly unitary lax 2-functor $\underline{2}^n \to \mathscr{B}$, and $\mathsf{Tot}(F \amalg F') = \mathsf{Tot}(F) \oplus \mathsf{Tot}(F')$.

Definition 5.4. If $F_1\colon \underline{2}^{n_1} \to \mathscr{B}$, $F_2\colon \underline{2}^{n_2} \to \mathscr{B}$ are two 2-functors, then the *product* $F_1 \times F_2\colon \underline{2}^{n_1+n_2} \to \mathscr{B}$ is defined as follows.

(1) For all $(v_1, v_2) \in \underline{2}^{n_1} \times \underline{2}^{n_2}$, $(F_1 \times F_2)((v_1, v_2)) = F_1(v_1) \times F_2(v_2)$.
(2) For all $(u_1, u_2) > (v_1, v_2)$, $(F_1 \times F_2)(\varphi_{(u_1,u_2),(v_1,v_2)}) = F_1(\varphi_{u_1,v_1}) \times F_2(\varphi_{u_2,v_2})$ with the source and target maps defined component-wise, with the understanding that if $u_i = v_i$ the correspondence $F_i(\varphi_{u_i,v_i})$ is the identity.
(3) For all $(u_1, u_2) > (v_1, v_2) > (w_1, w_2)$, the map $(F_1 \times F_2)_{(u_1,u_2),(v_1,v_2),(w_1,w_2)}$ is defined as
$$(F_1 \times F_2)_{(u_1,u_2),(v_1,v_2),(w_1,w_2)}(x_1, x_2) = \big((F_1)_{u_1,v_1,w_1}(x_1), (F_2)_{u_2,v_2,w_2}(x_2)\big),$$
with the understanding that if $u_i = v_i$ or $v_i = w_i$ then $(F_i)_{u_i,v_i,w_i}$ is the identity map.

Once again, it is straightforward to check that this defines a strictly unitary lax 2-functor $\underline{2}^{n_1+n_2} \to \mathscr{B}$. This time, $\mathsf{Tot}(F_1 \times F_2) = \mathsf{Tot}(F_1) \otimes \mathsf{Tot}(F_2)$: the sign assignment from Definition 2.1 translates into the Koszul sign convention on the tensor product.

Definition 5.5. A *face inclusion* ι is a functor $\underline{2}^n \to \underline{2}^N$ that is injective on objects, and preserves the relative gradings. Face inclusions can be described as functors of the following form: Fix $U \geq_n W$ in $\underline{2}^N$, and let $\{V_1,\dots,V_n\} = \{V \in \underline{2}^N \mid U \geq_{n-1} V \geq_1 W\}$. The functor that sends the object $(v_1,\dots,v_n) \in \underline{2}^n$ to the object $(W + \sum_i v_i(V_i - W)) \in \underline{2}^N$ is a face inclusion.

Let $|\iota| = |\iota(v)| - |v|$ for any $v \in \underline{2}^n$ be the grading of W.

Remark 5.6. The autoequivalences $\iota\colon \underline{2}^n \to \underline{2}^n$ are face inclusions. Note that the group of autoequivalences of $\underline{2}^n$ is the permutation group $\mathcal{S}_n$, where $\sigma \in \mathcal{S}_n$ corresponds to the autoequivalence that sends $(v_1,\dots,v_n) \in \underline{2}^n$ to $(v_{\sigma^{-1}(1)},\dots,v_{\sigma^{-1}(n)})$.

Definition 5.7. If $\iota\colon \underline{2}^n \hookrightarrow \underline{2}^N$ is a face inclusion, and $F\colon \underline{2}^n \to \mathscr{B}$ is a functor, then the induced functor $F_\iota\colon \underline{2}^N \to \mathscr{B}$ is uniquely defined by imposing $F = F_\iota \circ \iota$, and for all objects $v \in \underline{2}^N \setminus \iota(\underline{2}^n)$, $F_\iota(v) = \varnothing$.

Note that the chain complexes $\mathsf{Tot}(F_\iota)$ and $\Sigma^{|\iota|}\mathsf{Tot}(F)$ are canonically isomorphic except for signs. For all $u \geq_1 v$ in $\underline{2}^n$, the component of the differential from $\mathcal{A}(F(u))$ to $\mathcal{A}(F(v))$ has sign $(-1)^{s_{\iota(u),\iota(v)}}$ in $\mathsf{Tot}(F_\iota)$ and sign $(-1)^{|\iota|+s_{u,v}}$ in $\Sigma^{|\iota|}\mathsf{Tot}(F)$. To every $v \in \underline{2}^n$, assign $t_v \in \mathbb{Z}/2$, so that for all $u \geq_1 v$, $t_u + t_v = |\iota| + s_{u,v} + s_{\iota(u),\iota(v)}$. (Such an assignment exists, and is unique up to adding a constant to all the assignments.) Then the map $\mathsf{Tot}(F_\iota) \to \Sigma^{|\iota|}\mathsf{Tot}(F)$, defined to be $(-1)^{t_v}$ times the identity on the summand $\mathcal{A}(F(v))$ for all v is an isomorphism of chain complexes, and is canonical up to an overall sign.

Definition 5.8. A *natural transformation* $\eta\colon F \to F'$ between two 2-functors $F, F'\colon \underline{2}^n \to \mathscr{B}$ is a strictly unitary lax 2-functor $\eta\colon \underline{2}^{n+1} \to \mathscr{B}$ so that $\eta|_{\{1\}\times\underline{2}^n} = F$ and $\eta|_{\{0\}\times\underline{2}^n} = F'$ (with respect to the obvious identifications of $\{i\} \times \underline{2}^n$ with $\underline{2}^n$).

Note that a natural transformation η induces a chain map $\mathsf{Tot}(\eta)\colon \mathsf{Tot}(F) \to \mathsf{Tot}(F')$, and this is functorial in the following sense: If $\eta\colon F \to F'$ and $\theta\colon F' \to F''$ are two natural transformations, then $\mathsf{Tot}(\theta\circ\eta) = \mathsf{Tot}(\theta)\circ\mathsf{Tot}(\eta)$. Moreover, Tot of the functor $\underline{2}^{n+1} \to \mathscr{B}$ determined by η is the mapping cone of the chain map $\mathsf{Tot}(\eta)$.

Definition 5.9. Two stable functors $(E_1\colon \underline{2}^{m_1} \to \mathscr{B}, q_1)$ and $(E_2\colon \underline{2}^{m_2} \to \mathscr{B}, q_2)$ are defined to be *stably equivalent* if there is a sequence of stable functors $\{(F_i\colon \underline{2}^{n_i} \to \mathscr{B}, r_i)\}$ for $0 \leq i \leq \ell$, with $\Sigma^{q_1}E_1 = \Sigma^{r_0}F_0$ and $\Sigma^{q_2}E_2 = \Sigma^{r_\ell}F_\ell$, such that for every adjacent pair $\{\Sigma^{r_i}F_i, \Sigma^{r_{i+1}}F_{i+1}\}$, one of the following holds:

(1) $(n_i, r_i) = (n_{i+1}, r_{i+1})$, and there is a natural transformation η, either from F_i to F_{i+1} or from F_{i+1} to F_i, so that the induced map $\mathsf{Tot}(\eta)$ is a chain homotopy equivalence.
(2) $r_{i+1} = r_i - |\iota|$ and $F_{i+1} = (F_i)_\iota$ for some face inclusion $\iota\colon \underline{2}^{n_i} \hookrightarrow \underline{2}^{n_{i+1}}$; or $r_{i+1} = r_i + |\iota|$ and $F_i = (F_{i+1})_\iota$ for some face inclusion $\iota\colon \underline{2}^{n_{i+1}} \hookrightarrow \underline{2}^{n_i}$.

We call such a sequence $\{\Sigma^{r_i}F_i\}$ a *stable equivalence* between $\Sigma^{q_1}E_1$ and $\Sigma^{q_2}E_2$. Note that a stable equivalence induces a chain homotopy equivalence $\mathsf{Tot}(\Sigma^{q_1}E_1) \to \mathsf{Tot}(\Sigma^{q_2}E_2)$, well-defined up to choices of inverses of chain homotopy equivalences and an overall sign.

As is standard, instead of having to consider an arbitrary sequence of zig-zags of natural transformations, it is enough to consider a single zig-zag.

Proposition 5.10. *If stable functors $(E_1\colon \underline{2}^{m_1} \to \mathscr{B}, q_1)$ and $(E_2\colon \underline{2}^{m_2} \to \mathscr{B}, q_2)$ are stably equivalent, then there exist stable functors $(F_1\colon \underline{2}^n \to \mathscr{B}, r)$, $(F_2\colon \underline{2}^n \to \mathscr{B}, r)$, and $(G\colon \underline{2}^n \to \mathscr{B}, r)$, satisfying the following for all $i \in \{1,2\}$:*

(1) $F_i = (E_i)_{\iota_i}$ for some face inclusion $\iota_i\colon \underline{2}^{m_i} \hookrightarrow \underline{2}^n$, and $q_i = r + |\iota_i|$.
(2) There is a natural transformation $\eta_i\colon F_i \to G$, so that $\mathsf{Tot}(\eta_i)$ is a chain homotopy equivalence.

Proof. We can compose natural transformations $\eta\colon F \to F'$ and $\eta'\colon F' \to F''$ to get a natural transformation $\eta'\circ\eta\colon F \to F''$; and since $\mathsf{Tot}(\eta'\circ\eta) = \mathsf{Tot}(\eta')\circ\mathsf{Tot}(\eta)$, if $\mathsf{Tot}(\eta)$ and $\mathsf{Tot}(\eta')$ are chain homotopy equivalences, so is $\mathsf{Tot}(\eta'\circ\eta)$. Similarly, for any face inclusions $\iota\colon \underline{2}^n \hookrightarrow \underline{2}^{n'}$ and $\iota'\colon \underline{2}^{n'} \hookrightarrow \underline{2}^{n''}$, the composition $\iota'\circ\iota\colon \underline{2}^n \hookrightarrow \underline{2}^{n''}$ is a face inclusion with $|\iota'\circ\iota| = |\iota| + |\iota'|$; and for any $F\colon \underline{2}^n \to \mathscr{B}$, $(F_\iota)_{\iota'} = F_{\iota'\circ\iota}$. Finally, if $\eta\colon F \to F'$ is a natural transformation between two functors $F, F'\colon \underline{2}^n \to \mathscr{B}$, and $\iota\colon \underline{2}^n \hookrightarrow \underline{2}^N$ is a face inclusion, then there is an induced natural transformation $\eta_\iota\colon F_\iota \to F'_\iota$; and if $\mathsf{Tot}(\eta)$ is a chain homotopy equivalence, so is $\mathsf{Tot}(\eta_\iota)$.

Using these moves, we can convert any stable equivalence $\{\Sigma^{r_i} F_i\}$ from $\Sigma^{q_1} E_1$ to $\Sigma^{q_2} E_2$ to one of the form $\{\Sigma^{q_1} E_1, \Sigma^r F_1 = \Sigma^r G_0, \Sigma^r G_1, \dots, \Sigma^r G_{\ell-1}, \Sigma^r G_\ell = \Sigma^r F_2, \Sigma^{q_2} E_2\}$, where $G_0, \dots, G_\ell$ are all functors $\underline{2}^n \to \mathscr{B}$; $F_i = (E_i)_{\iota_i}$ for some face inclusion $\iota_i\colon \underline{2}^{m_i} \hookrightarrow \underline{2}^n$ and $q_i = r + |\iota_i|$; and there is a zig-zag of natural transformations connecting $\{G_0, \dots, G_\ell\}$, inducing chain homotopy equivalences among $\mathsf{Tot}(G_i)$.

So, in order to prove the proposition, it is enough to show that we can we can convert a zig-zag of the form $F \stackrel{\eta}{\leftarrow} G \stackrel{\eta'}{\rightarrow} F'$ with $\mathsf{Tot}(\eta)$ and $\mathsf{Tot}(\eta')$ chain homotopy equivalences to a zig-zag of the form $F \stackrel{\theta}{\rightarrow} H \stackrel{\theta'}{\leftarrow} F'$ with $\mathsf{Tot}(\theta)$ and $\mathsf{Tot}(\theta')$ chain homotopy equivalences (they then admit a calculus of left fractions in the sense of [GZ67]). We can achieve this by working on the cube $\underline{2}^{n+1}$ instead of $\underline{2}^n$. That is, we will construct $H\colon \underline{2}^{n+1} \to \mathscr{B}$, and $\theta\colon F_{\iota_0} \to H$, $\theta'\colon F'_{\iota_0} \to H$ so that $\mathsf{Tot}(\theta)$ and $\mathsf{Tot}(\theta')$ are chain homotopy equivalences (with ι_0 denoting the face inclusion $\underline{2}^n \cong \{0\} \times \underline{2}^n \hookrightarrow \underline{2}^{n+1}$).

To define H, note that the natural transformations η and η' are thought of as functors $\underline{2}^{n+1} \to \mathscr{B}$ satisfying $\eta|_{\{1\}\times\underline{2}^n} = \eta'|_{\{1\}\times\underline{2}^n} = G_{\iota_1}|_{\{1\}\times\underline{2}^n}$ (with ι_1 denoting the face inclusion $\underline{2}^n \cong \{1\} \times \underline{2}^n \hookrightarrow \underline{2}^{n+1}$), and $\eta|_{\{0\}\times\underline{2}^n} = F_{\iota_0}|_{\{0\}\times\underline{2}^n}$, and $\eta'|_{\{0\}\times\underline{2}^n} = F'_{\iota_0}|_{\{0\}\times\underline{2}^n}$. Define H to be the quotient of $\eta \amalg \eta'$ by identifying $\eta|_{\{1\}\times\underline{2}^n}$ with $\eta'|_{\{1\}\times\underline{2}^n}$. The natural transformations $F_{\iota_0} \stackrel{\theta}{\rightarrow} H$ and $F'_{\iota_0} \stackrel{\theta'}{\rightarrow} H$ come from inclusions. Since $\mathsf{Tot}(\eta)$ is a chain homotopy equivalence, so is $\mathsf{Tot}(\theta')$; and since $\mathsf{Tot}(\eta')$ is a chain homotopy equivalence, so is $\mathsf{Tot}(\theta)$. This concludes the proof. □

We conclude this section with some illustrative examples.

Example 5.11. Let $\mathbb{P}\colon \underline{2} \to \mathscr{B}$ denote the functor that assigns one-elements sets to 1 and 0, and a two-element correspondence to $\varphi_{1,0}$. Extend the data from Example 4.4 to a functor $F^{(1)}\colon \underline{2}^2 \to \mathscr{B}$ by declaring the matching $F^{(1)}_{11,10,01,00}$ to be the map

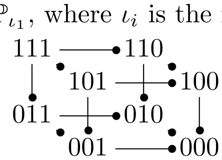
$$a_1b_1 \mapsto c_1d_1 \qquad a_1b_2 \mapsto c_2d_1 \qquad a_2b_1 \mapsto c_1d_2 \qquad a_2b_2 \mapsto c_2d_2.$$

Then $F^{(1)}$ is naturally isomorphic to $\mathbb{P}\times\mathbb{P}$.

Example 5.12. Consider once again the data from Example 4.4; this time define the functor $F^{(2)}\colon \underline{2}^2 \to \mathscr{B}$ by declaring the matching $F^{(2)}_{11,10,01,00}$ to be the map

$$a_1b_1 \mapsto c_1d_1 \qquad a_1b_2 \mapsto c_1d_2 \qquad a_2b_1 \mapsto c_2d_1 \qquad a_2b_2 \mapsto c_2d_2.$$

Then $F^{(2)}$ is stably equivalent to $\mathbb{P}_{\iota_0} \amalg \mathbb{P}_{\iota_1}$, where ι_i is the face inclusion $\underline{2} \cong \{i\} \times \underline{2} \hookrightarrow \underline{2}^2$. To see this, let G be the following diagram on the cube



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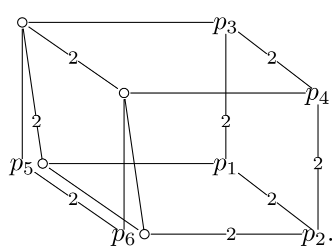

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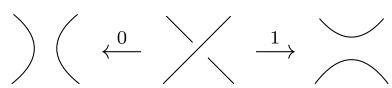
(We have only labeled some of the elements of some of the sets $G(v)$. We have not labeled the elements of the correspondences $G(\varphi_{u,v})$, but merely indicated their cardinalities if bigger than one.) Define the matching $G_{110,100,010,000}$ on the rightmost face to be isomorphic to the matching $F^{(2)}_{11,10,01,00}$ above. It is straightforward to verify that we can construct matchings on the other two-dimensional faces in a unique fashion so that condition (C-2) from Section 4 is satisfied. Therefore, this defines a functor $G\colon \underline{2}^3 \to \mathscr{B}$.

Let F be the diagram restricted to the objects $\{p_1, p_2, p_3, p_4\}$ and the correspondences between them, and let F' be the diagram restricted to the objects $\{p_1, p_2, p_5, p_6\}$ and the correspondences between them. It is easy to verify that both the inclusions $F \to G$ and $F' \to G$ are natural transformations that induce chain homotopy equivalences $\mathsf{Tot}(F) \to \mathsf{Tot}(G)$ and $\mathsf{Tot}(F') \to \mathsf{Tot}(G)$. Furthermore, F is naturally isomorphic to $F^{(2)}_\iota$ where ι is the face inclusion $\underline{2}^2 \cong \underline{2}^2 \times \{0\} \hookrightarrow \underline{2}^3$; and F' is naturally isomorphic to $(\mathbb{P}_{\iota_0} \amalg \mathbb{P}_{\iota_1})_{\iota'}$ where ι' is the face inclusion $\underline{2}^2 \cong \{0\} \times \underline{2}^2 \hookrightarrow \underline{2}^3$. Therefore, $F^{(2)}$ is stably equivalent to $\mathbb{P}_{\iota_0} \amalg \mathbb{P}_{\iota_1}$.

Remark 5.13. Properties of the topological realizations from Section 7 imply that the two functors $F^{(1)}$ and $F^{(2)}$ from Examples 5.11–5.12 are not stably equivalent. The reduced cohomology of $|\mathbb{P}|$ is $\mathbb{F}_2$, supported in grading zero (by property (Sp-1)(c)), and therefore, the spectrum $|\mathbb{P}|$ is homotopy equivalent to $\Sigma^{-1}\mathbb{RP}^2$. Since $F^{(1)}$ is naturally isomorphic to $\mathbb{P} \times \mathbb{P}$, the spectrum $|F^{(1)}|$ is homotopy equivalent to $\Sigma^{-2}(\mathbb{RP}^2 \wedge \mathbb{RP}^2)$ (by property (Sp-4)). On the other hand, since $F^{(2)}$ is stably equivalent to $\mathbb{P}_{\iota_0} \amalg \mathbb{P}_{\iota_1}$, the spectrum $|F^{(2)}|$ is homotopy equivalent to $(\Sigma^{-1}\mathbb{RP}^2) \vee \mathbb{RP}^2$ (by properties (Sp-3), (Sp-5), and (Sp-6)). However, the spectra $\Sigma^{-2}(\mathbb{RP}^2 \wedge \mathbb{RP}^2)$ and $(\Sigma^{-1}\mathbb{RP}^2) \vee \mathbb{RP}^2$ are not homotopy equivalent (being distinguished by the Steenrod square Sq^2 for instance), and therefore, (by property (Sp-6) once again) the diagrams $F^{(1)}$ and $F^{(2)}$ are not stably equivalent.

6. The Khovanov functor

We turn now to the refinement of Khovanov homology. Fix an oriented link diagram K with n crossings and an ordering of the crossings of K. Khovanov associated an n-dimensional cube of free Abelian groups to this data, as follows [Kho00]. Each crossing of K has a 0-resolution and a 1-resolution:



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So, given a vertex $v = (v_1, \dots, v_n) \in \{0,1\}^n$ of the cube, performing the v_i-resolution at the i^{th} crossing gives a collection K_v of disjoint, embedded circles in S^2. Further, for each edge $u \multimap v$ of the cube, K_u is obtained from K_v by either merging two circles into one or splitting one circle into two. Let $V = \mathbb{Z}\langle x_+, x_- \rangle$ be a free, rank-2 $\mathbb{Z}$-module, which we endow with a multiplication and comultiplication by

$$m(x_+ \otimes x_+) = x_+ \qquad m(x_+ \otimes x_-) = x_- \qquad m(x_- \otimes x_+) = x_- \qquad m(x_- \otimes x_-) = 0$$

$$\Delta(x_+) = x_+ \otimes x_- + x_- \otimes x_+ \qquad\qquad \Delta(x_-) = x_- \otimes x_-.$$

Define a functor $F_{Kh,Ab}\colon (\underline{2}^n)^{\mathrm{op}} \to \mathsf{Ab}$ on objects by setting $F_{Kh,Ab}(v) = \bigotimes_{S \in \pi_0(K_v)} V$. On morphisms, if $u \multimap v$ is an edge of the cube so that K_u is obtained from K_v by merging two circles then $F_{Kh,Ab}(\varphi^{\mathrm{op}}_{u,v})$ applies the multiplication map to the corresponding factors of $F_{Kh,Ab}(v)$ and the identity map to the remaining factors; if instead K_u is obtained from K_v by splitting one circle then $F_{Kh,Ab}(\varphi^{\mathrm{op}}_{u,v})$ applies the comultiplication map to the corresponding factor of $F_{Kh,Ab}(v)$ and the identity map to the remaining factors. It is straightforward to verify that the resulting diagram commutes. The total complex of this cube (i.e., multiplying the map on the edge $u \multimap v$ by $(-1)^{s_{u,v}}$ and summing over vertices of each grading, in a fashion similar to Definition 5.1) is the Khovanov complex $\mathcal{C}_{Kh}(K)$.

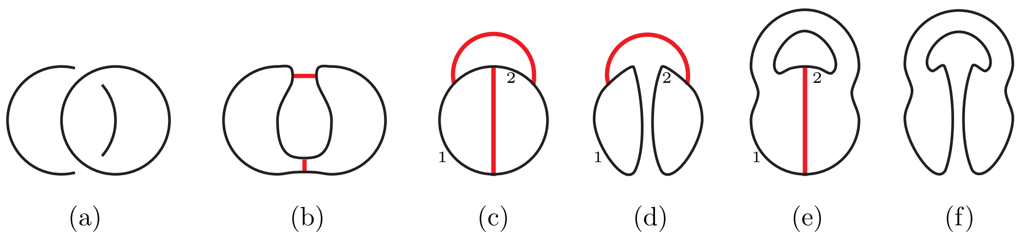


FIGURE 6.1. **The ladybug matching.** (a) A piece of a (perhaps) partially-resolved link diagram. (b) The 00-resolution, along with the arcs a_v and $a_{v'}$, drawn with thick lines. This is a ladybug configuration. (c) The same configuration, up to isotopy in S^2, looking more like a ladybug. The right pair of arcs is labeled 1 and 2. (d)–(e) The 10 and 01 resolutions. The circles in the 10 resolution and the circles in the 01 resolution can be identified via the induced numbering by $1, 2$. (f) The 11 resolution.

The Khovanov cube decomposes as a direct sum over *quantum gradings*. To define the quantum grading, notice that for $v \in \{0,1\}^n$, $F_{Kh,Ab}(v)$ comes with a preferred basis: the labelings of the circles in K_v by elements of $\{x_-, x_+\}$, i.e., functions $\pi_0(K_v) \to \{x_+, x_-\}$. The quantum grading of a basis element is

$$\mathrm{gr}_q(v, x\colon \pi_0(K_v) \to \{x_+, x_-\}) = n_+ - 2n_- + |v| + |\{Z \mid x(Z) = x_+\}| - |\{Z \mid x(Z) = x_-\}| \in \mathbb{Z},$$

where n_+ and n_- are the number of positive and negative crossings of K, respectively. There is also a formal homological grading shift of $-n_-$.

Our main goal is to refine $F_{Kh,Ab}$ to a functor $F_{Kh}\colon \underline{2}^n \to \mathscr{B}$, satisfying the following:

(a) For all v, $F_{Kh}(v) = \{x\colon \pi_0(K_v) \to \{x_+, x_-\}\}$ is the preferred basis of Khovanov generators.
(b) The above identification induces an isomorphism $\Sigma^{-n_-}\mathsf{Tot}(F_{Kh}) \cong (\mathcal{C}_{Kh})^*$.

By Proposition 4.3 it suffices to define F_{Kh} on vertices, edges, and 2-dimensional faces. For all vertices v, $F_{Kh}(v)$ is already defined. Next, notice that for each edge $u \to v$ and each element $y \in F_{Kh}(v)$, $F_{Kh,Ab}(\varphi^{\mathrm{op}}_{u,v})(y) = \sum_{x \in F_{Kh}(u)} \epsilon_{x,y}\, x$ where $\epsilon_{x,y} \in \{0,1\}$. (In other words, all of the entries of the matrix $F_{Kh,Ab}(\varphi^{\mathrm{op}}_{u,v})$ are 0 or 1.) Define $F_{Kh}(\varphi_{u,v}) = \{(y,x) \in F_{Kh}(v) \times F_{Kh}(u) \mid \epsilon_{x,y} = 1\}$, with the obvious source and target maps to $F_{Kh}(u)$ and $F_{Kh}(v)$.

So far, there is no information in F_{Kh} beyond that in the Khovanov complex. The (modest) new information is in the definition of $F_{u,v,v',w}\colon F_{Kh}(\varphi_{v,w}) \times_{F_{Kh}(v)} F_{Kh}(\varphi_{u,v}) \to F_{Kh}(\varphi_{v',w}) \times_{F(v')} F_{Kh}(\varphi_{u,v'})$ for the 2-dimensional faces $\begin{smallmatrix} & v & \\ u & & w \\ & v' & \end{smallmatrix}$. The fact that $F_{Kh,Ab}$ is a commutative diagram implies that there is a 2-isomorphism between $F_{Kh}(\varphi_{v,w}) \times_{F_{Kh}(v)} F_{Kh}(\varphi_{u,v})$ and $F_{Kh}(\varphi_{v',w}) \times_{F_{Kh}(v')} F_{Kh}(\varphi_{u,v'})$. Namely, for $x \in F_{Kh}(u)$ and $z \in F_{Kh}(w)$, the cardinalities of

$$A_{x,z} \coloneqq s^{-1}(x) \cap t^{-1}(z) \subseteq F_{Kh}(\varphi_{v,w}) \times_{F_{Kh}(v)} F_{Kh}(\varphi_{u,v}) \qquad \text{and}$$
$$A'_{x,z} \coloneqq s^{-1}(x) \cap t^{-1}(z) \subseteq F_{Kh}(\varphi_{v',w}) \times_{F_{Kh}(v)} F_{Kh}(\varphi_{u,v'})$$

are the (x,z) entries in the matrices $F_{Kh,Ab}(\varphi^{\mathrm{op}}_{u,v}) \circ F_{Kh,Ab}(\varphi^{\mathrm{op}}_{v,w})$ and $F_{Kh,Ab}(\varphi^{\mathrm{op}}_{u,v'}) \circ F_{Kh,Ab}(\varphi^{\mathrm{op}}_{v',w})$, respectively, and these two matrices are the same. Further, in most instances, these sets have 0 or 1 elements, so there is a unique isomorphism $F_{u,v,v',w}|_{A_{x,z}}\colon A_{x,z} \to A'_{x,z}$.

The exceptional case is when there is a circle C_w in K_w which splits to form two circles in each of K_v and $K_{v'}$, and these same two circles merge to form a single circle C_u in K_u; x labels C_u by x_-; and z labels

C_w by x_+. We call this configuration a *ladybug configuration* because of the following depiction. First, draw the circle C_w. Then, draw the arc a_v with endpoints on C_w which pinches along the edge $v \multimap w$ (i.e., performing embedded 1-surgery on K_w along a_v produces K_v). Similarly, draw another arc $a_{v'}$ with endpoints on C_w which pinches along the edge $v' \multimap w$. Up to isotopy in S^2, the result looks like the ladybug in Figure 6.1. Now, distinguish a pair of arcs in $(C_w, \partial a_v \cup \partial a_{v'})$, the *right pair*, as the two arcs you get to by walking along a_v or $a_{v'}$ to C_w and then turning right. (Again, see Figure 6.1.) Number the two arcs in the right pair as $1, 2$—the numbering will not matter. Label the two circles in K_v which come from C_w as C_v^1 and C_v^2, according to whether they contain the right arc numbered 1 or 2. Similarly, label the two circles in $K_{v'}$ which come from C_w as $C_{v'}^1$ and $C_{v'}^2$, according to whether they contain the right arc numbered 1 or 2. The two elements of $A_{x,z}$ are

$$\alpha = ((C_w \to x_+), (C_v^1, C_v^2) \to (x_-, x_+), (C_u \to x_-)), \;\; \beta = ((C_w \to x_+), (C_v^1, C_v^2) \to (x_+, x_-), (C_u \to x_-))$$

while the two elements of $A'_{x,z}$ are

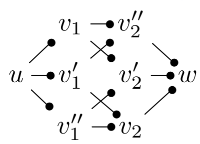
$$\alpha' = ((C_w \to x_+), (C_{v'}^1, C_{v'}^2) \to (x_-, x_+), (C_u \to x_-)), \;\; \beta' = ((C_w \to x_+), (C_{v'}^1, C_{v'}^2) \to (x_+, x_-), (C_u \to x_-))$$

The bijection $F_{u,v,v',w}$ sends α to α' and β to β'. (See also [LS14a, Section 5.4].)

Proposition 6.1. *The definitions above satisfy conditions (C-1)–(C-2) from Section 4, and so induce a strictly unitary 2-functor $F_{Kh} : \underline{2}^n \to \mathscr{B}$ by Proposition 4.3.*

Sketch of Proof. The proof is essentially the same as the proof of [LS14a, Proposition 5.19] and, indeed, the result follows from [LS14a, Proposition 5.20] and [LLS, Lemma 4.2], so we will only sketch the argument here.

Condition (C-1) is immediate, so we only need to check condition (C-2). Fix a three-dimensional face



. We have six correspondences from $F_{Kh}(u)$ to $F_{Kh}(w)$ coming from the six maximal chains from u to w. Fix Khovanov generators $x \in F_{Kh}(u)$ and $z \in F_{Kh}(w)$, and consider the subsets $s^{-1}(x) \cap t^{-1}(z)$ of these six correspondences. The various coherence maps $F_{u,v_1^{(i)},v_2^{(j)}}$ and $F_{v_1^{(i)},v_2^{(j)},w}$ produce six bijections connecting these six sets, and we need to check that these bijections agree.

Unless one of the two-dimensional faces of the above cube is a ladybug configuration, each of the above six sets will contain 0 or 1 element, and the check is vacuous or trivial, respectively. So we concentrate on configurations that contain ladybugs, and check them by hand. Using some underlying symmetries, we can reduce to just four configurations [LS14a, Figure 5.3. a–c, e]. The first three are similar, so there are essentially two case checks that need to be performed; and the proofs of [LS14a, Lemmas 5.14, 5.17] imply that the checks succeed. □

Hu-Kriz-Kriz gave an intrinsic description of the functor F_{Kh} [HKK], that does not require Proposition 4.3, as follows. On objects, F_{HKK} agrees with F_{Kh}. To define F_{HKK} on morphisms, first fix a checkerboard coloring for K. There is an induced checkerboard coloring of each resolution K_v. Further, each morphism $\varphi_{u,v}$ corresponds to a cobordism $\Sigma_{u,v}$ in $[0,1] \times S^2$ from K_u to K_v, and the checkerboard coloring of K induces a coloring of the components of $([0,1] \times S^2) \setminus \Sigma_{u,v}$. Let $B_{u,v} \subset ([0,1] \times S^2) \setminus \Sigma_{u,v}$ denote the closure of the black region. Then

$$H_1(B_{u,v})/H_1(B_{u,v} \cap \{0,1\} \times S^2) \cong \bigoplus_{\Sigma_0 \in \pi_0(\Sigma_{u,v})} \mathbb{Z}^{g(\Sigma_0)}. \tag{6.1}$$

If each connected component of $\Sigma_{u,v}$ has genus 0 or 1 then we call a vector $(\epsilon_1,\dots,\epsilon_k)$ in the group (6.1) such that each $\epsilon_i=\pm 1$ a *sign choice* for $\Sigma_{u,v}$. By a *valid boundary labeling* of $\Sigma_{u,v}$ we mean a labeling of the circles in $\partial\Sigma_{u,v}$ by x_+ or x_- satisfying the condition that every component Σ_0 of the cobordism has genus equal to

$$1-|\{C\in\pi_0(K_v\cap\partial\Sigma_0)\mid x(K_v)=x_-\}|-|\{C\in\pi_0(K_u\cap\partial\Sigma_0)\mid x(K_v)=x_+\}|.$$

(In particular, if some component Σ_0 has genus bigger than 1 then the set of valid boundary labelings is empty.) Then $F_{HKK}(\varphi_{u,v})$ is the product of the set of valid boundary labelings for $\Sigma_{u,v}$ and the set of sign choices for $\Sigma_{u,v}$. The source and target maps for $F_{HKK}(\varphi_{u,v})$ are given by restricting the boundary labelings of $\Sigma_{u,v}$ to K_u and K_v, respectively. The coherence map $F_{HKK}(\varphi_{v,w})\times_{F_{HKK}(v)}F_{HKK}(\varphi_{u,v})\to F_{HKK}(\varphi_{u,w})$ is obvious except when a genus 0 component Σ_0 of $F_{HKK}(\varphi_{u,v})$ and a genus 0 component Σ_1 of $F_{HKK}(\varphi_{v,w})$ glue to give a genus 1 component Σ of $F_{HKK}(\varphi_{u,w})$. In this last case, there is a single circle C of $\Sigma_0\cap\Sigma_1$, that is non-separating in Σ and is labeled x_+. Orient C and Σ as the boundaries of the black regions in S^2 and $[0,1]\times S^2$, respectively, and choose a circle D in $\Sigma_{u,w}$ with intersection number $D\cdot C=1$. Either the pushoff of C or the pushoff of D into $B_{u,w}$ is a generator of the new $\mathbb{Z}$-summand of (6.1); use this generator to extend the sign choice to $\Sigma_{u,w}$. This finishes the construction of F_{HKK}, and verifying that this does, in fact, define a 2-functor is straightforward. The functor F_{HKK} is naturally isomorphic to the functor F_{Kh} defined via the ladybug matching [LLS, Lemma 8.1].

Definition 6.2. Associated to an oriented n-crossing link diagram K, let $F_{Kh}(K)\colon \underline{2}^n\to\mathscr{B}$ be the functor constructed in Proposition 6.1. Define the *Khovanov functor* to be the stable functor $\Sigma^{-n_-}F_{Kh}(K)$, where n_- is the number of negative crossings in K.

There is also a reduced Khovanov functor associated to a pointed link (K,p): the basepoint p chooses a preferred circle $C_{v,p}$ in each resolution K_v, and we declare that

$$F_{\widetilde{Kh}}(v)=\{x\in F_{Kh}(v)\mid x(C_{v,p})=x_-\},$$
$$F_{\widetilde{Kh}}(\varphi_{u,v})=s^{-1}(F_{\widetilde{Kh}}(u))\cap t^{-1}(F_{\widetilde{Kh}}(v))\subseteq F_{Kh}(\varphi_{u,v}),$$

and similarly the coherence maps $(F_{\widetilde{Kh}})_{u,v,w}$ are restrictions of the coherence maps $(F_{Kh})_{u,v,w}$ for F_{Kh}. It is straightforward to see that $F_{\widetilde{Kh}}$ does define a strictly unitary 2-functor. (Replacing x_- with x_+ in the definition would give a naturally isomorphic functor $F'_{\widetilde{Kh}}$.)

Definition 6.3. Let $F_{\widetilde{Kh}}(K,p)\colon\underline{2}^n\to\mathscr{B}$ be the above functor associated to a pointed, oriented n-crossing link diagram K. Define the *reduced Khovanov functor* to be the stable functor $\Sigma^{-n_-}F_{\widetilde{Kh}}(K)$, where n_- is the number of negative crossings in K.

Since the chain complexes $\mathcal{C}_{Kh}$ and $\widetilde{\mathcal{C}}_{Kh}$ decompose according to quantum gradings, so do the functors F_{Kh} and $F_{\widetilde{Kh}}$:

$$F_{Kh}=\coprod_{j\in\mathbb{Z}}F^j_{Kh}\qquad\qquad F_{\widetilde{Kh}}=\coprod_{j\in\mathbb{Z}}F^j_{\widetilde{Kh}},$$

where $F^j_{Kh}(v)=\{x\in F_{Kh}(v)\mid \mathrm{gr}_q(v,x)=j\}$, and $F^j_{\widetilde{Kh}}(v)=F_{\widetilde{Kh}}(v)\cap F^{j-1}_{Kh}(v)$.

Theorem 1. *If K_1 and K_2 are oriented link diagrams, with n^1_- and n^2_- negative crossings, respectively, representing isotopic oriented links then the Khovanov functors $\Sigma^{-n^1_-}F^j_{Kh}(K_1)$ and $\Sigma^{-n^2_-}F^j_{Kh}(K_2)$ are stably equivalent stable functors. Similarly, if (K_1,p_1) and (K_2,p_2) are pointed, oriented link diagrams, with n^1_- and n^2_- negative crossings, respectively, representing isotopic pointed, oriented links then the reduced Khovanov functors $\Sigma^{-n^1_-}F^j_{\widetilde{Kh}}(K_1,p_1)$ and $\Sigma^{-n^2_-}F^j_{\widetilde{Kh}}(K_2,p_2)$ are stably equivalent stable functors.*

Sketch of Proof. The proof is essentially the same as the proof of invariance of the Khovanov spectrum [LS14a, Theorem 1.1], so we only give a sketch. We start with a general principle. Since $\mathcal{C}_{Kh}(K)$ comes with a preferred basis, so does the dual complex $(\mathcal{C}_{Kh}(K))^*$. Suppose that S is a subset of the preferred basis for $(\mathcal{C}_{Kh}(K))^*$ so that the span of S is a subcomplex of $(\mathcal{C}_{Kh}(K))^*$. Then there is a functor $F^S_{Kh}\colon \underline{2}^n \to \mathscr{B}$ defined by

$$\begin{aligned} F^S_{Kh}(v) &= S \cap F_{Kh}(v), \\ F^S_{Kh}(\varphi_{u,v}) &= s^{-1}(F^S_{Kh}(u)) \cap t^{-1}(F^S_{Kh}(v)) \subseteq F_{Kh}(\varphi_{u,v}), \end{aligned} \tag{6.2}$$

with coherence maps induced by the coherence maps for $F_{Kh}(K)$. Further, inclusion induces a natural transformation $\eta\colon F^S_{Kh} \to F_{Kh}(K)$, and if $\mathrm{span}(S) \to (\mathcal{C}_{Kh}(K))^*$ is a quasi-isomorphism then, for any $m \in \mathbb{Z}$, η is a stable equivalence between the stable functors $\Sigma^m F^S_{Kh}$ and $\Sigma^m F_{Kh}(K)$. Similarly, if S is a subset of the preferred basis for $(\mathcal{C}_{Kh}(K))^*$ so that the complement of S spans a subcomplex of $(\mathcal{C}_{Kh}(K))^*$ then Formula (6.2) still defines a functor $F^S_{Kh}\colon \underline{2}^n \to \mathscr{B}$, and there is a natural transformation $\eta\colon F_{Kh}(K) \to F^S_{Kh}$ defined as follows. Recall that a natural transformation is a functor $\eta\colon \underline{2}^1 \times \underline{2}^n \to \mathscr{B}$. Define $\eta(\varphi_{1,0} \times \mathrm{Id}_v) = F^S_{Kh}(v) = S \cap F_{Kh}(v)$, with source map given by the inclusion $F^S_{Kh}(v) \hookrightarrow F_{Kh}(v)$ and target map the identity map $F^S_{Kh}(v) \to F^S_{Kh}(v)$; the definition extends in an obvious way to all of $\underline{2}^1 \times \underline{2}^n$. If the quotient map $(\mathcal{C}_{Kh}(K))^* \to \mathrm{span}(S)$ is a quasi-isomorphism then η is a stable equivalence between the stable functors $\Sigma^m F_{Kh}(K)$ and $\Sigma^m F^S_{Kh}$.

Now, to prove the theorem, it suffices to check invariance under the three Reidemeister moves. (For pointed links, we need to check invariance under the Reidemeister moves in the complement of the basepoint.) By the previous paragraph, it suffices to show that the (duals of the) isomorphisms on Khovanov homology induced by Reidemeister moves come from sequences of:

- Inclusions of subcomplexes spanned by Khovanov generators, inducing isomorphisms on homology.
- Projections to quotient complexes spanned by Khovanov generators, inducing isomorphisms on homology.
- Face inclusions of cubes.

For Reidemeister I and II moves, Bar-Natan's formulation [Bar02] of Khovanov's invariance proof [Kho00] has this form, so proves invariance of the stable functor as well; see also [LS14a, Propositions 6.2 and 6.3]. For Reidemeister III moves, it suffices to prove invariance under the braid-like Reidemeister III move, which locally has the form $\sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1} \sim \mathrm{Id}$ (where the σ_i are braid generators) [Bal11, Section 7.3]. For this braid-like Reidemeister III, one can again reduce to the identity braid by a sequence of subcomplex inclusions and quotient complex projections, though the sequence is somewhat tedious [LS14a, Proposition 6.4]. The result follows. ☐

The following properties are strightforward from the definitions. (See [LLS, Proposition 9.1].)

(X-1) If K_1 and K_2 are oriented links then

$$F^j_{Kh}(K_1 \amalg K_2) \cong \coprod_{j_1+j_2=j} F^{j_1}_{Kh}(K_1) \times F^{j_2}_{Kh}(K_2).$$

(X-2) If (K_1, p_1) is a pointed, oriented link, and K_2 is an oriented link then

$$F^j_{\widetilde{Kh}}(K_1 \amalg K_2, p_1) \cong \coprod_{j_1+j_2=j} F^{j_1}_{\widetilde{Kh}}(K_1, p_1) \times F^{j_2}_{Kh}(K_2).$$

(X-3) If (K_1, p_1) and (K_2, p_2) are pointed, oriented links and $(K_1\#K_2, p)$ is the connected sum of K_1 and K_2 at the basepoints and the new basepoint p is chosen on one of the connected sum strands, then

$$F^j_{\widetilde{Kh}}(K_1\#K_2, p) \cong \coprod_{j_1+j_2=j} F^{j_1}_{\widetilde{Kh}}(K_1, p_1) \times F^{j_2}_{\widetilde{Kh}}(K_2, p_2).$$

Section 7 describes a recipe for turning a stable functor $\Sigma^r F$ into a spectrum $|\Sigma^r F|$. Applying that recipe to the Khovanov functor associated to a link K (respectively, reduced Khovanov functor associated to a pointed link (K,p)) gives the Khovanov stable homotopy type $\mathcal{X}_{Kh}(K)$ (respectively, $\widetilde{\mathcal{X}}_{Kh}(K,p)$).

7. Spaces

Finally, we return to the connection with topological spaces. Given a diagram $F\colon \underline{2}^n \to \mathscr{B}$, one can associate an essentially well-defined spectrum $|F|$. To give a concrete construction of $|F|$, following [LLS, Section 5], start by fixing an integer $\ell \geq n$. We build a diagram $G\colon \underline{2}^n \to \mathsf{CW}$ of based CW complexes, as follows. Given a vertex v, let

$$G(v) = (F(v) \times [0,1]^\ell)/(F(v) \times \partial([0,1]^\ell)) \simeq \bigvee_{x \in F(v)} S^\ell.$$

For each $v > w$, choose an embedding

$$\Phi(\varphi_{v,w})\colon F(\varphi_{v,w}) \times [0,1]^\ell \to F(v) \times [0,1]^\ell$$

satisfying the following conditions:

(1) For each $a \in F(\varphi_{v,w})$, $\Phi(\{a\} \times [0,1]^\ell) \subseteq \{s(a)\} \times [0,1]^\ell$; that is, the following diagram commutes:

$$\begin{array}{ccc} F(\varphi_{v,w}) \times [0,1]^\ell & \xhookrightarrow{\Phi(\varphi_{v,w})} & F(v) \times [0,1]^\ell \\ \downarrow & & \downarrow \\ F(\varphi_{v,w}) & \xrightarrow{\quad s \quad} & F(v). \end{array}$$

(2) For each $a \in F(\varphi_{v,w})$, the embedding $(\{a\} \times [0,1]^\ell) \hookrightarrow (\{s(a)\} \times [0,1]^\ell)$ is an inclusion as a sub-box; that is, the lower arrow in the following diagram

$$\begin{array}{ccc} \{a\} \times [0,1]^\ell & \hookrightarrow & \{s(a)\} \times [0,1]^\ell \\ \downarrow \cong & & \cong \downarrow \\ [0,1]^\ell & \hookrightarrow & [0,1]^\ell \end{array}$$

is a map of the form

$$(x_1, \dots, x_\ell) \mapsto (c_1 + d_1 x_1, \dots, c_\ell + d_\ell x_\ell)$$

for some non-negative constants $c_1, d_1, \dots, c_\ell, d_\ell$.

Define $G(\varphi_{v,w})$ to be the composition

$$F(v) \times [0,1]^\ell/\partial \to F(\varphi_{v,w}) \times [0,1]^\ell/\partial \to F(w) \times [0,1]^\ell/\partial,$$

where the ∂ symbols denote the unions of the boundaries of the cubes, the first map sends a point of the form $\Phi(a,x)$ to $(a,x) \in F(\varphi_{v,w}) \times [0,1]^\ell$ and all other points to the collapsed boundary, and the second map sends (a,x) to $(t(a),x)$.

The resulting map $G\colon \underline{2}^n \to \mathsf{CW}$ does not commute on the nose: the maps $G(\varphi_{v,w}) \circ G(\varphi_{u,v})$ and $G(\varphi_{u,w})$ are (probably) defined using different choices of embeddings. However, the spaces of embeddings Φ are $(\ell-2)$-connected, so since any sequence of composable morphisms in $\underline{2}^n$ has length at most $n \leq \ell$, the cube commutes up to coherent homotopies, i.e., is a *homotopy coherent diagram* in the sense of Vogt [Vog73]. Homotopy coherent diagrams are essentially as good as commutative diagrams: they appear naturally whenever one takes a commutative diagram of spaces and replaces all of the objects by homotopy equivalent ones, and conversely every homotopy coherent diagram can be replaced by a homotopy equivalent honestly commutative diagram.

Next, we add a single object $*$ to the category $\underline{2}^n$, and a single morphism $v \to *$ for each non-terminal object v of $\underline{2}^n$, to get a bigger category $\underline{2}^n_+$. Extend G to a functor $G_+ : \underline{2}^n_+ \to \mathsf{CW}$ by declaring that $G_+(*)$ is a single point. Finally, take the homotopy colimit of G_+ (as defined by Vogt for homotopy coherent diagrams [Vog73]) to define

$$\|F\|_\ell = \operatorname{hocolim}(G_+).$$

This is an iterated version of the mapping cone construction. In particular, if $G : \underline{2}^1 \to \mathsf{CW}$ were a diagram on the one-dimensional cube, then G would consist of the data of a cellular map between two CW complexes, $G(\varphi_{1,0}) : G(1) \to G(0)$, and $\operatorname{hocolim}(G_+)$ would simply be the mapping cone of $G(\varphi_{1,0})$. This iterated mapping cone of G is a space-level version of the totalization construction from Definition 5.1. So, unsurprisingly, $\|F\|_\ell$ carries a CW complex structure so that its reduced cellular chain complex $\widetilde{C}^{cell}_\bullet(\|F\|_\ell)$ can be identified with $\Sigma^\ell \mathsf{Tot}(F)$, implying $\widetilde{H}_*(\|F\|_\ell) \cong \Sigma^\ell H_*(\mathsf{Tot}(F))$.

Similarly, given any natural transformation $\eta : F \to F'$ between two diagrams $F, F' : \underline{2}^n \to \mathscr{B}$, and an $\ell \geq n+1$, one can construct a based cellular map $\|\eta\|_\ell : \|F\|_\ell \to \|F'\|_\ell$ so that the induced map on the reduced cellular chain complexes is the map $\Sigma^\ell \mathsf{Tot}(\eta) : \Sigma^\ell \mathsf{Tot}(F) \to \Sigma^\ell \mathsf{Tot}(F')$. This is not functorial on the nose, since the construction depends on the choices of embeddings Φ. However, the space of choices of the coherent homotopies in the construction of the homotopy coherent diagrams $\underline{2}^n \to \mathsf{CW}$ is an $(\ell - n - 2)$-connected space, and the space of choices in the construction of the map is an $(\ell - n - 1)$-connected space. By allowing ℓ to be arbitrarily large, we can make these constructions *essentially canonical*, namely, parametrized by a contractible space.

Therefore, to make the space independent of ℓ and the choices of embeddings, and to make the construction functorial, we replace the CW complex $\|F\|_\ell$ by its suspension spectrum, and then formally desuspend ℓ times:

$$|F| = \Sigma^{-\ell}(\Sigma^\infty \|F\|_\ell).$$

Alternately, one could replace the diagram G by its suspension spectrum before taking the homotopy colimit. Finally for any stable functor $\Sigma^r F$, define $|\Sigma^r F|$ to be the formal suspension $\Sigma^r |F|$.

There are several other ways of turning a diagram $F : \underline{2}^n \to \mathscr{S}$ into a space:

(1) Produce a diagram $G : \underline{2}^n \to \mathscr{S}$ by viewing F as a functor $\widetilde{F}$ to the category of permutative categories, by letting $\widetilde{F}(v) = \mathsf{Sets}/F(v)$ be the category of sets over $F(v)$, and then applying K-theory. Then take the iterated mapping cone, as above. This is the procedure used by Hu-Kriz-Kriz [HKK].
(2) Turn F into a (rather special) flow category, in the sense of Cohen-Jones-Segal, and then apply the Cohen-Jones-Segal realization [CJS95]. This is essentially the approach taken in our previous work [LS14a].
(3) Without making any choices, produce a diagram $\widehat{F} : \widehat{\underline{2}^n} \to \mathscr{S}$ from a slightly larger category $\widehat{\underline{2}^n}$, and then take an appropriate iterated mapping cone of $\widehat{F}$ [LLS, Section 4].

These constructions all give homotopy equivalent spectra [LLS, Theorem 3].

For convenience, we summarize some of the properties of this functor

$$|\cdot| : \mathscr{B}^{\underline{2}^n} \to \mathscr{S}.$$

(Sp-1) For any stable functor $(F : \underline{2}^n \to \mathscr{B}, r)$,
  (a) $|\Sigma^r F|$ is the formal suspension $\Sigma^r |F|$.
  (b) The spectrum $|F|$ is homotopy equivalent to a formal desuspension of the suspension spectrum of some finite CW complex $\|F\|_\ell$.
  (c) There is an identification of $\mathsf{Tot}(F)$ with the reduced cellular chain complex of $|F|$ from (Sp-1)(b). In particular, $\widetilde{H}_*(|F|) \cong H_*(\mathsf{Tot}(F))$ and $\widetilde{H}^*(|F|) \cong H^*(\mathsf{Tot}(F))$.

(Sp-2) For any natural transformation $\eta\colon F \to F'$ between any two diagrams $F, F'\colon \underline{2}^n \to \mathscr{B}$, and any integer r,
  (a) The map $|\Sigma^r \eta|$ is the formal suspension $\Sigma^r|\eta|$.
  (b) The map $|\eta|$ is homotopic to a formal desuspension of some cellular map $\|\eta\|_\ell\colon \|F\|_\ell \to \|F'\|_\ell$.
  (c) With respect to the identification from (Sp-1)(c) and (Sp-2)(b), the map $\mathsf{Tot}(\eta)$ can be identified with map induced by $|\eta|$ on the reduced cellular chain complex.

(Sp-3) For any two functors $F, F'\colon \underline{2}^n \to \mathscr{B}$, there is a canonical homotopy equivalence $|F \amalg F'| \simeq |F| \vee |F'|$.

(Sp-4) For any two functors $F_1\colon \underline{2}^{n_1} \to \mathscr{B}$ and $F_2\colon \underline{2}^{n_2} \to \mathscr{B}$, there is a canonical homotopy equivalence $|F_1 \times F_2| \simeq |F_1| \wedge |F_2|$.

(Sp-5) If $\iota\colon \underline{2}^n \hookrightarrow \underline{2}^N$ is a face inclusion, then there is a canonical homotopy equivalence $|F| \simeq |\Sigma^{-|\iota|} F_\iota|$.

(Sp-6) In particular, from (Sp-5) and (Sp-2)(c), stably equivalent stable functors give homotopy equivalent spectra.

See [LLS], particularly Section 4, for more details about the construction of $|F|$, and for proofs of these properties.

8. Some questions

We end with a few questions. First, the Khovanov homotopy type $\mathcal{X}^j_{Kh}(K)$ contains more information that the Khovanov homology $Kh^{i,j}(K) = \widetilde{H}^i(\mathcal{X}^j_{Kh}(K))$. Specifically, the Khovanov spaces $\mathcal{X}^j_{Kh}(K)$ induce Steenrod operations on $Kh^{i,j}(K)$. One can give a combinatorial formula for $\mathrm{Sq}^2\colon Kh^{i,j}(K;\mathbb{F}_2) \to Kh^{i+2,j}(K;\mathbb{F}_2)$ [LS14c]; using this, Seed showed that there are knots with isomorphic Khovanov homologies but distinct Steenrod squares [See]. Further, using $\mathrm{Sq}^2(K)$, one can give refinements $s^{\mathrm{Sq}^2}_{\pm}$ [LS14b] of Rasmussen's s-invariant [Ras10] and, using direct computations and the connected sum formula from Properties (X-1) and (Sp-4), one obtains (modest) new concordance results [LLS, Corollary 1.5].

The formula for Sq^2 is combinatorial and not hard to formulate in terms of $F_{Kh}(K)$, but does not seem particularly obvious in this language. We do not know how to generalize the formula to higher Steenrod squares, or (reduced) p^{th} powers. So:

Question 8.1. Are there nice formulations of the action of the Steenrod algebra on $Kh(K)$, purely in terms of the stable Khovanov functor from Definition 6.2? Of other algebro-topological invariants of $\mathcal{X}_{Kh}(K)$ (such as the K-theory or bordism groups)? Do these give additional concordance invariants?

As described in Section 7, one can turn stable functors $\underline{2}^n \to \mathscr{B}$ into spectra. Perhaps this operation loses useful information:

Question 8.2. Are there stable functors $\Sigma^k F$ and $\Sigma^\ell G$ so that the spectra $|\Sigma^k F|$ and $|\Sigma^\ell G|$ are homotopy equivalent but $\Sigma^k F$ and $\Sigma^\ell G$ are not stably equivalent? If so, are there useful knot invariants that can be obtained from the Khovanov functor which are lost on passing to $\mathcal{X}_{Kh}(K)$?

There are other formulations of Khovanov homology, including Cautis-Kamnitzer's formulation via algebraic geometry [CK08], formulations by Webster [Weba, Webb], Lauda-Queffelec-Rose [LQR], and others via categorified quantum groups, and Seidel-Smith's formulation via Floer homology [SS06] (see also [AS]). In the Floer homology case, the Cohen-Jones-Segal program [CJS95] is expected to produce a Floer spectrum.

Question 8.3. Does the Cautis-Kamniter or Webster formulation of Khovanov homology have a natural extension along the lines of stable functors to the Burnside category?

Question 8.4. Is the Floer spectrum (conjecturally) given by applying the Cohen-Jones-Segal construction to the Seidel-Smith formulation of Khovanov homology homotopy equivalent to $\mathcal{X}_{Kh}(K)$?

Finally, it would be nice to have an explicit description of the stable Khovanov functor or the Khovanov homotopy type in more cases. One possible direction would be to understand the stable invariant in the sense of [Sto07, GOR13]:

Question 8.5. Do the Khovanov functors for (m, N) torus knots admit a limit as $N \to \infty$, and if so, is there a simple description of the limit functor or the associated spectrum?

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