Synthetic spectra are (usually) cellular

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Abstract

If E is a connective ring spectrum, then Pstrągowski's category Syn_E of E-synthetic spectra is generated by the bigraded spheres $S^{i,j}$. In particular, it is equivalent to the category of modules over a filtered ring spectrum.

1 Introduction

Our goal in this short note is to prove the following result.

{thm:syntheticcellular}

Theorem 1.1. *If E is connective then the category Syn_E of E-synthetic spectra from [Pst22] is* cellular*: it is generated under homotopy colimits by the bigraded spheres $S^{i,j}$.*

The bigraded spheres then serve as a set of compact generators for Syn_E, and thus the Schwede–Shipley theorem applies; this allows us to exhibit the synthetic category as a category of left modules over a $\mathbb{Z}$-graded ring spectrum $R_\bullet$ whose bigraded coefficient groups are $\pi_{*,*}(S^{0,0})$.

Our argument roughly parallels that of [Pst22, 6.2] for the case of MU-modules. It relies relatively heavily on the classification of finitely generated abelian groups; more specifically, it would apply to a category of "E-synthetic R-modules" for any connective commutative ring spectrum R such that $\pi_0 R$ is a principal ideal domain. (It specifically does *not* apply to the spectra of Patchkoria–Pstrągowski [PP23]; those are no longer generated by finite X with E_*X projective.) We are grateful to Pstrągowski for their comments.

Throughout this paper we assume that E is an associative ring spectrum and that E_* is connective. Associated to this, there is a natural retraction of graded rings $E_0 \to E_* \to E_0$, the latter map realized by a map $E \to HE_0$ of ring spectra. Finally, when we refer to a category of modules over a ring, graded ring, or ring spectrum, we are referring to *left* modules.

2 Graded projective modules

The coefficient ring E_* is a connective graded ring, and so the following is a standard result.

{prop:detectprojectivity}

Proposition 2.1. *The following are equivalent for a bounded-below graded E_*-module M:*

- *M is projective as an E_*-module.*
- *M is of the form $E_* \otimes_{E_0} P$ for a projective graded E_0-module P.*
- *The graded E_0-module $\overline{M} = E_0 \otimes_{E_*} M$ is projective, and M is isomorphic to $E_* \otimes_{E_0} \overline{M}$.*
- *The graded E_0-module $\overline{M} = E_0 \otimes_{E_*} M$ is projective, and the bigraded Tor-groups $\mathrm{Tor}^{E_*}_{p,q}(M; E_0)$ are zero if the homological degree p is positive.*

The following is a relatively immediate consequence.

{cor:splitinjection}

Corollary 2.2. *Suppose $f\colon M \to N$ is a map of projective graded E_*-modules, and the map $\bar{f}\colon \overline{M} \to \overline{N}$ is a split injective map of graded E_0-modules. Then f is a split injective map of graded E_*-modules. In particular, N/M is also projective.*

tiyahhirzebruchprojective}

Proposition 2.3. *Suppose X is a bounded-below spectrum. Then E_*X is a projective E_*-module if and only if:*

- *the ordinary homology $H_*(X; E_0)$ is a projective E_0-module, and*
- *the map $E_*(X) \to H_*(X; E_0)$ is surjective.*

Proof. If E_*X is projective, then the Künneth spectral sequence

$$\mathrm{Tor}^{E_*}_{**}(E_0, E_*X) \Rightarrow H_*(X; E_0),$$

induced by $HE_0 \wedge X \simeq HE_0 \wedge_E (E \wedge X)$, degenerates to an isomorphism

$$H_*(X; E_0) \cong E_0 \otimes_{E_*} E_*X,$$

and so the conditions are clearly necessary. We now show that they are sufficient.

Smashing X with the Postnikov tower of E gives, by boundedness of X and E, a convergent Atiyah–Hirzebruch spectral sequence

$$H_p(X; E_q) \Rightarrow E_{p+q}X$$

compatible with the E_*-module structure. If $H_*(X; E_0)$ is a projective graded E_0-module, then the Künneth spectral sequence

$$\mathrm{Tor}^{E_0}_{**}(E_q, H_*(X; E_0)) \Rightarrow H_*(X; E_q)$$

degenerates to an isomorphism

$$H_p(X; E_q) \cong E_q \otimes_{E_0} H_p(X; E_0),$$

and so the spectral sequence is of the form

$$E_q \otimes_{E_0} H_p(X; E_0) \Rightarrow E_{p+q}X.$$

If the map $E_*X \to H_*(X; E_0)$ is surjective, then the elements of $H_*(X; E_0)$ are permanent cycles which generate the left-hand side as an E_*-module, and hence the spectral sequence collapses. By projectivity, there are no extension problems as E_*-modules. □

3 Moore spectra

{prop:mooresurjective}

Proposition 3.1. *For any abelian group A with associated Moore spectrum MA, the map $E_*MA \to H_*(MA;E_0)$ is surjective.*

Proof. The map $E_*MA \to H_*(MA;E_0)$ is the edge morphism in the Atiyah–Hirzebruch spectral sequence $H_*(MA;E_*) \Rightarrow E_*MA$. This degenerates due to sparsity; for a Moore spectrum MA, only $H_0(MA;-)$ and $H_1(MA;-)$ can be nontrivial. □

{prop:mooreprojective}

Proposition 3.2. *If A is finitely generated, then E_*MA is a projective E_*-module if and only if $E_0 \otimes A$ is a projective E_0-module.*

Proof. If E_*MA is a projective E_*-module, then it is also flat and this induces a Künneth isomorphism $H_*(MA;E_0) \cong E_0 \otimes_{E_*} E_*MA$. As a result, $H_*(MA;E_0)$ is a projective graded E_0-module, and in particular $H_0(MA,E_0) \cong E_0 \otimes A$ is a projective E_0-module.

Now we need to prove the converse. Since $MA \oplus MB \simeq M(A \oplus B)$, by the classification of finitely generated abelian groups it suffices to prove the case where A is cyclic.

If $A \cong \mathbb{Z}$, then $E_*M\mathbb{Z} \cong E_*$ is a projective E_*-module and $E_0 \otimes \mathbb{Z} \cong E_0$ is a projective E_0-module.

If $A \cong \mathbb{Z}/m$, with associated Moore spectrum $\mathbb{S}/m$, then we have an exact sequence

$$0 \to H_1(\mathbb{S}/m;E_0) \to E_0 \xrightarrow{m} E_0 \to H_0(\mathbb{S}/m;E_0) \to 0$$

of E_0-modules. If $E_0 \otimes \mathbb{Z}/m$ is projective, this last term splits: there is an idempotent $e \in E_0$ such that $me = 0$ and $e \equiv 1 \mod m$. This gives us a splitting

$$E_0 \cong E_0[1/m] \times E_0/m$$

of left E_0-modules. Our exact sequence therefore determines isomorphisms

$$H_0(\mathbb{S}/m;E_0) \cong H_1(\mathbb{S}/m;E_0) \cong E_0/m.$$

In particular, both are projective E_0-modules.

Further, this idempotent gives us a splitting

$$E_* \cong E_*[1/m] \oplus E_*/m$$

of left E_*-modules, and the long exact sequence

$$\cdots \to E_* \xrightarrow{m} E_* \to E_*(\mathbb{S}/m) \to \dots$$

determines a short exact sequence

$$0 \to E_*/m \to E_*(\mathbb{S}/m) \to \Sigma E_*/m \to 0$$

of E_*-modules. In particular, the outside terms are projective and hence so is $E_*(\mathbb{S}/m)$. (Alternatively, we could be less explicit and apply Propositions 2.3 and 3.1 to conclude that $E_*(\mathbb{S}/m)$ is projective.) □

4 Filtrations

{prop:projectivefiltration}

Proposition 4.1. *Let $Y^0 \to Y^1 \to Y^2 \to \cdots \to Y^n = *$ be a sequence of maps of spectra, and write F^k for the fiber of $Y^k \to Y^{k+1}$. Suppose that we have the following properties:*

- *$E_*(Y^0)$ is a projective E_*-module; and*
- *the maps $H_*(Y^k; E_0) \to H_*(Y^{k+1}; E_0)$ on ordinary homology are split surjections of graded E_0-modules.*

Then the sequences $0 \to H_*(F^k; E_0) \to H_*(Y^k; E_0) \to H_*(Y^{k+1}; E_0) \to 0$ are split exact sequences of projective graded E_0-modules. If, in addition, we have that

- *the maps $E_*F^k \to H_*(F^k; E_0)$ are surjective,*

then the sequences $0 \to E_*F^k \to E_*Y^k \to E_*Y^{k+1} \to 0$ are split exact sequences of projective E_*-modules.

Proof. Proposition 2.3 shows that $H_*(Y^0; E_0)$ is a projective graded E_0-module, and the split surjection criterion implies that

$$0 \to H_*(F^k; E_0) \to H_*(Y^k; E_0) \to H_*(Y^{k+1}; E_0) \to 0$$

is always a split exact sequence of graded E_0-modules. In particular, by induction on k we find that $H_*(F^k; E_0)$ and $H_*(Y^k; E_0)$ are projective graded E_0-modules.

If we additionally know that $E_*(F^k) \to H_*(F^k; E_0)$ is surjective, then the modules $E_*(F^k)$ are projective by Proposition 2.3. Applying Corollary 2.2 inductively, we find that $0 \to E_*F^k \to E_*Y^k \to E_*Y^{k+1} \to 0$ are split exact sequences of projective E_*-modules. □

Corollary 4.2. *Suppose E is commutative and $Y^0 \to Y^1 \to Y^2 \to \cdots \to Y^n = *$ is a sequence of maps of spectra satisfying the above three criteria. Then the synthetic analogues form cofiber sequences $\nu(F^k) \to \nu(Y^k) \to \nu(Y^{k+1})$ in Syn_E.*

Proof. By [Pst22, 4.23], a fiber sequence $F^k \to Y^k \to Y^{k+1}$ which is E_*-exact becomes a fiber sequence of $\nu(F^k) \to \nu(Y^k) \to \nu(Y^{k+1})$ in Syn_E. □

Any finite spectrum X, with integral homology concentrated in degrees n through m, has a "Moore filtration"

$$X = X^n \to X^{n+1} \to \cdots \to X^m \to *$$

such that the fiber $F^k = fib(X^k \to X^{k+1})$ is a Moore spectrum $\Sigma^k M(H_k(X))$.

{cor:moorefiltration}

Corollary 4.3. *Let X be a finite spectrum with a Moore filtration $X^n \to X^{n+1} \to \cdots \to X^m \to *$, with fibers $F^k \simeq \Sigma^k M(H_k X)$.*

If $E_*(X^0)$ is a projective E_*-module, then the sequences $0 \to E_*F^k \to E_*X^k \to E_*X^{k+1} \to 0$ are split exact sequences of projective E_*-modules, and $\nu(F^k) \to \nu(X^k) \to \nu(X^{k+1})$ are cofiber sequences in Syn_E.

Proof. On integral homology, the map $H_*(X^k) \to H_*(X^{k+1})$ is a split surjection (explicitly, the kernel is the degree-k part $H_k(X)$). The universal coefficient theorem (in particular, that its splitting is natural in the coefficient group) then implies that $H_*(X^k; E_0) \to H_*(X^{k+1}; E_0)$ is a split surjection of E_0-modules. Applying Proposition 4.1 and Proposition 3.1, we arrive at the result. □

5 Cellularity of Moore spectra

Throughout this section we assume that E is commutative so that we can work with the category Syn_E of synthetic spectra.

{lem:localizationcellular}

Lemma 5.1. *For any flat $\mathbb{Z}$-module A, the synthetic analogue $\nu(MA)$ of a Moore spectrum is cellular.*

Proof. We can construct MA from a resolution $0 \to \oplus_J \mathbb{Z} \to \oplus_I \mathbb{Z} \to A \to 0$ of abelian groups by lifting it to a cofiber sequence $\oplus_J \mathbb{S} \to \oplus_I \mathbb{S} \to MA$. On E-homology, we get a long exact sequence

$$\cdots \to \oplus_J E_* \to \oplus_I E_* \to E_* MA \to \dots$$

However, the kernel of the first map is $\mathrm{Tor}(E_*, A)$, which is trivial, and so the above is actually a short exact sequence. Therefore, the cofiber sequence is preserved by ν by [Pst22, 4.23], and so we have a cofiber sequence

$$\oplus_I \nu(\mathbb{S}) \to \oplus_J \nu(\mathbb{S}) \to \nu(MA)$$

As a result, $\nu(MA)$ is cellular. □

{lem:quotientcellular}

Lemma 5.2. *Suppose that m is an integer such that E_0/m is a projective E_0-module. Then the synthetic spectrum $\nu(\mathbb{S}/m)$ is cellular.*

Proof. The object $\nu(\mathbb{S}[1/m])$ is cellular by Lemma 5.1. The fiber sequence

$$\Sigma^{-1}\mathbb{S}/m^\infty \to \mathbb{S} \to \mathbb{S}[1/m],$$

upon E_*, becomes a long exact sequence including the maps

$$\Sigma^{-1} E_*(\mathbb{S}/m^\infty) \to E_* \to E_*[1/m].$$

However, recall from Proposition 3.2 that we have a splitting

$$E_* \cong E_*[1/m] \times E_*/m.$$

The map $E_* \to E_*[1/m]$ is then identified with the projection onto a split summand; hence the first term is identified with the complementary summand E_*/m, and this is a short exact sequence on E_*. Therefore, the sequence

$$\nu(\Sigma^{-1}\mathbb{S}/m^\infty) \to \nu(\mathbb{S}) \to \nu(\mathbb{S}[1/m])$$

is a fiber sequence. The second two terms are cellular, and hence so is the first.

Finally, the fiber sequence

$$\Sigma^{-1}\mathbb{S}/m^\infty \to \mathbb{S}/m \to \mathbb{S}/m^\infty$$

becomes, on E_*, a short exact seqence

$$0 \to E_*/m \to E_*(\mathbb{S}/m) \to \Sigma E_*/m \to 0$$

and therefore by [Pst22, 4.23] we have a fiber sequence

$$\nu(\Sigma^{-1}\mathbb{S}/m^\infty) \to \nu(\mathbb{S}/m) \to \nu(\mathbb{S}/m^\infty).$$

The outer two terms have just been shown to be cellular, and hence so is the middle term, as desired. □

{cor:fingen}

Corollary 5.3. *Suppose that A is a finitely generated abelian group such that $E_0 \otimes A$ is projective over E_0. Given a Moore spectrum MA, then $E_*(MA)$ is a projective E_*-module and $\nu(\Sigma^k MA)$ is cellular.*

Proof. To prove that $\nu(MA)$ is cellular, we can use the classification of finitely generated abelian groups and apply the previous lemmas summand-by-summand. To prove that $\nu(\Sigma^k MA)$ is cellular, we recall that $\nu(\Sigma^k MA) \simeq \Sigma^{k,k}\nu(MA)$, and so this follows from cellularity of $\nu(MA)$. □

6 Cellularity of synthetic spectra

We are now ready to prove that synthetic spectra are generated by bigraded spheres.

Proof of Theorem 1.1. (cf. [Pst22, 6.2]) The category Syn_E is a sheaf category, and so generated under homotopy colimits by the Yoneda image: objects of the form $\nu(X)$ where X is finite and E_*X is projective. It therefore suffices to prove that such $\nu(X)$ are cellular.

By Corollary 4.3, every such synthetic spectrum spectrum $\nu(X)$ has a finite filtration whose subquotients are of the form $\nu(\Sigma^k MA)$ where A is a finitely generated abelian group with $E_0 \otimes A$ a projective graded E_0-module.

Finally, by Corollary 5.3, such Moore spectra $\nu(\Sigma^k MA)$ are cellular. □

Corollary 6.1. *The category of E-synthetic spectra is equivalent to the category of left modules over a $\mathbb{Z}$-graded spectrum $R_\bullet \simeq \mathrm{map}_{\mathrm{Syn}_E}(S^{0,\bullet}, S^{0,0})$.*

Proof. There is a lax symmetric monoidal functor $\mathbb{Z} \to \mathrm{Syn}_E$, given by $n \mapsto S^{0,n}$. (The synthetic spectrum $S^{0,n}$ is the sheafification of the presheaf $X \mapsto \tau_{\geq -n}\,\mathrm{map}(X, \mathbb{S})$, and so this is implied by lax symmetric monoidality of the Whitehead tower.) The set of $S^{0,n}$ are invertible compact generators for Syn_E as a stable category, and so the functors

$$X \mapsto \mathrm{map}_{\mathrm{Syn}_E}(S^{0,n}, X)_{n\in\mathbb{Z}}$$

determine a conservative functor $\mathrm{Syn}_E \to \mathrm{Fun}(\mathbb{Z}, \mathrm{Sp})$ which preserves homotopy limits and colimits. The monadicity theorem thus applies. The left adjoint is $(Y_n)_{n\in\mathbb{Z}} \mapsto \bigoplus S^{0,n} \otimes Y_n$, and so the associated monad sends $(Y_n)_{n\in\mathbb{Z}}$ to

$$\left(\bigoplus_n \mathrm{map}_{\mathrm{Syn}_E}(S^{0,m}, S^{0,n}) \otimes Y_n\right)_{m\in\mathbb{Z}} \simeq \left(\bigoplus_n \mathrm{map}_{\mathrm{Syn}_E}(S^{0,m-n}, S^{0,0}) \otimes Y_n\right)_{m\in\mathbb{Z}}$$

However, this is equivalent to a monad

$$Y_\bullet \mapsto \mathrm{map}_{\mathrm{Syn}_E}(S^{0,\bullet}, S^{0,0}) \circledast Y_\bullet$$

where $\circledast$ is the Day convolution on $\mathbb{Z}$-graded spectra, as desired. □

Remark 6.2. The functor $n \mapsto S^{0,n}$ is actually a strong monoidal functor $(\mathbb{Z}, \leq) \to \mathrm{Syn}_E$, and the functor

$$X \mapsto \mathrm{map}_{\mathrm{Syn}_E}(S^{0,\bullet}, X)$$

is thus a lax symmetric monoidal functor $\mathrm{Syn}_E \to \mathrm{Fun}((\mathbb{Z}, \leq)^{op}, \mathrm{Sp})$. The ring spectrum $R_\bullet$ is the image of the unit, and thus has the structure of a commutative ring object in filtered spectra.

References

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