LOCALIZATION OF ENRICHED CATEGORIES AND CUBICAL SETS

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ABSTRACT. The invertibility hypothesis for a monoidal model category **S** asks that localizing an **S**-enriched category with respect to an equivalence results in an weakly equivalent enriched category. This is the most technical among the axioms for **S** to be an *excellent model category* in the sense of Lurie, who showed that the category $\mathcal{C}at_{\mathbf{S}}$ of **S**-enriched categories then has a model structure with characterizable fibrant objects. We use a universal property of cubical sets, as a monoidal model category, to show that the invertibility hypothesis is a consequence of the other axioms.

Topological categories, simplicial categories, and differential graded categories are special types of enriched categories: the enriching category has a notion of weak equivalence and its own homotopy theory. These have played a prominent role a diverse array of subjects.

Getting control over the homotopy theory of some of these enriched categories and homotopical constructions in them (such as pushouts, pullbacks, and other derived limit and colimit constructions) is easier in the presence of model structures. If **S** is a monoidal model category, Lurie gave conditions for the existence of a model structure with many useful properties on the collection $\mathcal{C}at_{\mathbf{S}}$ of **S**-enriched categories [Lur09, A.3.2.4]. (In the terminology of [BM13], this allows Lurie to assert that the *canonical* model structure exists.) The cofibrations and weak equivalences in $\mathcal{C}at_{\mathbf{S}}$ have a relatively straightforward description (see §2), but in order to get a useful characterization of the fibrations more assumptions are required. With this goal, Lurie defined an *excellent* model category as a model category **S**, with a symmetric monoidal structure, satisfying additional axioms labeled (A1) through (A5). The first four of these axioms are all relatively standard concepts or are straightforward to verify.

Axiom (A5) is called the invertibility hypothesis. It is more technical—it roughly asserts that inverting a weak equivalence results in a weakly equivalent enriched category—and is more difficult to verify in practice. The fact that the category $\mathcal{S}et_{\Delta}$ of simplicial sets satisfies the invertibility hypothesis is an important result of Dwyer and Kan [DK80, 10.4]. The invertibility hypothesis for differential graded categories is a consequence of work of Toën [Toë07, 8.7], and for enrichment in simplicial model categories it is a theorem of Dundas [Dun01, 0.9].

Our main result is the following.

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0.1. Theorem. *Let $\boldsymbol{S}$ be a combinatorial monoidal model category. Assume that every object of $\boldsymbol{S}$ is cofibrant and that the collection of weak equivalences in $\boldsymbol{S}$ is stable under filtered colimits. Then $\boldsymbol{S}$ satisfies the invertibility hypothesis of [Lur09, A.3.2.12].*

Combinatoriality is Lurie's axiom (A1), cofibrance of all objects is a consequence of axiom (A2), stability of weak equivalences under filtered colimits is axiom (A3), and the model structure being monoidal is axiom (A4). (Lurie also asks as part of the definition that the model category be symmetric monoidal.)

Our method is the following. We will first show that the category $\mathcal{S}\mathrm{et}_\square$ of cubical sets, with a model structure due to Cisinski [Cis06], admits a monoidal left Quillen functor out to essentially *any* monoidal model category $\mathbf{S}$ with cofibrant unit: the choice of such a functor is essentially a choice of cylinder object for the monoidal unit. Second, we will show (in a method adapted from [Lur09, A.3.2.20, A.3.2.21]) that this left Quillen functor allows the model category $\mathbf{S}$ to inherit the invertibility hypothesis from $\mathcal{S}\mathrm{et}_\square$.

This gives cubical sets a useful universal property. However, some good properties of simplicial sets are lost: the monoidal structure on cubical sets is not symmetric, and understanding the homotopy theory of cubical sets requires hard theorems.

Approaches to the homotopy theory of enriched categories other than Lurie's have also been studied, e.g. by Berger–Moerdijk [BM13] and Muro [Mur15]. Due to the complementary nature of their work and assumptions with Lurie's, it does not seem that the results of this paper bring new results to their framework.

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1. Cubical sets

We'll begin with preliminaries on the category of cubical sets; much of what we discuss in the following appears in Cisinski's work [Cis06, §8.4].

- The cube category $\square$ has, as objects, the n-cubes $\square^n$ for $n \geq 0$.
- The maps $\square^n \to \square^m$ are in bijective correspondence with the set of maps $[0,1]^n \to [0,1]^m$ which are composites of coordinate projections $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, \widehat{x_i}, \ldots, x_n)$ and face inclusions $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, \epsilon, x_i, \ldots, x_n)$ for $\epsilon \in \{0, 1\}$.
- There is a monoidal product $\otimes\colon \square \times \square \to \square$ such that $\square^n \otimes \square^m = \square^{n+m}$, corresponding to the isomorphism $[0,1]^n \times [0,1]^m \cong [0,1]^{n+m}$. (This monoidal product is not symmetric.)

Write $\square^{\leq 1}$ for the full subcategory of $\square$ spanned by $\square^0$ and $\square^1$. In this subcategory, there are two maps $j^0, j^1 \colon \square^0 \to \square^1$ and a map $r \colon \square^1 \to \square^0$, and these satisfy $rj^0 = rj^1 = id_{\square^0}$; moreover, these maps and these relations generate all maps and relations in $\square^{\leq 1}$.

The category $\square$ is generated by $\square^{\leq 1}$ and the monoidal structure in the following sense.

1.1. Proposition. [Cis06, 8.4.6] *Suppose $\mathcal{C}$ is a monoidal category. For any functor $F \colon \square^{\leq 1} \to \mathcal{C}$ such that $F(\square^0)$ is the monoidal unit $\mathbb{I}$, there exists an extension to a functor $\tilde{F} \colon \square \to \mathcal{C}$ which is monoidal. This monoidal extension is unique up to natural isomorphism.*

A cubical set is a functor $\square^{op} \to \mathcal{S}\mathrm{et}$, and $\mathcal{S}\mathrm{et}_\square$ is the category of cubical sets. The covariant Yoneda embedding $\theta \colon \square \to \mathcal{S}\mathrm{et}_\square$ satisfies the following property.

1.2. Proposition. *Suppose $\mathcal{C}$ is cocomplete. For any functor $F \colon \square \to \mathcal{C}$, the "singular cubical set" functor* $\mathrm{Hom}_{\mathcal{C}}(F(\square^\bullet), -)$ *has a left adjoint $\tilde{F} \colon \mathcal{S}\mathrm{et}_\square \to \mathcal{C}$ (the associated realization functor) extending F. This extension is unique up to natural isomorphism.*

The monoidal structure on $\square$ gives rise to a Day convolution product $\otimes$ on $\mathcal{S}\mathrm{et}_\square$. More specifically, given $X, Y \colon \square^{op} \to \mathcal{S}\mathrm{et}$, the tensor $X \otimes Y$ is the left Kan extension of $X \times Y \colon \square^{op} \times \square^{op} \to \mathcal{S}\mathrm{et} \times \mathcal{S}\mathrm{et} \to \mathcal{S}\mathrm{et}$ along the functor $\otimes \colon \square^{op} \times \square^{op} \to \square^{op}$. The universal property of left Kan extension gives this (nonsymmetric) monoidal product a universal property as well.

1.3. Proposition. [Cis06, 8.4.23] *Suppose $\mathcal{C}$ is a cocomplete category with a monoidal structure that preserves colimits in each variable separately. For any monoidal functor $F \colon \square \to \mathcal{C}$, the colimit-preserving extension $\tilde{F} \colon \mathcal{S}\mathrm{et}_\square \to \mathcal{C}$ is also monoidal.*

We now consider monoidal model categories. Recall that for maps $f_1 \colon A_1 \to B_1$ and $f_2 \colon A_2 \to B_2$ in a cocomplete monoidal category $\mathcal{C}$, the pushout-product $f_1 \boxtimes f_2$ is the map

$$A_1 \otimes B_2 \coprod_{A_1 \otimes A_2} B_1 \otimes A_2 \to B_1 \otimes B_2.$$

Write i for the map $j^0 \amalg j^1 \colon \square^0 \coprod \square^0 \to \square^1$ in $\mathcal{S}\mathrm{et}_\square$. The pushout-product allows us to define cubical sets $\partial \square^n$ (for $n \geq 0$) and $\sqcap^n_{(k,\epsilon)}$ (for $1 \leq k \leq n$ and $\epsilon \in \{0,1\}$) as the sources of the following pushout-product maps:

$$(i \boxtimes \cdots \boxtimes i) \colon \partial \square^n \to \square^n \tag{1}$$

$$(i \boxtimes \cdots \boxtimes j^\epsilon \boxtimes \cdots \boxtimes i) \colon \sqcap^n_{(k,\epsilon)} \to \square^n \tag{2}$$

(We will use the convention that, when $n = 0$, the map $\partial \square^0 \to \square^0$ of Equation (1) is the map $\emptyset \to \square^0$ from the initial object.)

We will require the following result of Cisinski.

1.4. Theorem. [Cis06, 8.4.38], [Jar06, §7] *There exists a combinatorial, left proper model structure on* $\mathcal{S}\mathrm{et}_{\square}$ *with generating cofibrations the maps of Equation* (1) *and generating acyclic cofibrations the maps of Equation* (2). *There is a monoidal left Quillen equivalence* $\mathcal{S}\mathrm{et}_{\square} \to \mathcal{S}\mathrm{et}_{\Delta}$, *given by the cubical realization functor which sends* $\square^n$ *to* $(\Delta^1)^n$.

1.5. Corollary. *Suppose* $\mathcal{C}$ *is a monoidal model category in the sense of [Hov99, §4]. Let* $F\colon \square^{\leq 1} \to \mathcal{C}$ *be a functor sending* $\square^0$ *to the unit, and let* $\tilde{F}\colon \mathcal{S}\mathrm{et}_{\square} \to \mathcal{C}$ *be an extension to a monoidal left adjoint. Then* $\tilde{F}$ *is a left Quillen functor if and only if the unit of* $\mathcal{C}$ *is cofibrant and the maps* $F(j^0)$, $F(j^1)$ *express* $F(\square^1)$ *as a cylinder object for the unit of* $\mathcal{C}$. *In particular, such a monoidal left Quillen functor exists.*

Proof. In order for $\tilde{F}$ to be a left Quillen functor, it must take the acyclic cofibrations j^0 and j^1 to acyclic cofibrations and the cofibration i to a cofibration; this happens precisely when $F(\square^1)$ is expressed as a cylinder object. It also must take the map $\emptyset \to \square^0$ to a cofibration, so the unit must be cofibrant.

Conversely, suppose that F expresses $F(\square^1)$ as a cylinder object, so that the map $\tilde{F}(i)$ is a cofibration and that the maps $\tilde{F}(j^\epsilon)$ are both acyclic cofibrations. Then the fact that $\tilde{F}$ is monoidal and colimit-preserving implies that $\tilde{F}(f_1 \boxtimes \cdots \boxtimes f_n)$ is the pushout-product $\tilde{F}(f_1) \boxtimes \cdots \boxtimes \tilde{F}(f_n)$ for $n \geq 1$. This makes the map $\tilde{F}(i \boxtimes \cdots \boxtimes i)$ into an iterated pushout-product of cofibrations, and makes the map $\tilde{F}(i \boxtimes \cdots \boxtimes j^\epsilon \boxtimes \cdots \boxtimes i)$ into an iterated pushout-product of several cofibrations and one acyclic cofibration. When $n = 0$, the map $F(\emptyset) \to F(\partial\square^0)$ is sent to the map from the initial object to the unit. As $\mathcal{C}$ is a monoidal model category with cofibrant unit, $\tilde{F}$ then preserves the generating cofibrations and generating acyclic cofibrations, and hence is a left Quillen functor. ■

2. Enriched categories

Suppose that $\mathbf{S}$ is a monoidal model category with unit $\mathbb{I}$, that every object of $\mathbf{S}$ is cofibrant, and that the collection of weak equivalences in $\mathbf{S}$ is stable under filtered colimits. For such $\mathbf{S}$, Lurie constructs a left proper combinatorial model structure on $\mathcal{C}\mathrm{at}_{\mathbf{S}}$ in [Lur09, A.3.2.4] which we will review now.

Following the notation of [Lur09, §A.3.2], we define the following four special examples of enriched categories:

- Let $\emptyset$ be the trivial enriched category with no objects.
- Let $[0]_{\mathbf{S}}$ be the category with a single object 0 and $\mathrm{Map}_{\mathcal{C}}(0, 0) = \mathbb{I}$.
- Let $[1]_A$ be the category with objects 0 and 1, such that $\mathrm{Map}_{[1]_A}(0, 1) = A$, $\mathrm{Map}_{[1]_A}(i, i) = \mathbb{I}$, and $\mathrm{Map}_{[1]_A}(1, 0) = \emptyset$. If $A = \mathbb{I}$, we simply write $[1]_{\mathbf{S}}$ for $[1]_{\mathbb{I}}$.

- Let $[1]_{\tilde{\mathbf{S}}}$ be the category with objects 0 and 1, such that $\mathrm{Map}_{[1]_{\tilde{\mathbf{S}}}}(i,j) = \mathbb{I}$ for all i and j.

The model structure on $\mathcal{C}\mathrm{at}_{\mathbf{S}}$ is defined by the following requirements:

- An enriched functor $F\colon \mathcal{C} \to \mathcal{D}$ is a weak equivalence if the map $h\mathcal{C} \to h\mathcal{D}$ of homotopy categories, obtained by applying $[\mathbb{I}, -]_{h\mathbf{S}}$ to morphism objects, is an equivalence, and if for all $c, c' \in \mathcal{C}$ the map $\mathrm{Map}_{\mathcal{C}}(c,c') \to \mathrm{Map}_{\mathcal{D}}(Fc, Fc')$ is an equivalence in $\mathbf{S}$.
- The set
$$\Big\{[1]_S \to [1]_{S'} \mid S \to S' \text{ a generating cofibration}\Big\} \cup \Big\{\emptyset \to [0]_{\mathbf{S}}\Big\}$$
is a set of generating cofibrations.

As a consequence, a monoidal left Quillen functor $F\colon \mathbf{S} \to \mathbf{S}'$ between such categories gives rise to a left Quillen functor $\mathcal{C}\mathrm{at}_{\mathbf{S}} \to \mathcal{C}\mathrm{at}_{\mathbf{S}'}$, which is a left Quillen equivalence if F was [Lur09, A.3.2.6].

In the following, we will write $\mathcal{C}\mathrm{at}_{\Box}$ for the category $\mathcal{C}\mathrm{at}_{\mathcal{S}\mathrm{et}_{\Box}}$ of categories enriched in cubical sets, and similarly $\mathcal{C}\mathrm{at}_{\Delta}$ for the category $\mathcal{C}\mathrm{at}_{\mathcal{S}\mathrm{et}_{\Delta}}$ of categories enriched in simplicial sets.

2.1. Definition. *Let $\mathcal{C} \in \mathcal{C}\mathrm{at}_{\Box}$ be a category enriched in cubical sets. Given morphisms $f, g\colon c \to c'$ in the underlying category $\mathcal{C}$, classified by maps $f, g\colon \Box^0 \to \mathrm{Map}_{\mathcal{C}}(c,c')$, a homotopy from f to g identity is a morphism $H\colon \Box^1 \to \mathrm{Map}_{\mathcal{C}}(c,c')$ such that $H \circ j^0 = f$ and $H \circ j^1 = g$.*

Let $\mathcal{H} \in \mathcal{C}\mathrm{at}_{\Box}$ be universal among cubically enriched categories possessing morphisms $u\colon c \to c'$ and $v\colon c' \to c$ together with a homotopy from $v \cdot u$ to the identity id_c. We refer to $\mathcal{H}$ as the homotopy inverse *category.*

The category $\mathcal{H}$ is sometimes called the *walking deformation retract.* It can be described as an iterated pushout diagram in $\mathcal{C}\mathrm{at}_{\Box}$ as follows.

$$\begin{array}{ccc} [1]_{\emptyset} & \longrightarrow & [1]_{\Box} \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle u} \\ [1]_{\Box} & \xrightarrow[v]{} & P \end{array} \qquad\qquad \begin{array}{ccc} [1]_{\partial\Box^1} & \longrightarrow & [1]_{\Box^1} \\ {\scriptstyle vu \coprod id}\downarrow & & \downarrow{\scriptstyle H} \\ P & \longrightarrow & \mathcal{H} \end{array}$$

(In the left-hand square, the functors $[1]_{\emptyset} \to [1]_{\Box}$ are the standard functor and the unique "twist" functor τ that switches the objects.) The category P classifies a pair of morphisms u and v in opposing directions, and the pushout defining $\mathcal{H}$ adjoins a homotopy from $v \cdot u$ to id_c. These pushouts allow us to deduce the following properties.

2.2. Proposition. *The functors $\ell\colon [1]_\square \to \mathcal{H}$ and $r\colon [1]_\square \to \mathcal{H}$, classifying the functions u and v respectively, are cofibrations. A functor of enriched categories $[1]_\square \to \mathcal{C}$, classifying a map $f\colon c \to c'$ in $\mathcal{C}$, has an extension along ℓ if and only if f admits a left homotopy inverse, and has an extension along r if and only if f admits a right homotopy inverse.*

2.3. Definition. *We define the category $\mathcal{E} \in \mathcal{C}\mathrm{at}_\square$ as the pushout in the diagram*

$$\begin{array}{ccc} [1]_\square & \xrightarrow{\ell} & \mathcal{H} \\ {\scriptstyle r}\downarrow & & \downarrow \\ \mathcal{H} & \longrightarrow & \mathcal{E}. \end{array}$$

The category $\mathcal{E}$ is sometimes called the *walking homotopy equivalence*. There is a unique factorization $[1]_\square \to \mathcal{E} \to [1]_{\tilde{\square}}$, which expresses that invertible maps have homotopy inverses.

2.4. Lemma. *Suppose $\mathcal{C} \in \mathcal{C}\mathrm{at}_\square$ is fibrant, and $f\colon [1]_\square \to \mathcal{C}$ classifies a map which becomes an isomorphism in the homotopy category $h\mathcal{C}$. Then f extends to a map $\mathcal{E} \to \mathcal{C}$.*

Proof. We need to show that f extends along the maps $\ell, r\colon [1]_\square \to \mathcal{H}$. First assume that f has a left inverse g: we will show that it extends over ℓ.

We note that since $\mathcal{C}$ is a fibrant $\mathcal{S}\mathrm{et}_\square$-enriched category, the objects $\mathrm{Map}_\mathcal{C}(c, c')$ are fibrant for all c and c' (cf. the proof of [Lur09, A.3.2.24]). Therefore, because $f\colon \square^0 \to \mathrm{Map}_\mathcal{C}(c, c')$ has a left inverse in the homotopy category, the left inverse has a representative g in the form of a map $g\colon \square^0 \to \mathrm{Map}_\mathcal{C}(c', c)$. More, since the composite $g \cdot f$ becomes equal to id_c as maps $\square^0 \to \mathrm{Map}_\mathcal{C}(c, c)$ in the homotopy category of $\mathbf{S}$, these maps are left homotopic. In particular, as $\square^1$ is a cylinder object for $\square^0$ in $\mathcal{S}\mathrm{et}_\square$, there exists an extension in the following diagram:

$$\begin{array}{ccc} \partial\square^1 & \xrightarrow{g\cdot f \coprod id_c} & \mathrm{Map}_\mathcal{C}(c, c). \\ \downarrow & \overset{H}{\nearrow} & \\ \square^1 & & \end{array}$$

The description of the map $\ell\colon [1]_\square \to \mathcal{H}$ by its universal property then produces precisely the desired extension.

The extension over r is symmetric. (Note that it is the proof itself that is symmetric: we cannot, for example, argue by taking the opposite category of $\mathcal{H}$ because cubically enriched categories do not have a natural opposite category.) ∎

We now consider what happens when the map f is inverted.

2.5. Definition. *Suppose we have a map g in an $\boldsymbol{S}$-enriched category $\mathcal{C}$, represented by a functor $i\colon [1]_{\boldsymbol{S}} \to \mathcal{C}$. The* localization $\mathcal{C}\langle g^{-1}\rangle$ *is the pushout in the diagram of $\boldsymbol{S}$-enriched categories*

$$\begin{array}{ccc} [1]_{\boldsymbol{S}} & \xrightarrow{i} & \mathcal{C} \\ \downarrow & & \downarrow{\scriptstyle j} \\ [1]_{\tilde{\boldsymbol{S}}} & \longrightarrow & \mathcal{C}\langle g^{-1}\rangle. \end{array}$$

We refer to $j\colon \mathcal{C} \to \mathcal{C}\langle g^{-1}\rangle$ as the localization functor.

2.6. Proposition. *Let $\mathcal{E}\langle f^{-1}\rangle$ be the localization of $\mathcal{E}$ obtained by inverting the map f. Then in the diagram*

$$\mathcal{E} \to \mathcal{E}\langle f^{-1}\rangle \to [1]_{\tilde{\square}},$$

which is determined by the universal property of the localization, both maps are weak equivalences in $\mathcal{C}\mathrm{at}_{\square}$.

Proof. Because there is a monoidal left Quillen equivalence $L\colon \mathcal{C}\mathrm{at}_{\square} \to \mathcal{C}\mathrm{at}_{\Delta}$, which preserves pushouts and localizations, it suffices to show that these maps of cubically enriched categories becomes equivalences of simplicially enriched categories.

The category $L(\mathcal{E})$ is the universal simplicial category with a morphism f together with a homotopy left inverse and a homotopy right inverse; moreover, the map $[1]_{\Delta} \to L(\mathcal{E})$ classifying f is a cofibration (as the image of a cofibration under L).

We may then apply work of Dwyer and Kan [DK80, 10.4], which shows that the localization map $L(\mathcal{E}) \to L(\mathcal{E})\langle f^{-1}\rangle$ that inverts f is a weak equivalence of simplicially enriched categories.

We now need to show that the map $L(\mathcal{E})\langle f^{-1}\rangle \to [1]_{\tilde{\Delta}}$ is an equivalence. This localization is still an iterated pushout, but as its two objects are now isomorphic it may be reinterpreted: it is the universal example of a simplicial category with two objects c and c', an isomorphism $c \to c'$, two maps $g_1, g_2\colon c \to c$, and two homotopies $H_1\colon g_1 \simeq id_c$ and $H_2\colon g_2 \simeq id_c$. To show that this is equivalent to $[1]_{\tilde{\Delta}}$, we must show that the mapping spaces in $L(\mathcal{E})\langle f^{-1}\rangle$ are all weakly equivalent to Δ^0.

The full subcategory $\mathcal{E}' \subset L(\mathcal{E})\langle f^{-1}\rangle$ spanned by the object c is equivalent to this one. As it has one object, $\mathcal{E}'$ is determined completely by the simplicial monoid $\mathrm{Map}(c, c)$, which is the free simplicial monoid with two elements g_i and paths from g_i to the identity. This monoid is the James construction $J(\Delta^1 \vee \Delta^1)$ on the based simplicial set $\Delta^1 \vee \Delta^1$, and as such it is weakly equivalent to Δ^0: its realization $J([0, 1] \vee [0, 1])$ as a topological space is contractible. ∎

We note that for this result to hold, it is important that the construction of $\mathcal{E}$ not ask for the left and right inverses of f to be the same map: in the final step we would instead obtain $J(S^1) \simeq \Omega S^2$ rather than a contractible space if we did so.

2.7. Corollary. *The map $[1]_\square \to \mathcal{E}$ is a cofibrant replacement for $[1]_\square \to [1]_{\tilde{\square}}$ in $\mathcal{C}\mathrm{at}_\square$.*

3. The invertibility hypothesis

We now recall the precise statement of the invertibility hypothesis [Lur09, A.3.2.12].

3.1. Definition. *Let $\boldsymbol{S}$ be a combinatorial monoidal model category. Assume that every object of $\boldsymbol{S}$ is cofibrant and that the collection of weak equivalences in $\boldsymbol{S}$ is stable under filtered colimits. We say that $\boldsymbol{S}$ satisfies the invertibility hypothesis when, for any isomorphism in the homotopy category $h\mathcal{C}$ classified by a cofibration $f\colon [1]_{\boldsymbol{S}} \to \mathcal{C}$, the localization map $j\colon \mathcal{C} \to \mathcal{C}\langle f^{-1}\rangle$ is a weak equivalence.*

We will now prove our main result.

Proof of Theorem 0.1. Let $f\colon [1]_{\mathbf{S}} \to \mathcal{C}$ be a cofibration as in Definition 3.1. As noted in [Lur09, A.3.2.13], it suffices to verify that this condition is satisfied in the case where $\mathcal{C}$ is a fibrant $\mathbf{S}$-enriched category: in particular, we may assume that the mapping objects in $\mathcal{C}$ are fibrant.

By Corollary 1.5, there exists a monoidal left Quillen functor $L\colon \mathcal{S}\mathrm{et}_\square \to \mathbf{S}$ from the category of cubical sets, sending $\square^1$ to a cylinder object for the monoidal unit. This induces a left Quillen functor $L\colon \mathcal{C}\mathrm{at}_\square \to \mathcal{C}\mathrm{at}_{\mathbf{S}}$ with right adjoint R. The right adjoint preserves homotopy categories:

$$\begin{aligned}
\mathrm{Hom}_{h\mathcal{C}}(c,c') &= \mathrm{Hom}_{h\mathbf{S}}([1]_{\mathbf{S}}, \mathrm{Map}_{\mathcal{C}}(c,c')) \\
&= \mathrm{Hom}_{h\mathbf{S}}(L[1]_\square, \mathrm{Map}_{\mathcal{C}}(c,c')) \\
&\cong \mathrm{Hom}_{h\mathcal{S}\mathrm{et}_\square}([1]_\square, R\mathrm{Map}_{\mathcal{C}}(c,c')) \\
&= \mathrm{Hom}_{hR\mathcal{C}}(c,c')
\end{aligned}$$

The map f becomes an isomorphism in the homotopy category of $\mathcal{C}$, so Lemma 2.4 implies that the map $[1]_\square \to R\mathcal{C}$ classifying f extends to a map $\mathcal{E} \to R\mathcal{C}$. The adjoint is an extension $[1]_{\mathbf{S}} \to L(\mathcal{E}) \to \mathcal{C}$.

Now factor $L(\mathcal{E}) \to \mathcal{C}$ into a cofibration $L(\mathcal{E}) \to \mathcal{C}'$ followed by an acyclic fibration $\mathcal{C}' \to \mathcal{C}$. The functor $[1]_{\mathbf{S}} \to L(\mathcal{E}) \to \mathcal{C}'$ is a composite of cofibrations, and the functor $[1]_{\mathbf{S}} \to \mathcal{C}$ is a cofibration by assumption. In any model category, pushouts of $A \to B$ along cofibrations $A \to X$ preserve weak equivalences in X (for example, by Ken Brown's lemma), and hence the functor $\mathcal{C}'\langle f^{-1}\rangle \to \mathcal{C}\langle f^{-1}\rangle$ induced by pushing out along $[1]_{\mathbf{S}} \to [1]_{\tilde{\mathbf{S}}}$ is an equivalence.

Thus, by replacing $\mathcal{C}$ with $\mathcal{C}'$ we may assume without loss of generality that $L(\mathcal{E}) \to \mathcal{C}$ is a cofibration.

Consider the following diagram.

$$\begin{array}{ccccc} [1]_{\mathbf{S}} & \longrightarrow & L(\mathcal{E}) & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow & & \downarrow \\ [1]_{\tilde{\mathbf{S}}} & \longrightarrow & L(\mathcal{E}\langle f^{-1}\rangle) & \longrightarrow & \mathcal{C}\langle f^{-1}\rangle \end{array}$$

The top right map is a cofibration. As L is a left Quillen functor, it preserves pushouts and hence localizations; this means that both squares are pushout squares. Proposition 2.6 implies that the center vertical map is an equivalence. As the model structure on $\mathcal{C}\mathrm{at}_{\mathbf{S}}$ is left proper, the right-hand vertical map is also an equivalence. ∎

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