Truncated Brown-Peterson spectra

T. Lawson[1] N. Naumann[2]

[1]University of Minnesota

[2]Universität Regensburg

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Algebraic K-theory

Conjecture (Ausoni-Rognes)

For any prime p and $n \geq 0$, there exist localization sequences

$$K(BP\langle n-1\rangle_p^\wedge)_p^\wedge \to K(BP\langle n\rangle_p^\wedge)_p^\wedge \to K(E(n)_p^\wedge)_p^\wedge.$$

Here $BP\langle k\rangle$ and $E(n)$ are p-local truncated Brown-Peterson spectra and Johnson-Wilson spectra respectively.

- $n = 0$ is a devissage result of Quillen
- $n = 1$ is a theorem of Blumberg-Mandell

Things to understand

- More multiplicative structure on R gives more structure to $K(R)$ and to a zoo of related objects: TC, TR, TF, THH
- This makes computations in algebraic K-theory easier by imposing multiplication and power operations
- Can we understand these truncated Brown-Peterson spectra $BP\langle n\rangle$?
- Can they be equipped with extra multiplicative structure?

Formal group laws are pointless groups

Definition

A formal group law $\mathbb{G}$ over a ring R is a power series $x +_{\mathbb{G}} y$ in $R[[x, y]]$ satisfying power series identities:

unitality:	$x +_{\mathbb{G}} 0$	$\equiv$	x
commutativity:	$x +_{\mathbb{G}} y$	$\equiv$	$y +_{\mathbb{G}} x$
associativity:	$(x +_{\mathbb{G}} y) +_{\mathbb{G}} z$	$\equiv$	$x +_{\mathbb{G}} (y +_{\mathbb{G}} z)$

- The existence of an "inverse" is automatic
- Underlying any formal group law is a *formal group*, which remembers only the isomorphism type

p-typical formal group laws

Definition

Suppose R is a torsion-free ring. A formal group law $\mathbb{G}$ over R is p-typical if there is a power series

$$\ell(x) = x + \ell_1 x^p + \ell_2 x^{p^2} + \cdots ,$$

with coefficients in $R \otimes \mathbb{Q}$, such that

$$x +_{\mathbb{G}} y \equiv \ell^{-1}(\ell(x) + \ell(y)).$$

Such a power series is called a *logarithm* for $\mathbb{G}$.

- This has an intrinsic definition, applicable over any ring
- Every formal group law over a p-local ring is isomorphic to a p-typical one

Complex oriented cohomology theories

Definition

A *complex oriented cohomology theory* is

- a cohomology theory E^*,
- with an associative and commutative multiplication,
- and an element $x \in \tilde{E}^2(\mathbb{CP}^\infty)$ which restricts to the element $1 \in \tilde{E}^2(S^2)$.

- Since $\mathbb{CP}^\infty$ classifies complex line bundles, E^* gets a natural characteristic class $c_1(\mathcal{L})$ for complex line bundles
- This automatically extends to a full theory of Chern classes

Chern classes produce formal group laws

Proposition

Given a complex oriented cohomology theory, there is a formal group law $\mathbb{G}_E$ over E^* such that the first Chern class satisfies a natural identity

$$c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) +_{\mathbb{G}_E} c_1(\mathcal{L}').$$

- Different choices of orientation produce different, but isomorphic, formal group laws
- If we are feeling energetic, we can throw gradings into the story

Example: Complex cobordism

The cohomology theory MU^*, associated to bordism of stably almost-complex manifolds, is complex orientable.

Theorem (Milnor)

$MU^* \cong \mathbb{Z}[b_1, b_2, \ldots]$, where b_i is in grading $2i$.

Theorem (Quillen)

The formal group law $\mathbb{G}_{MU}$ is universal.

This means that, for any ring R, there is a natural bijection

$$\{\text{homomorphisms } MU^* \to R\} \longrightarrow \{\text{formal group laws over } R\}$$

Brown-Peterson cohomology

- Brown-Peterson cohomology BP^* is a p-local, complex orientable cohomology theory
- The coefficient ring is isomorphic to $\mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ with v_i in degree $(2p^i - 2)$
- $\mathbb{G}_{BP}$ is p-typical
- $\mathbb{G}_{BP}$ is universal among p-local, p-typical formal group laws
- There is a map $MU^* \to BP^*$ classifying the inclusion

$$\{p\text{-typical formal group laws}\} \subset \{\text{formal group laws}\}$$

Truncated Brown-Peterson cohomology

Proto-definition

$BP\langle n\rangle^*$ is a complex oriented cohomology theory whose underlying coefficient ring is the quotient $\mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_n]$ of BP^*.

- Classically constructed using Baas-Sullivan theory of manifolds with singularity
- Newer constructions, with more structure, using more machinery
- The v_i are **not intrinsically defined** and so the definition depends (at least) on a choice of sequence of generators (e.g. Hazewinkel vs. Araki)

Generalized $BP\langle n\rangle$

A cohomology theory R^* is represented by a *spectrum* R.

Proposition

The following are equivalent for a spectrum R representing a cohomology theory R^* with a commutative and associative multiplication.

1. *R admits a p-typical orientation so that the map $BP^* \to R^*$ maps the (intrinsic) subring $\mathbb{Z}_{(p)}[v_1, \ldots, v_n] \subset BP^*$ isomorphically to R^*.*
2. *R is a p-local, connective, finite type spectrum such that $H^*(R; \mathbb{F}_p)$ is isomorphic to the quotient $\mathcal{A}^*/(Q^0, Q^1, \ldots, Q^n)$ of the Steenrod algebra.*

We will call such a spectrum a *generalized* truncated Brown-Peterson spectrum.

Questions about generalized $BP\langle n\rangle$

Things we don't appear to know about such spectra:

- For a given quotient $\mathbb{Z}_{(p)}[v_1, \ldots, v_n]$ of MU^*, how many distinct complex oriented cohomology theories are there with this given formal group law?
- Given distinct such formal group laws, when do the spectra have the same underlying homotopy type?
- Does any (p-local, finite type) spectrum with this homology automatically have a ring structure?

Standard technology (the Adams spectral sequence) appears to be very messy as soon as $n > 1$.

Power operations

- Multiplication on the cohomology theory might be lifted to a strictly commutative multiplication on the spectrum level
- This provides extra power operation structure
- Given $\alpha\colon X \to R$, we get a factorization

$$\begin{array}{ccccc}
X & \xrightarrow{\Delta} & X^{\wedge p} & & \\
\downarrow & & \downarrow & \searrow^{\alpha^{\wedge p}} & \\
X \times B\Sigma_p & \longrightarrow & X^{\wedge p}/\Sigma_p & \longrightarrow & R
\end{array}$$

Subgroups produce power operations

- Study of these power operations and formal group data initiated in Ando's thesis
- If R is strictly commutative and complex orientable, the power operations equip $\mathbb{G}_R$ with *quotient operations*
- Given a ring map $f\colon R^* \to S$ and a subgroup $H \subset f^*(\mathbb{G}_R)$ of the formal group, we get a new ring homomorphism $f_H\colon R^* \to S$ and a map $f^*(\mathbb{G}_R) \to (f_H)^*(\mathbb{G}_R)$ with kernel H
- In practice, if R^* parametrizes some type of object X with an attached formal group, then this means that there is a canonical way to take quotients of the formal group while producing new objects of type X

Situations of interest

- *MU* has a strictly commutative structure, and MU^* parametrizes formal group *laws*; Ando calculated how the power operations give a canonical formal group *law* on any quotient formal group
- BP^* parametrizes p-typical formal group laws; a quotient is isomorphic to a p-typical law, but there is no reason to expect compatibility with the canonical law on a quotient (*MU* and p-typical *BP* are incompatible — work of Noel-Johnson)
- Generalized $BP\langle n\rangle$ parametrizes formal group laws of a restricted "shape"; there is no intrinsic description, so we don't even know if these types of formal groups are closed under quotients!

Realizability for $BP\langle 2\rangle$

There are some positive results for $BP\langle 2\rangle$.

Theorem (L.-Naumann)

Let $\mathbb{G}$ be a formal group law over the ring $R^* = \mathbb{Z}_{(p)}[v_1, v_2]$ which might come from a generalized $BP\langle 2\rangle$. Then there is a strictly commutative ring spectrum R realizing this formal group law data if and only if the subring

$$\mathbb{Z}[v_1^{p+1}/v_2]_p^\wedge \subset \mathbb{Z}((v_2/v_1^{p+1}))_p^\wedge$$

is closed under a certain algebraic power operation θ on the right-hand side.
Any two such commutative objects are equivalent.

The proof is mainly $K(1)$-local obstruction theory along the lines of the "old" construction of *tmf*.

Elliptic curves with level 3 structure

Theorem (L.-Naumann)

There exists a strictly commutative generalized $BP\langle 2\rangle$ at the prime 2.

- To get this, consider the elliptic curve

$$y^2 + v_1 xy + v_2 y = x^3$$

  over $\mathbb{Z}_{(2)}[v_1, v_2]$, which parametrizes elliptic curves with a choice of 3-torsion point (after Rezk, Mahowald-Rezk)
- This elliptic curve produces a formal group law
- The universal property of this moduli object **forces** the existence of the power operation data

Realizing diagrams over the Steenrod algebra

Theorem (L.-Naumann)

There exists a commutative diagram of strictly commutative ring spectra realizing a classical diagram of modules over the Steenrod algebra:

$$\begin{array}{ccc} tmf_{(2)} & \longrightarrow & ko_{(2)} \\ \downarrow & & \downarrow \\ BP\langle 2\rangle & \longrightarrow & ku_{(2)} \end{array} \qquad \begin{array}{ccc} \mathcal{A}^*/A(2) & \longleftarrow & \mathcal{A}^*/A(1) \\ \uparrow & & \uparrow \\ \mathcal{A}^*/E(2) & \longleftarrow & \mathcal{A}^*/E(1) \end{array}$$

- The maps come from the interpretation in terms of moduli of elliptic curves
- Horizontal maps come from evaluating at a “ramified” cusp

Realization with a form of K-theory

- While we're at it, there's also an unramified cusp and we could use that instead to construct a similar diagram

$$\begin{array}{ccc} tmf_{(2)} & \longrightarrow & ko_{(2)} \\ \downarrow & & \downarrow \\ BP\langle 2\rangle & \longrightarrow & ku^{\tau}_{(2)} \end{array}$$

- Here ku^{τ} is the form of K-theory associated with the formal group law

$$x +_{\mathbb{G}} y = (x + y + 3xy)/(1 - 3xy)$$

- This becomes isomorphic to the multiplicative formal group after adjoining a third root of unity