Calculating obstruction groups for E_∞ ring spectra

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Abstract

We describe a special instance of the Goerss–Hopkins obstruction theory, due to Senger, for calculating the moduli of E_∞ ring spectra with given mod-p homology. In particular, for the 2-primary Brown–Peterson spectrum we give a chain complex that calculates the first obstruction groups, locate the first potential genuine obstructions, and discuss how some of the obstruction classes can be interpreted in terms of secondary operations.

1 Introduction

The mod-p homology of an E_∞ ring spectrum R comes equipped with operations called the (Araki–Kudo–)Dyer–Lashof operations. At the prime 2, these take the form of additive natural transformations

$$Q^s\colon\ H_n(R) \to H_{n+s}(R)$$

that satisfy a Cartan formula, have their own Adem relations, and interact in a concrete way with homology operations via the Nishida relations. These operations give extra structure which can be used to classify existing E_∞ rings and exclude certain phenomena. For example, Hu–Kriz–May used relations between these operations in the dual Steenrod algebra to show that the natural splitting $BP \to MU_{(p)}$ cannot be a map of E_∞ ring spectra [HKM01].

Unfortunately, these primary operations alone are not enough information to determine whether the Brown–Peterson spectrum BP admits the structure of an E_∞ ring, a long-standing problem in the field. However, more is available in H_*R: secondary operations that arise from relations between primary operations. For example, the Adem relation $Q^{2n+1}Q^n = 0$ gives rise to a secondary operation that increases degree by $3n + 2$, but it is only defined on $\ker(Q^n)$ and its value is only well-defined mod the image of Q^{2n+1}. This extra data provides more information that can enhance our understanding of the objects that we have and the objects that we cannot.

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It it easiest to describe the obstructions we might obtain in a constructive fashion. The Goerss–Hopkins method attempts to construct BP or $BP\langle n\rangle$ by starting with the sphere $\mathbb{S}$ and then iteratively killing off maps from finite complexes Z. Let us examine one way that this procedure might fail:

1. We start with the sphere $\mathbb{S}$, whose homology is $\mathbb{F}_2$, and call it R_0.

2. The homology is not yet correct: it is missing ξ_1^2, which is connected to the unit by the Steenrod operation Sq^2. We attach an E_∞-cell in a way that produces it. In this case, we can do this by coning off the map $\eta\colon S^1 \to \mathbb{S}$, forming a pushout along a map of free E_∞ ring spectra. Specifically, we take the diagram $CS^1 \leftarrow S^1 \to \mathbb{S}$, use the free E_∞ ring spectrum functor $\mathbb{P}$ to build a diagram $\mathbb{P}(CS^1) \leftarrow \mathbb{P}(S^1) \to \mathbb{S}$, and form the pushout in E_∞ ring spectra to construct R_1.

3. It turns out that we are no longer missing any homology classes, but the homology is still not correct: there are relations that are not yet satisfied. There is a relation involving the Dyer–Lashof operation Q^6 (this relation turns out to be $Q^6(\xi_1^2) = \xi_1^8$ in H_8BP), and we know that $Q^n(\xi_1^2) = 0$ for n odd, and so on. We, again, can cone off finite complexes in a way that imposes these relations on homology. The element $Q^6(\xi_1^2) + \xi_1^8$ does not lift to the homotopy of R_1 (the Steenrod operation P_1 is nontrivial on it), so we can't simply cone off a map $S^8 \to R_1$. Instead, we need to map in a finite complex $Z \to R_1$ that hits this class in homology (and hence the Steenrod operations on it). We construct the pushout $\mathbb{P}(CZ) \leftarrow \mathbb{P}(Z) \to R_1$ to cone this map off (and repeat for the other relations that we must impose) to construct a new ring R_2.

4. We have correctly imposed relations that should hold now, but the homology is still not correct: new classes have appeared in homology when we imposed these relations. These come from "relations between relations" and are called secondary Dyer–Lashof operations. For example, there is an Adem relation saying that $Q^{22}Q^6(y) = Q^{17}Q^{11}(y) + Q^{15}Q^{13}(y)$ for elements $y \in H_2$, the Cartan formula implies that $Q^{22}(\xi_1^8) = 0$, and there are relations $Q^{11}(\xi_1^2) = Q^{13}(\xi_1^2) = 0$. These relations between relations glue together into a secondary operation θ that can be applied to ξ_1^2. This new element $\theta(\xi_1^2) \in H_{31}BP$ needs to be eliminated because $H_{31}BP$ is supposed to be zero. We map in a finite complex $W \to R_2$ to clamp down on this spurious new class $\theta(\xi_1^2)$ (and the Steenrod operations on it), coning it off to construct a new ring R_3.

5. Now we arrive at our potential problem. When we eliminated $\theta(\xi_1^2)$, we were forced to kill off the Steenrod operations on it, such as the action of the Milnor primitive M_4: the element $M_4\theta(\xi_1^2) \in H_0(R_2)$ must map to zero in $H_0(R_3)$. However, it's possible that $M_4\theta(\xi_1^2)$ is the unit 1, in which case we have been forced to make H_*R_3 into the zero ring.

This particular problem doesn't really happen. The reason is not particularly interesting: *all the relations we used in this argument also take place in H_*MU*, and we know that MU admits an E_∞ ring structure. This secondary operation we've written down can't actually satisfy $M_4\theta(\xi_1^2) = 1$ or this argument would equally well exclude the existence of the E_∞ ring structure on MU that we already know is there.

In [Law17], we used the calculation of a different secondary operation in the dual Steenrod algebra to show that the 2-primary Brown–Peterson spectrum BP does not admit the structure of an E_∞ ring. Just as above, this secondary operation exists because of a relation between primary operations. Unfortunately, the relation in question is much more complicated than a simple Adem relation like $Q^{2n+1}Q^n = 0$.

Proposition 1.1 ([Law17, 5.4.1])**.** *Suppose that A is an E_∞-algebra over the Eilenberg–Mac Lane spectrum $H\mathbb{F}_2$ and $x \in \pi_2(A)$. Define the following classes:*

$$\begin{aligned}
y_5 &= Q^3 x\\
y_7 &= Q^5 x\\
y_9 &= Q^7 x\\
y_{13} &= Q^{11} x\\
y_8 &= Q^6 x + x^4\\
y_{10} &= Q^8 x + x^2 Q^4 x\\
y_{12} &= Q^{10} x + (Q^4 x)^2
\end{aligned}$$

Then there is a relation

$$\begin{aligned}
0 = Q^{20} y_{10} + Q^{18} y_{12} + Q^{17} y_{13} + x^4 (Q^{12} y_{10}) + y_9^2 (Q^4 x)^2 +\\
y_7^2 Q^9 Q^5 x + y_8^2 Q^8 Q^4 x + (Q^9 y_9)(Q^4 x)^2 + (Q^{10} y_8)(Q^4 x)^2 +\\
y_5^2 (Q^{11} Q^7 x + Q^{10} Q^8 x + x^4 Q^6 Q^4 x)
\end{aligned}$$

in $\pi_{30}(A)$.

In particular, if R is an E_∞ ring then $A = H\mathbb{F}_2 \wedge R$ is an E_∞-algebra over $H\mathbb{F}_2$, and for $x \in H_2(R)$ this gives a relation in $H_{30}R$ between Dyer–Lashof operations on x.

This relation gives rise to a secondary operation defined on a subset of $H_2(R)$ (those elements x for which the elements y_k described above all vanish) that takes values in a quotient of $H_{31}(R)$. A calculation of this secondary operation found that its value on the generator $\xi_1^2 \in H_2 H\mathbb{F}_2$ is unambiguously ξ_5 mod decomposables, which makes it impossible for the map $H_*BP \to H_*H\mathbb{F}_2$ to preserve secondary operations. This method has now been generalized by Senger to show that BP cannot have an E_∞ ring structure at odd primes [Sen17].

While this gives a rough description of the main result of [Law17], and the size of this relation provides some mild amusement value, most people who see this relation are inclined to ask *where on earth it came from*. There are no obvious indications why we should focus attention on this particular secondary operation. There are certainly many simpler ones—including ones based solely on the Adem relations—that can be applied to elements in H_*BP, and we could attempt to obtain simpler obstructions using those. However, the author's experience has been that most more straightforward attempts either fail to work or are difficult to calculate. Most of the blame for this can be directed at MU: the subring H_*BP of the dual Steenrod algebra is the same as the image of H_*MU. If our goal is to show that H_*BP cannot be closed under secondary operations, then we have to find a secondary operation built out of a relation that holds in H_*BP *but not in* H_*MU. The first such relation appears in the list above: it is the relation $0 = Q^8(\xi_1^2) + \xi_1^4 Q^4(\xi_1^2)$ that proves Hu–Kriz–May's nonsplitting result at the prime 2. The nonlinear nature of this relation forces much of the rest of the mess.

The goal of this paper is to describe the obstruction-theoretic method we used, built to find secondary operations that might genuinely produce a problem. Our method is based on an attempt to repair Kriz's paper [Kri95], which circulated quietly in preprint form but was not published: it assumed an interaction between Steenrod operations and certain operations in topological André–Quillen cohomology that could not be verified. Trying at length to understand the exact interaction between these operations, and how the obstruction theory could be enhanced to one that took this interaction into account, led to the direction we discuss in this paper. It is difficult to emphasize adequately how much debt we owe to Kriz's work.

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2 Postnikov-based obstructions to commutativity

For context, we will begin by discussing a revisionist version of Kriz's technique based on Postnikov towers [Kri95, Bas99].

In rough, for a commutative (or E_∞) ring spectrum R and an R-module M, Basterra constructed topological André–Quillen cohomology groups

$$\mathrm{TAQ}^*(R, M),$$

by analogy with the André–Quillen cohomology groups of ordinary commutative rings [Qui70]. There is a restriction map

$$\mathrm{TAQ}^*(R, M) \to [R, \Sigma^* M]$$

from TAQ-cohomology with coefficients in M to ordinary cohomology with coefficients in M. Just as a connective spectrum X has a Postnikov tower $\{\cdots \to P_2X \to P_1X \to P_0X\}$ of spectra which is determined by k-invariants

$$k_n \in [P_nX, \Sigma^{n+2} H\pi_{n+1}X],$$

Basterra showed that a connective E_∞ ring spectrum R has a Postnikov tower $\{\cdots \to P_2R \to P_1R \to P_0R\}$ of E_∞ ring spect4ra which is determined by k-invariants

$$\tilde{k}_n \in \mathrm{TAQ}^{n+2}(P_nR, H\pi_{n+1}R)$$

that restrict to the k-invariants of the underlying spectrum. Thus, if we have the ability to show that all the k-invariants of a homotopy ring spectrum R lift from cohomology to TAQ-cohomology, we can lift R from a spectrum to an E_∞ ring spectrum.

Of course, this method is not particularly useful if we can't calculate anything about TAQ. Kriz also developed a spectral sequence for calculating TAQ-cohomology, in particular with coefficients in $\mathbb{F}_p$, based on the Miller spectral sequence [Mil78]. Kriz's spectral sequence can be interpreted as being of the form

$$\mathrm{Der}^s_{\mathcal{A}}(H_*R, \Omega^t\mathbb{F}_p) \Rightarrow \mathrm{TAQ}^{s-t}(R, H\mathbb{F}_p),$$

where the groups $\mathrm{Der}^s_{\mathcal{A}}$ are André–Quillen cohomology groups in a certain category $\mathcal{A}$ of simplicial graded-commutative rings equipped with Dyer–Lashof operations. From knowledge of $\mathrm{TAQ}^*(R, H\mathbb{F}_p)$, one can deduce $\mathrm{TAQ}^*(R, HM)$ for a $\pi_0(R)$-module M using Bockstein spectral sequences.

One of the main difficulties with the Postnikov-based technique is essentially the same as the difficulty that occurs when using Serre's Postnikov-based technique for computing homotopy groups of spheres. To use the Postnikov technique, you use the coaction of the dual Steenrod algebra A_* on H_*R to compute the homotopy groups π_*R, and then use the Dyer–Lashof operations on H_*R to compute the TAQ-cohomology groups $\mathrm{TAQ}^*(R, \pi_*R)$ where k-invariants live. This division into homotopy and homology uses knowledge of the Steenrod coaction and Dyer–Lashof action on the homology groups separately, but it does not track the fact that there are strong and concrete relationships between them.

3 Background: Goerss–Hopkins obstruction theory

In [GH], Goerss and Hopkins developed a very general obstruction theory for the construction of algebras over operads. In this section, we will give a brief discussion of their method.

The Goerss–Hopkins obstruction theory requires two main ingredients.

- We need a multiplicative homology theory E_*, represented by a ring spectrum E.
- We need a simplicial operad $\mathcal{O}_\bullet$.

These are required to satisfy two main constraints: an Adams–Atiyah condition [GH, 1.4.1], and a "homotopically adapted" condition [GH, 1.4.16]. Some of the main consequences are the following.

- The ring E_*E is flat over E_*. This ensures that the pair (E_*, E_*E) form a Hopf algebroid and that E_*X naturally takes values in its category of comodules.
- There is a large library of "basic cells" $\{X_\alpha\}$ with X_α finite and E_*X_α projective over E_*. This is large enough so that for any spectrum Y, there are enough maps $X_\alpha \to Y$ so that the images of the maps $E_*X_\alpha \to E_*Y$ are jointly surjective.

- If we take a simplicial $\mathcal{O}_\bullet$-algebra and apply E-homology, the resulting simplicial E_*E-comodule lands in some algebraic category $\mathcal{A}$ of simplicial E_*E-comodules with extra structure.[1]
- If we take a set of basic cells X_α and form the free simplicial $\mathcal{O}_\bullet$-algebra on them, on E-homology we must get the free object on E_*X_α in our algebraic category $\mathcal{A}$.

With these ingredients, they built an obstruction theory based on Dwyer–Kan–Stover's resolution model categories [DKS93, Bou03]. This asks whether we can construct an algebra over the geometric realization $|\mathcal{O}_\bullet|$ whose E-homology is a prescribed algebra B_* over $\pi_0(E_*\mathcal{O}_\bullet)$, and produces obstructions to it. These live in obstruction groups: Ext groups calculated in the homotopical category $\mathcal{A}$. In rough, one tries to build a simplicial $\mathcal{O}_\bullet$-algebra $X_\bullet$, one degree at a time, so that the chain complex $E_*(X_\bullet)$ is a resolution of B_*. The Ext-groups in question involve coefficients, and part of determining the algebra in this obstruction theory is determining what type of object Ext takes coefficients in.

Here are some specializations of the Goerss–Hopkins obstruction theory.

1. We could let $\mathcal{O}_\bullet$ be the trivial operad, and $E = \mathbb{S}$ so that $E_*(X) = \pi_*(X)$. A simplicial $\mathcal{O}_\bullet$-algebra is simply a simplicial spectrum, our "basic cells" are the shifts $\Sigma^t\mathbb{S}$, and the algebraic category $\mathcal{A}$ is the category of simplicial modules over the stable homotopy groups of spheres $\pi_*\mathbb{S}$—which is equivalent, via the Dold–Kan correspondence, to chain complexes of $\pi_*\mathbb{S}$-modules. The Ext-groups here are classical Ext-groups in $\pi_*\mathbb{S}$-modules. Given a $\pi_*\mathbb{S}$-module M_*, the Goerss–Hopkins obstruction theory for the existence of a spectrum X with $\pi_*X \cong M_*$ takes place in the groups

   $$\mathrm{Ext}^s_{\pi_*\mathbb{S}}(M_*, \Omega^t M_*).$$

   Specifically, obstructions to existence occur when $t - s = -2$, and to uniqueness occur when $t - s = -1$.

   The first obstruction occurs in $\mathrm{Ext}^3_{\pi_*\mathbb{S}}(M_*, \Omega M_*)$, and we can be relatively explicit about it. Start with a free resolution

   $$0 \leftarrow M_* \leftarrow \oplus\Sigma^{m_i}\pi_*\mathbb{S} \xleftarrow{d_1} \oplus\Sigma^{n_j}\pi_*\mathbb{S} \leftarrow \dots$$

   and lift d_1 to a map of spectra $\bigvee \Sigma^{m_i}\mathbb{S} \leftarrow \bigvee \Sigma^{n_j}\mathbb{S}$, with cofiber $X^{(1)}$. This cofiber $X^{(1)}$ is our first approximation, and there is an exact sequence

   $$0 \to M_* \to \pi_*X^{(1)} \to \Sigma\ker(d_1) \to 0.$$

   If $X^{(1)}$ is going to eventually map to a spectrum X so that $M_* \subset \pi_*X^{(1)}$ maps isomorphically to π_*X, this short exact sequence must be split over $\pi_*\mathbb{S}$. Thus, a first obstruction takes place in the group

   $$\mathrm{Ext}^1_{\pi_*\mathbb{S}}(\Sigma\ker(d_1), M_*) \cong \mathrm{Ext}^1_{\pi_*\mathbb{S}}(\ker(d_1), \Omega M_*).$$

[1] Namely, $\mathcal{A}$ is a category of algebras over some simplicial monad.

Applying the isomorphisms on Ext induced by the exact sequences

$$0 \to \ker(d_1) \to \oplus\Sigma^{n_j}\pi_*\mathbb{S} \to \mathrm{Im}(d_1) \to 0 \text{ and}$$
$$0 \to \mathrm{Im}(d_1) \to \oplus\Sigma^{m_i}\pi_*\mathbb{S} \to M_* \to 0$$

identifies this with an obstruction in $\mathrm{Ext}^3_{\pi_*\mathbb{S}}(M_*, \Omega M_*)$.

If the sequence is split, then we use the splitting to construct a map $\bigvee \Sigma^{p_k+1}\mathbb{S} \to X^{(1)}$ whose image in homotopy groups is the complementary summand, take the cofiber $X^{(2)}$, and again examine the resulting homotopy groups to get a splitting obstruction in $\mathrm{Ext}^4_{\pi_*\mathbb{S}}(M_*, \Omega^2 M_*)$. This process then continues.[2]

2. Again letting $\mathcal{O}_\bullet$ be the trivial operad, we could let E be the Eilenberg–Mac Lane spectrum $H = H\mathbb{F}_p$. A simplicial $\mathcal{O}_\bullet$-algebra is a simplicial spectrum, our "basic cells" are all the finite spectra, and the algebraic category $\mathcal{A}$ is the category of simplicial comodules over the mod-p dual Steenrod algebra A_*—which is equivalent to chain complexes of A_*-comodules. The Ext-groups here are classical Ext-groups in A_*-comodules. Given an A_*-comodule M_*, the Goerss–Hopkins obstruction theory for the existence of a spectrum X with $H_*X \cong M_*$ as an A_*-comodule takes place in the groups

$$\mathrm{Ext}^s_{A_*}(M_*, \Omega^t M_*).$$

Specifically, obstructions to existence occur when $t - s = -2$, and obstructions to uniqueness occur when $t - s = -1$.

A handy interpretation for uniqueness is as follows: given two such spectra X and Y, these Ext groups form the (-1)-line of the Adams spectral sequence for calculating maps $X \to Y$. If X and Y are inequivalent, then any isomorphism $\phi\colon H_*X \to H_*Y$ doesn't lift to a spectrum map, and so the corresponding element $\phi \in \mathrm{Hom}_{A_*}(H_*X, H_*Y)$ on the 0-line must support a differential that hits an element on the -1-line. Similar techniques are commonly used in the calculation of Picard groups [HMS94].

3. Complementary to this, we could let $\mathcal{O}_\bullet$ be a nontrivial operad: the associative operad (viewed as a constant simplicial operad), but return to choosing $E = \mathbb{S}$. Then a simplicial $\mathcal{O}_\bullet$-algebra is a simplicial object in associative ring spectra, our basic cells are shifts of the sphere, and the algebraic category $\mathcal{A}$ is the category of simplicial algebras over $\pi_*\mathbb{S}$.[3] The Ext-groups of a $\pi_*\mathbb{S}$ algebra B_* here have coefficients in a B_*-bimodule. Given a $\pi_*\mathbb{S}$-algebra B_*, the Goerss–Hopkins obstruction theory for the existence of an associative ring spectrum R with $\pi_*R \cong B_*$ (as rings) takes place in certain groups

$$\mathrm{Der}^s_{\mathrm{assoc}}(B_*, \Omega^t B_*).$$

[2] We note that there is absolutely nothing special about the stable homotopy category here, and we could carry this procedure out in the category of modules over an associative ring spectrum, the category of modules over a differential graded algebra, and many other triangulated categories that have generators.

[3] Proving that this is homotopically adapted is now more work, and fails if we try to replace "associative" with "commutative" because the homotopy groups of the free commutative ring spectrum on X are a more confusing functor of π_*X.

Again, obstructions to existence occur when $t - s = -2$, and obstructions to uniqueness occur when $t - s = -1$. These are often called André–Quillen cohomology groups of an associative algebra with coefficients in a bimodule [Qui70], and are the nonabelian derived functors of derivations. They are also closely related to Hochschild cohomology: there is an exact sequence

$$0 \to HH^0(B_*, M_*) \to M_* \to \mathrm{Der}^0(B_*, M_*) \to HH^1(B_*, M_*) \to 0$$

that identifies the first two Hochschild cohomology groups with central elements and derivations modulo principal derivations, and there are isomorphisms

$$\mathrm{Der}^s(B_*, M_*) \to HH^{s+1}(B_*, M_*)$$

for $s > 0$.[4]

4. We could mix these procedures, getting an obstruction theory for associative ring spectra based on mod-p homology that lives in André–Quillen cohomology groups—for algebras in the category of A_*-comodules.

5. We should mention, at least in passing, the possibility of using a nonconstant operad $\mathcal{O}_\bullet$—for example, $\mathcal{O}_\bullet$ could be a simplicial resolution of the commutative operad. In the commutative case this tends to lead to an obstruction theory closely related to Robinson's obstruction theory, whose obstruction groups are Γ-cohomology groups [Rob03]. These obstruction groups have been examined in detail for BP in [Ric06], and do not take the Dyer–Lashof operations into account. Part of the goal of this paper is to develop an obstruction theory which does.

Remark 3.1. The reader might wonder why we even bother to mention cohomology groups other than those containing obstructions. It is worth pointing out that these groups do more: the groups

$$\mathrm{Ext}^s(E_*X, \Omega^t E_*Y)$$

serve as a tool for calculating the homotopy groups $\pi_{t-s}\,\mathrm{Map}(X, Y)$ for spaces of maps between two realizations [Bou03]. In the above discussion, these specialize to things such as the universal coefficient spectral sequence and the Adams spectral sequence.

4 Homology-based obstructions to commutativity

In this section we will discuss a specialization of the Goerss–Hopkins obstruction theory developed by Senger [Sen], whose full writeup is still forthcoming. Just as Serre's method is improved to Adams' by switching from a technique that proceeds one homotopy group at a time to one that uses all the cohomology information simultaneously, the Postnikov-based obstruction theory is sometimes improved by the Goerss–Hopkins method that can use both the Dyer–Lashof and Steenrod information simultaneously.

[4] Again, there is nothing special about $\mathbb{S}$ here, and we could apply this to create an obstruction theory for algebras over a commutative ring spectrum R or differential graded algebras over a commutative ring.

To set up this obstruction theory, we need a simplicial operad $\mathcal{O}_\bullet$ (which we choose to be a constant E_∞-operad) and a homology theory (which we choose to be mod-p homology H_*). A simplicial $\mathcal{O}_\bullet$-algebra is then a simplicial E_∞ ring spectrum, and our "basic cells" are free algebras on finite spectra. Mod-p homology satisfies the Adams–Atiyah condition, and the fact that this operad is homotopically adapted amounts to the following theorems.

Theorem 4.1 ([BMMS86, §III.1]). *For an E_∞ ring spectrum R, the mod-p homology H_*R has the following structure.*

1. *It is a comodule over the mod-p dual Steenrod algebra.*
2. *It is a graded-commutative ring.*
3. *It has Dyer–Lashof operations that satisfy the Cartan formula, Adem relations, and instability relations.*
4. *It satisfies the Nishida relations.*

Following the literature, we will refer to such algebras as AR-algebras.[5]

Theorem 4.2 ([BMMS86, §IX.2]). *The following results about AR-algebras hold:*

1. *The forgetful functor from AR-algebras to graded comodules has a left adjoint $\mathbb{Q}$.*
2. *If a comodule M has a basis over $\mathbb{F}_p$ of elements e_i in degrees n_i, then $\mathbb{Q}(M)$ is a free graded-commutative algebra on the elements $P(e_i)$ such that P is an admissible monomial in the Dyer–Lashof algebra of excess at least n_i.*
3. *If we write $\mathbb{P}(X)$ for the free E_∞ ring spectrum on X, then the homology $H_*\mathbb{P}(X)$ is a free AR-algebra: the natural map $H_*X \to H_*\mathbb{P}(X)$ induces a natural isomorphism*

$$\mathbb{Q}(H_*X) \to H_*\mathbb{P}(X).$$

As a result, the mod-p homology of simplicial E_∞ ring spectra takes place in the category of simplicial AR-algebras. The Ext-groups of an AR-algebra B_* here have coefficients in an AR-B_*-module: a B_*-module in A_*-comodules, with compatible Dyer–Lashof operations Q^s such that $Q^s(x) = 0$ for $s \leq |x|$.

Theorem 4.3 ([Sen]). *Given B_*, an AR-algebra, there are Goerss–Hopkins obstruction groups*

$$\mathrm{Der}^s_{AR}(B_*, \Omega^t B_*)$$

calculated in the category of simplicial AR-algebras. The groups with $t - s = -2$ contain an iterative sequence of obstructions to realizing B_* by an E_∞ ring spectrum R such that $H_*R \cong B_*$, and the groups with $t - s = -1$ contain obstructions to uniqueness.

From this point forward, it will be our goal to give methods to calculate the nonabelian Ext-groups Der^s_{AR} in specific cases and to interpret elements in them as concrete obstructions.

[5] Because the origin of this subject is in studying the homology of infinite loop spaces rather than the homology of E_∞ ring spectra, the definitions of AR-algebras in the literature often involve connectivity assumptions and only discuss Dyer–Lashof operations of nonnegative degree.

5 Tools for calculation

In this section, we will begin discussing how [Sen] reduces the Goerss–Hopkins obstruction theory for the Brown–Peterson spectrum BP and its truncated versions $BP\langle n\rangle$ to more straightforward calculations.[6] We need to calculate the obstruction groups

$$\mathrm{Der}^s_{AR}(H_*BP, \Omega^t H_*BP).^7$$

We begin by recalling the structure of H_*BP as a comodule over the dual Steenrod algebra. The dual Steenrod algebra A_* has quotient Hopf algebras, given by exterior algebras

$$E(n)_* = \Lambda[\tau_0, \tau_1, \ldots, \tau_n]$$

with $|\tau_i| = 2p^i - 1$. (At the prime 2, τ_i is the image of ξ_i.) When $n = \infty$, we get a Hopf algebra E_*. The homology $H_*BP\langle n\rangle$ can be identified with a coextended comodule:

$$H_*BP\langle n\rangle \cong A_* \square_{E(n)_*} \mathbb{F}_p.$$

This coextension functor is right adjoint to the forgetful functor from A_*-comodules to $E(n)_*$-comodules. To proceed, we need to know that this adjunction is compatible with Dyer–Lashof operations.

Proposition 5.1 ([Bak15, 7.3])**.** *Let $p = 2$, and let M_r be the Milnor primitive of degree $2^r - 1$ dual to ξ_r in the Milnor basis of the dual Steenrod algebra. Then the Nishida relations imply*

$$M_r Q^s = (s+1) Q^{s-2^r+1} + \sum_{0 \le k < r} Q^{s-2^r+2^k} M_k. \tag{5.1}$$

The existence of this formula means that it is possible to define a category of $E(n)_*$-comodule algebras or E_*-comodule algebras with Dyer–Lashof operations, which we will refer to as $E(n)R$-algebras or ER-algebras respectively. These are compatible across n.

Proposition 5.2. *The forgetful functors from A_*-comodules to E_*-comodules and $E(n)_*$-comodules lift to ones from AR-algebras to ER-algebras and $E(n)R$-algebras. These functors have exact right adjoints, given by $A_* \square_{E(n)_*}(-)$ and $A \square_{E_*}(-)$ on the underlying comodules.*

Remark 5.3. There are analogous equations to (5.1) at odd primes:

$$\begin{aligned} M_r Q^s &= \pm \beta Q^{s-2p^r+1} \pm \sum_{0 \le k < r} Q^{s-2p^r+2p^k} M_k, \\ M_r \beta Q^s &= \pm \sum_{0 \le k < r} \beta Q^{s-2p^r+2p^k} M_k, \end{aligned}$$

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[6] Because these calculations only depend on mod-p homology, they apply to the generalized $BP\langle n\rangle$ as defined in [LN12, §3].

[7] From here forward, rather than using the shift operator Ω^t we will often regard the obstruction groups as bigraded. The number s is the *filtration* degree, and the number $t - s$ is the *total* degree.

We have not verified the signs ourselves and so are not comfortable stating them here. We are unable to locate this result in the literature but have been assured that both it and Proposition 5.1 have been known for some time.

Due to exactness of both functors in this free-forgetful adjunction, the adjunction carries forward to an isomorphism of Ext-groups.

Proposition 5.4. *For any AR-algebra R with an augmentation $R \to A_* \square_{E_*} \mathbb{F}_p$, there is an isomorphism*

$$\mathrm{Der}^s_{AR}(R, A_* \square_{E_*} \mathbb{F}_p) \cong \mathrm{Der}^s_{ER}(R, \mathbb{F}_p).$$

Similarly, for any AR-algebra R with an augmentation $R \to A_* \square_{E(n)_*} \mathbb{F}_p$, there is an isomorphism

$$\mathrm{Der}^s_{AR}(R, A_* \square_{E(n)_*} \mathbb{F}_p) \cong \mathrm{Der}^s_{E(n)R}(R, \mathbb{F}_p).$$

Remark 5.5. It's worth noting that there are two ER-module (or $E(n)R$-module) structures on $\mathbb{F}_p$: one has Q^0 acting as the identity, and the other has Q^0 acting trivially. The group appearing above is the former. Only the latter has *deloopings*: the shifts $\Sigma^t \mathbb{F}_p$ for $t \geq 0$ must have $Q^0 = 0$ to satisfy the instability relations. Therefore, the obstruction groups $\mathrm{Der}^s(R, \Omega^t A_s \square_{E_*} \mathbb{F}_p)$ do not, a priori, extend to an integer grading in t. We will later show that, in the range of interest, both modules have the same cohomology, and the added integer grading is very useful in assembling a systematic calculation.

The groups $\mathrm{Der}^s_{E(n)R}(R, \mathbb{F}_p)$ can be calculated as follows: resolve R by a simplicial $E(n)R$-algebra which is free in each simplicial degree on some free $E(n)_*$-comodule,[8] take indecomposables in each simplicial degree, apply $\mathrm{Hom}_{E(n)R\text{-mod}}(-, \mathbb{F}_p)$ in the category of $E(n)_*$-*comodules* with Dyer–Lashof operations to get a cosimplicial abelian group, and take the cohomology groups of the associated cochain complex. This description as a composite allows us to get a Grothendieck spectral sequence.

Proposition 5.6. *For any $E(n)R$-algebra R with an augmentation $R \to \mathbb{F}_p$, there is a spectral sequence*

$$\mathrm{Ext}^p_{E(n)R\text{-mod}}(AQ_q(R), \mathbb{F}_p) \Rightarrow \mathrm{Der}^{p+q}_{E(n)R}(R, \mathbb{F}_p),$$

where $AQ_*(R)$ are the ordinary André–Quillen homology groups of R (the nonabelian derived functors of the indecomposables functor Q). In particular, if the underlying algebra of R is a free graded-commutative algebra, this degenerates to an isomorphism

$$\mathrm{Ext}^s_{E(n)R\text{-mod}}(QR, \mathbb{F}_p) \cong \mathrm{Der}^s_{E(n)R}(R, \mathbb{F}_p).$$

In particular, these propositions can be applied to the $BP\langle n \rangle$.

Corollary 5.7. *For any $n \geq m \geq -1$, there is an isomorphism*

$$\mathrm{Der}^s_{AR}(H_*BP\langle n \rangle, H_*BP\langle m \rangle) \cong \mathrm{Ext}^s_{E(m)R\text{-mod}}(QH_*BP\langle n \rangle, \mathbb{F}_p).$$

[8] As we will see shortly, because $E(n)_*$ is finite-dimensional its comodules are equivalent to $E(n)^*$-modules, and so there is an ample supply of free comodules. This is more problematic for A_*-comodules or E_*-comodules.

For later calculations, it will be helpful to know the primitives in H_*BP.

Proposition 5.8. *The ER-module QH_*BP has a basis consisting of the elements $[\xi_i^2]$ in degree $2^{i+1}-2$ at the prime 2, and $[\xi_i]$ in degree $2p^i-2$ at odd primes. These are acted on trivially by the Milnor primitives. The Dyer–Lashof operations satisfy $Q^{2^j-2^i}[\xi_i^2] = [\xi_j^2]$ for all $j > i$ at the prime 2.*

Proof. The identification of H_*BP as $\mathbb{F}_2[\xi_1^2, \xi_2^2, \ldots]$ or $\mathbb{F}_p[\xi_1, \xi_2, \ldots]$ makes the identification of the indecomposables clear. Since the generating classes are all in even degrees and the Milnor primitives are of odd degree, the Milnor primitives must act trivially. The remaining formula is a consequence of the calculations of Steinberger [BMMS86, §III]. □

6 Koszul duality

Now that we have reduced to calculations of Ext groups in a category $E(n)R$-mod of graded modules, we should examine the structure on these modules: it is now much simpler. Because $E(n)_*$ is finite-dimensional, an $E(n)_*$-comodule structure is precisely the same as an $E(n)^*$-module structure—an action of the exterior algebra $\Lambda[M_0, \ldots, M_n]$ generated by the Milnor primitives. Thus, an $E(n)R$-module N is almost the same as a module over a graded ring: the graded ring with generators M_k and Q^s subject to quadratic relations.

We now begin systematically specializing to the prime 2, where these relations take the following form:

$$\begin{aligned}
Q^r Q^s &= \sum \binom{i-s-1}{2i-r} Q^{r+s-i} Q^i && \text{if } r > 2s,\\
M_r M_s &= M_s M_r && \text{if } r > s,\\
M_r^2 &= 0,\\
M_r Q^s &= (s-1) Q^{s-2^r+1} + \sum_{0 \le k < r} Q^{s-2^r+2^k} M_k.
\end{aligned}$$

However, $E(n)R$-modules satisfy one further *instability relation*:

$$Q^s x = 0 \text{ if } s \le |x|.$$

The relations above mean that the operators on $E(n)R$-modules have a canonical basis of monomials of the form $M_0^{\epsilon_0} M_1^{\epsilon_1} \ldots M_n^{\epsilon_n} Q^{a_1} \ldots Q^{a_m}$, where $a_i \le 2a_{i+1}$ and $\epsilon_i \in \{0, 1\}$. If $E(n)R$-modules were actually modules over a graded ring, this graded ring would be a quadratic algebra and this basis would be a PBW-basis in the sense of [Pri70]. Under these circumstances, there would be a Koszul complex calculating Ext, based on a “Koszul dual” quadratic algebra with differential.

Despite the fact that we are not quite in the case of a quadratic algebra, Senger showed that Priddy’s proof still works. (This technique was originally carried out for the nonnegative-degree Dyer–Lashof algebra by Miller [Mil78].)

Proposition 6.1 ([Sen]). *For N an $E(n)R$-module, there is a Koszul complex $C^*(N)$ calculating $\mathrm{Ext}_{E(n)R\text{-mod}}(N, \mathbb{F}_p)$. Let N have basis $\{y^i\}$ with dual basis $\{y_i\}$. The Koszul complex $C^*(N)$ has a basis consisting of monomials*

$$\lambda v_0^{k_0} v_1^{k_1} \dots v_n^{k_n} R^{a_1} R^{a_2} \dots R^{a_m} y^i$$

where $k_i \geq 0$, $a_i \geq 2a_{i+1}$, and $-|y^i| + 2 \leq a_m$. We refer to such monomials as admissible. *Here the operators v_i of total degree $2^{i+1} - 2$ are dual to M_i, R^a of total degree $-a$ is dual to Q^{a-1}, and λ of total degree 0 is dual to the unit of $\mathbb{F}_2$. These are subject to the following relations:*

$$\begin{aligned} R^a R^b &= \sum \binom{b-1-c}{a-2c} R^{a+b-c} R^c \ \textit{if } a < 2b \\ R^a v_i &= \sum_{i<k\leq n} v_k R^{a-2^i+2^k} \\ v_i v_j &= v_j v_i \end{aligned}$$

The operators R^a are also subject to the instability constraint*: the operator R^a can only be applied to an element z if $-|z| < a + 1$.*

The differential in the Koszul complex is determined by relations

$$\begin{aligned} dR^a(x) &= (a+1) \sum_{0\leq k\leq n} v_k R^{a+2^k-1}(x) + R^a(dx) \ \textit{and} \\ dv_i x &= v_i dx, \\ d\lambda x &= \begin{cases} \lambda R^1 x + \lambda(dx) & \textit{if } |x| > 0, \\ \lambda(dx) & \textit{if } |x| \leq 0, \end{cases} \end{aligned}$$

and the fact that the differential on the y^i is dual to the action of the Q^s and M_k on N.

The element λ, with its differential, appears precisely due to the fact that Q^0 acts by the identity on the coefficient group $\mathbb{F}_p$. However, we have the following simplifying result.

Theorem 6.2. *For N an $E(n)R$-module, the groups $\mathrm{Der}^s(N; \mathbb{F}_p)$ agree in total degree $t - s < 0$ for the two different actions of Q^0 on $\mathbb{F}_p$.*

For the zero action on $\mathbb{F}_p$, there is a Koszul complex $C^*(N)$ calculating $\mathrm{Ext}_{E(n)R\text{-mod}}(N, \mathbb{F}_p)$. Let N have basis $\{y^i\}$ with dual basis $\{y_i\}$. The Koszul complex $C^*(N)$ has a basis consisting of monomials

$$v_0^{k_0} v_1^{k_1} \dots v_n^{k_n} R^{a_1} R^{a_2} \dots R^{a_m} y^i$$

where $k_i \geq 0$, $a_i \geq 2a_{i+1}$, and $-|y^i| + 2 \leq a_m$. We refer to such monomials as admissible.

These are subject to the following relations:

$$\begin{aligned} R^a R^b &= \sum \binom{b-1-c}{a-2c} R^{a+b-c} R^c \ \textit{if } a < 2b \\ R^a v_i &= \sum_{i<k\leq n} v_k R^{a-2^i+2^k} \\ v_i v_j &= v_j v_i \end{aligned}$$

The operators R^a are also subject to the instability constraint*: the operator R^a can only be applied to an element z if $-|z| < a+1$.*

The differential in the Koszul complex is determined by relations

$$
\begin{aligned}
dR^a(x) &= (a+1)\sum_{0\le k\le n} v_k R^{a+2^k-1}(x) + R^a(dx) \text{ and} \\
dv_i x &= v_i dx
\end{aligned}
$$

and the fact that the differential on the y^i is dual to the action of the Q^s and M_k on N.

Proof. This follows because λ preserves the differential except on classes in total degree $t-s>0$, and hence does not alter the Goerss–Hopkins obstruction groups in question. □

This gives us a large but *explicit* cochain complex that calculates our Goerss–Hopkins obstruction groups for the realization of $BP\langle n\rangle$. The groups where potential obstructions live are in total grading -2.

7 Filtrations and stability

The Koszul complex calculating these obstruction groups for $BP\langle n\rangle$ is relatively large and involves complicated interaction between the R^a and v_i. In addition, the obstruction groups that we described in the previous section depend on n very strongly. This is part and parcel of how we're working: we're using the Adams filtration rather than the Postnikov filtration. For any two different values of n, the Adams towers for $BP\langle n\rangle$ are quite different as soon as one reaches filtration 1, and so our multiplicative obstruction theory doesn't really stabilize as n grows.

Fortunately, both of these problems can be addressed to some degree. There are several natural filtrations on this complex obtained by assigning degrees to the v_i, and this allows us to calculate by an inductive method.

Proposition 7.1. *For any $0\le k\le n$ and any $E(n)R$-module N, the Koszul complex has quotient complexes*

$$D^k = C^*(N)/(v_{k+1},\ldots,v_n)$$

and for $0\le k<n$ there are Bockstein spectral sequences

$$H^*(D^k)\otimes \mathbb{F}_p[v_{k+1},\ldots,v_m] \Rightarrow H^*(D^m).$$

Proof. The quotient complex exists because the v_i preserve the differential. We obtain the Bockstein spectral sequence in question from a filtration on the Koszul complex, obtained by giving the elements $v_{k+1}\ldots v_n$ grading 1 and $v_0\ldots v_k$, R^a, and y_m grading zero. The associated graded is $D^k\otimes\mathbb{F}_p[v_{k+1},\ldots,v_n]$ with differential induces by that on D^k, giving the E_1-term in question. □

In terms of Ext-groups, these have concrete interpretations. For any $k\le n$ and any $E(n)R$-module N, we can view N as an $E(k)R$-module, and D^k is a Koszul complex calculating Ext in $E(k)R$-modules. Thus, this can be viewed as a collection of Bockstein spectral sequences

$$\mathrm{Ext}_{E(k)R\text{-mod}}(N, \mathbb{F}_p) \otimes \mathbb{F}_p[v_{k+1}, \ldots, v_m] \Rightarrow \mathrm{Ext}_{E(m)R\text{-mod}}(N, \mathbb{F}_p).$$

We now specialize to what happens when we consider indecomposables.

Consider the sequence of maps

$$H_*BP \to \cdots \to H_*BP\langle 2\rangle \to H_*BP\langle 1\rangle \to H_*BP\langle 0\rangle \to H_*BP\langle -1\rangle.$$

This creates an array of Ext-groups:

$$\begin{tikzcd}
\mathrm{Ext}_{AR}(H_*BP\langle -1\rangle, H_*BP\langle -1\rangle) \arrow[d] & & \\
\mathrm{Ext}_{AR}(H_*BP\langle 0\rangle, H_*BP\langle -1\rangle) \arrow[d] & \mathrm{Ext}_{AR}(H_*BP\langle 0\rangle, H_*BP\langle 0\rangle) \arrow[l] \arrow[d] & \\
\mathrm{Ext}_{AR}(H_*BP\langle 1\rangle, H_*BP\langle -1\rangle) \arrow[d] & \mathrm{Ext}_{AR}(H_*BP\langle 1\rangle, H_*BP\langle 0\rangle) \arrow[l] \arrow[d] & \mathrm{Ext}_{AR}(H_*BP\langle 1\rangle, H_*BP\langle 1\rangle) \arrow[l] \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

After our chain of isomorphisms, these can be re-identified:

$$\begin{tikzcd}
\mathrm{Ext}_{E(-1)R\text{-mod}}(QH_*BP\langle -1\rangle, \mathbb{F}_p) \arrow[d] & & \\
\mathrm{Ext}_{E(-1)R\text{-mod}}(QH_*BP\langle 0\rangle, \mathbb{F}_p) \arrow[d] & \mathrm{Ext}_{E(0)R\text{-mod}}(QH_*BP\langle 0\rangle, \mathbb{F}_p) \arrow[l] \arrow[d] & \\
\mathrm{Ext}_{E(-1)R\text{-mod}}(QH_*BP\langle 1\rangle, \mathbb{F}_p) \arrow[d] & \mathrm{Ext}_{E(0)R\text{-mod}}(QH_*BP\langle 1\rangle, \mathbb{F}_p) \arrow[l] \arrow[d] & \mathrm{Ext}_{E(1)R\text{-mod}}(QH_*BP\langle 1\rangle, \mathbb{F}_p) \arrow[l] \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

Each Bockstein spectral sequence requires one of the terms in this diagram and converges to the one immediately to its right. Because the degrees of the elements in the Koszul complex in any column involve only a finite list $v_0, \ldots, v_n$ of positive-degree operators, and $QH_*BP\langle n\rangle \to QH_*BP\langle m\rangle$ is always an isomorphism in large degrees, the vertical towers *do* stabilize to the tower of groups

$$\mathrm{Ext}_{E(-1)R\text{-mod}}(QH_*BP, \mathbb{F}_p) \leftarrow \mathrm{Ext}_{E(0)R\text{-mod}}(QH_*BP, \mathbb{F}_p) \leftarrow \mathrm{Ext}_{E(1)R\text{-mod}}(QH_*BP, \mathbb{F}_p) \leftarrow \ldots,$$

or equivalently the tower

$$\mathrm{Ext}_{AR}(H_*BP, H_*BP\langle -1\rangle) \leftarrow \mathrm{Ext}_{AR}(H_*BP, H_*BP\langle 0\rangle) \leftarrow \mathrm{Ext}_{AR}(H_*BP, H_*BP\langle 1\rangle) \leftarrow \ldots$$

whose limit is roughly $\mathrm{Ext}_{AR}(H_*BP, H_*BP)$.[9]

Since we're interested in BP anyway, computing in this grid gives us a workaround for problems that don't show up for $E(n)R$-algebras, with the Koszul complex and with convergence, when using ER-algebras. In practice this means that we will be calculating

$$\mathrm{Ext}_{AR}(H_*BP, H_*BP\langle n\rangle) \cong \mathrm{Ext}_{E(n)R\text{-mod}}(QH_*BP, \mathbb{F}_p).$$

and calculating Bockstein spectral sequences

$$\mathrm{Ext}_{AR}(H_*BP, H_*BP\langle n\rangle) \otimes \mathbb{F}_p[v_{n+1}] \Rightarrow \mathrm{Ext}_{AR}(H_*BP, H_*BP\langle n+1\rangle)$$

to get at the limiting value.

8 The critical group and secondary operations

We now consider the critical group: the first Goerss–Hopkins obstruction group

$$\mathrm{Ext}^3_{E(n)R\text{-mod}}(QH_*BP, \Omega\mathbb{F}_2)$$

that could support an obstruction class. The Koszul complex, in this degree, had a basis of those monomials of the form

$$v_iv_jv_ky_m, v_iv_jR^ay_m, v_iR^aR^by_m, \text{ and } R^aR^bR^cy_m$$

of total degree -2, where y_k is dual to $[\xi_k^2] \in QH_*BP$. Our calculations in the remainder of this paper will determine exactly what has survived to Ext and what has not. The first type supports a differential if $m > 1$ and is usually in degree greater than -2 if $m = 1$; the second type only survives if it is in odd total degree; the fourth type is always in a large negative degree. This leaves us only with the third type to carefully check.

Theorem 8.1. *The first obstruction group*

$$\mathrm{Ext}^3_{E(n)R\text{-mod}}(QH_*BP, \Omega\mathbb{F}_2)$$

has a basis of classes of the following forms: the class $v_0^3y_1$, and those of form $v_iR^aR^by_1$ where $i \geq 3$, a and b are odd, $b \geq 7$, $a > 2b$, and $a + b = 2^{i+1} - 2$.

The remaining sections will be dedicated to proving this result.

Corollary 8.2. *The minimal value of i such that an admissible monomial $v_iR^aR^by_1$ appears is when $i = 4$; the only monomials of this type are $v_4R^{23}R^7y_1$ and $v_4R^{21}R^9y_1$.*

It falls to us now to determine what these obstructions mean. In the Koszul complex, these are basis elements that detect the elements $[M_4|Q^{22}|Q^6|\xi_1^2]$ and $[M_4|Q^{20}|Q^8|\xi_1^2]$ in a bar resolution for QH_*BP in $E(n)R$-mod. How do we interpret these?

The first of these, $v_4R^{23}R^7y_1$, relies on a secondary operation based on the Adem relations for $Q^{22}Q^6$ and a relation satisfied by $Q^6\xi_1^2$. This potential obstruction is

[9] Modulo a $\lim^1$-issue.

precisely the one described in the introduction, and it is zero because it would also be an obstruction to the existence of an E_∞-ring structure on MU.

Fortunately, we have another potential basis element $v_4 R^{21} R^9 y_1$, which would detect some kind of problem involving a relation satisfied by $Q^8(\xi_1^2)$ in $H_* BP$, the Adem relation for $Q^{20} Q^8$, and the Milnor primitive M_4. It gives us a place to look, and looking here leads us to the start of [Law17].

9 Calculation setup for BP

It's time to get down to the business of calculation.[10] In this section we'll begin the process of calculating $\mathrm{Ext}_{AR}(H_* BP, H_* BP\langle n\rangle)$ inductively at the prime 2. This uses knowledge of the structure of the indecomposables $QH_* BP$ as an ER-module. We begin by writing down the Koszul complexes that calculate Ext.

Proposition 9.1. *Let y_k be dual to $[\xi_k^2] \in QH_* BP$. The Koszul complex calculating $\mathrm{Ext}_{E(n)R\text{-mod}}(QH_* BP, \mathbb{F}_2)$ has a basis of monomials*

$$v_0^{k_0} v_1^{k_1} \dots v_n^{k_r} R^{a_1} R^{a_2} \dots R^{a_m} y_k$$

ranging over k_i, a_i, and k such that $k_i \geq 0$, $k \geq 1$, $a_i \geq 2a_{i+1}$, and $2^{k+1} \leq a_m$.

The differential is determined by

$$\begin{aligned} dy_k &= \sum_{j<k} R^{2^{k+1}-2^{j+1}+1} y_j, \\ dv_i x &= v_i dx, \\ dR^a(x) &= \begin{cases} R^a(dx) & \text{if } a \text{ is odd,} \\ \sum_{0\leq k\leq n} v_k R^{a+2^k-1}(x) + R^a(dx) & \text{if } a \text{ is even,} \end{cases} \end{aligned}$$

and the relation

$$R^a v_i x = \sum_{i<j\leq n} v_j R^{a-2^i+2^j} x.$$

Proof. This is a direct application of Theorem 6.2 to the module $N = QH_* BP$. $\square$

We can draw several immediate conclusions from the relations in Proposition 9.1.

Corollary 9.2. *Any admissible monomials of the form $v_0^{k_0} v_1^{k_1} \dots v_n^{k_n} R^{a_1} R^{a_2} \dots R^{a_m} y_1$, where the $a_1, \dots, a_m$ are odd, are permanent cycles in the $E(n)$-Koszul complex.*

Corollary 9.3. *The Koszul complex has a filtration under "degree in R": if we say that an admissible monomial $v_0^{k_0} v_1^{k_1} \dots v_n^{k_n} R^{a_1} R^{a_2} \dots R^{a_m} y_k$ has weight m, then the Koszul differential d increases weight, and preserves it on classes ending in y_1.*

[10]The calculations we will describe in the following sections determine many more of the Goerss–Hopkins obstruction groups than the critical group that we need. We have this calculation available, and it helps to illuminate the critical group by examining the algebraic structure in the large.

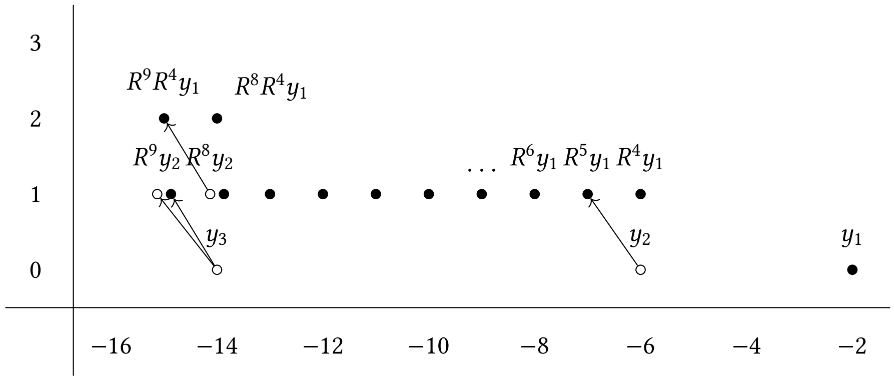

Figure 1: Part of the Koszul complex for $n = -1$

10 First calculations: $n = -1$

Our base case for calculation is when $n = -1$, where things simplify greatly: there is no interference from the v_i or differentials on R^a.

Proposition 10.1. *The Koszul complex calculating* $\mathrm{Ext}_{E(-1)R\text{-mod}}(QH_*BP, \mathbb{F}_2)$ *has a basis of monomials*

$$R^{a_1}R^{a_2}\ldots R^{a_m}y_k$$

ranging over a_i and k such that $k \geq 1$, $a_i \geq 2a_{i+1}$, and $2^{k+1} \leq a_m$.

The differential is determined by

$$dR^a(x) = R^a(dx)$$

and

$$dy_k = \sum_{j<k} R^{2^{k+1}-2^{j+1}+1}y_j.$$

This allows us to start charting things up and doing the work of calculating the result. The first portion of the Koszul complex appears in Figure 1, with operations on y_1 indicated by classes in black.

Proposition 10.2. *The* Ext*-groups* $\mathrm{Ext}_{E(-1)R\text{-mod}}(QH_*BP, \mathbb{F}_2)$ *have a basis of admissible monomials*

$$R^{a_1}R^{a_2}\ldots R^{a_m}y_1$$

(meaning, monomials such that $a_i \geq 2a_{i+1}$ and $a_m \geq 4$) which do not end in any of the sequences R^5y_1, $R^9R^4y_1$, $R^{17}R^8R^4y_1$, …

Remark 10.3. An examination of the Adem relations for the operations R^a finds that this result means that we essentially have a "free unstable module" over the R^a subject to one relation: $R^5y_1 = 0$. For instance, the Adem relations imply $R^8R^5 = R^9R^4$, $R^{16}R^8R^5 = R^{17}R^8R^4$, and so on.

Proof sketch. We filter the Koszul complex by putting y_i into filtration i so that in the associated graded complex the differential becomes $d(y_i) = R^{2^i+1} y_{i-1}$. The claim is then that, on the associated graded, this is exact except in filtration zero.

In degree zero of the associated graded, we have everything of the form $R^A y_1 = R^{a_1} R^{a_2} \ldots R^{a_k} y_1$, and all of those are permanent cycles.

In degree one of the associated graded, we have $d(R^A y_2) = R^A R^5 y_1$. (We remark that the elements $R^A y_2$ are only defined when the last term is R^8 or higher.) The map $R^A \mapsto R^A R^5$ has image consisting of the right multiples of R^5. It is also injective on anything where the admissible R^A ends with R^{10} or higher, because then $R^A R^5$ is still admissible. Thus we only have to see what happens to the admissible monomials that end in R^9 or R^8.

The Adem relations say $R^9 R^5 = 0$, so all admissible monomials ending in R^9 are in the kernel.

The Adem relations also say that $R^8 R^5 = R^9 R^4$. The admissible monomials $R^B R^8$ where B ends in R^{18} or higher map isomorphically to admissible monomials of the form $R^B R^9 R^4$. That now just leaves us checking admissible monomials that end in $R^{17} R^8$ or $R^{16} R^8$.

The Adem relations say $R^{17} R^9 = 0$, so anything ending in $R^{17} R^8 y_2$ is in the kernel. The Adem relations also say that $R^{16} R^9 = R^{17} R^8$. We inductively repeat this pattern.

We ultimately find that the kernel in grading one consists of any admissible monomials $R^A y_2$ where R^A ends in R^9, $R^{17} R^8$, $R^{33} R^{16} R^8$, $R^{65} R^{32} R^{16} R^8$, and so on. These are precisely all the right multiples of R^9.

We then look at grading two, where we have $d(y_3) = R^9 y_2$; all the right multiples of R^9 are in the image. We run the exact same computation and find that the kernel consists of all the multiples of R^{17}. This procedure continues indefinitely. $\square$

Remark 10.4. This is very similar to a computation of the topological André–Quillen cohomology of $H\mathbb{F}_2$ stated in [Laz01] and determined by alternative means in Hoyer's thesis [Hoy14].

The final answer provides us with a bit of relief: the cohomology of the Koszul complex is actually a lot *less* complicated to describe than the Koszul complex itself.

Remark 10.5. It is tempting to hope that the Koszul complex is quasi-isomorphic to its quotient by all the y_i for $i > 1$ and by the relation $R^5 y_1 = 0$. Unfortunately, this is not the case. For example, the differentials $d(y_2) = R^5 y_1$ and $d(y_3) = R^9 y_2 + R^{13} y_1$ in the Koszul complex show that the Koszul complex has an identity of secondary operations $\langle R^9, R^5, y_1 \rangle = R^{13} y_1$ not satisfied in the quotient complex.

11 Further calculations: $n = 0$

To calculate the next Ext-groups, we can feed our previous Ext-calculation into a Bockstein spectral sequence.

Proposition 11.1. *The E_1-term of the Bockstein spectral sequence*

$$\mathrm{Ext}_{E(-1)R\text{-mod}}(QH_*BP, \mathbb{F}_2) \otimes \mathbb{F}_2[v_0] \Rightarrow \mathrm{Ext}_{E(0)R\text{-mod}}(QH_*BP, \mathbb{F}_2)$$

has a basis of admissible monomials

$$v_0^k R^{a_1} R^{a_2} \dots R^{a_m} y_1$$

(except for those which end in R^5, $R^9 R^4$, ...) with v_0-linear differential satisfying

$$dR^a(x) = (a+1) v_0 R^{a+1} x + R^a(dx)$$

and the rule

$$R^a v_0 = 0.$$

In particular, these rules make it easy to apply the differential to any element in our basis:

$$d(v_0^k R^{a_1} R^{a_2} \dots R^{a_m} y_1) = \begin{cases} 0 & \text{if } a_1 \text{ is odd} \\ v_0^{k+1} R^{a_1+1} R^{a_2} \dots R^{a_m} y_1 & \text{if } a_1 \text{ is even} \end{cases}$$

In essence, our admissible monomials that start with R^{even} are attempting to make the admissible monomials that start with R^{odd} into v_0-torsion elements. Calculating the d_1-differential is an exercise in being careful about edge cases.

Proposition 11.2. *The* Ext*-groups*

$$\mathrm{Ext}_{E(0)R\text{-mod}}(QH_* BP, \mathbb{F}_2)$$

are a direct sum of copies of two types of terms:

1. *v_0-torsion copies of $\mathbb{F}_2$ indexed by those admissible monomials $R^{a_1} R^{a_2} \dots R^{a_m} y_1$ with a_1 odd which do not end in any of the sequences $R^5 y_1$, $R^9 R^4 y_1$, $R^{17} R^8 R^4 y_1$, and so on, and*

2. *copies of $\mathbb{F}_2[v_0]$ indexed by the admissible monomials y_1, $R^4 y_1$, $R^8 R^4 y_1$, $R^{16} R^8 R^4 y_1$, and so on.*

The initial portion of these Ext-groups is sketched in Figure 2.

Proof sketch. The first observation is that if $R^{a_1} R^{a_2} \dots R^{a_m}$ is an admissible monomial and a_1 is odd, then $R^{a_1-1} R^{a_2} \dots R^{a_m}$ is also an admissible monomial. Therefore, every admissible monomial starting with an *odd* term is a permanent cycle that becomes annihilated by v_0.

Almost all of the admissible monomials $R^{a_1} R^{a_2} \dots R^{a_m}$ where a_1 is even, by contrast, support differentials and do not survive the spectral sequence. The only exception is when the admissible monomial $R^{a_1+1} R^{a_2} \dots R^{a_m}$ is already zero, and this occurs only when it ends with one of the sequences R^5, $R^9 R^4$, $R^{17} R^8 R^4$, and so on. In this case, there are two possibilities: either $R^{a_1} R^{a_2} \dots R^{a_m}$ *also* ends with this sequence and it was already the zero monomial, or it is one of the monomials R^4, $R^8 R^4$, $R^{16} R^8 R^4$, and so on. These monomials are permanent cycles and produce infinite v_0-towers. ◻

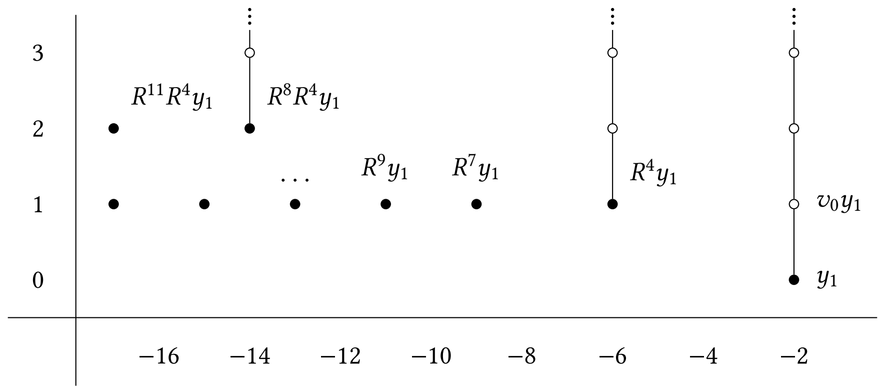

Figure 2: Part of the Ext-groups for $n = 0$

12 Further calculations: $n = 1$

There are two useful spectral sequences for computing $\mathrm{Ext}_{E(1)R\text{-mod}}(QH_*BP, \mathbb{F}_2)$ based on our previous work. Playing the information in these two spectral sequences off each other provides a useful technique for resolving hidden extensions.

The first spectral sequence is the Bockstein spectral sequence

$$\mathrm{Ext}_{E(0)R\text{-mod}}(QH_*BP, \mathbb{F}_2) \otimes \mathbb{F}_2[v_1] \Rightarrow \mathrm{Ext}_{E(1)R\text{-mod}}(QH_*BP, \mathbb{F}_2)$$

The second spectral sequence is the filtration on the Koszul complex that puts both v_0 and v_1 in filtration 1 simultaneously:

$$\mathrm{Ext}_{E(-1)R\text{-mod}}(QH_*BP, \mathbb{F}_2) \otimes \mathbb{F}_2[v_0, v_1] \Rightarrow \mathrm{Ext}_{E(1)R\text{-mod}}(QH_*BP, \mathbb{F}_2)$$

This has an E_1-term with a basis of monomials

$$v_0^k v_1^l R^{a_1} R^{a_2} \dots R^{a_m} y_1$$

(except for those which are right multiples of R^5) with $\mathbb{F}_2[v_0, v_1]$-linear differential satisfying the following relations:

$$\begin{aligned} dR^a(x) &= (a+1)(v_0 R^{a+1}x + v_1 R^{a+3}x) + R^a(dx) \\ R^a(v_1 x) &= 0 \\ R^a(v_0 x) &= v_1 R^{a+2} x \end{aligned}$$

In particular, these rules make it mechanical to apply the differential to any element in our basis:

$$d(R^{a_1}R^{a_2}x) = \begin{cases} 0 & \text{if } a_1 \text{ and } a_2 \text{ are odd} \\ v_1 R^{a_1+2} R^{a_2+1} x & \text{if } a_1 \text{ is odd and } a_2 \text{ is even} \\ v_0 R^{a_1+1} R^{a_2} x + v_1 R^{a_1+3} R^{a_2} x & \text{if } a_1 \text{ is even and } a_2 \text{ is odd} \\ v_0 R^{a_1+1} R^{a_2} x + v_1 R^{a_1+3} R^{a_2} x + v_1 R^{a_1+2} R^{a_2+1} x & \text{if } a_1 \text{ and } a_2 \text{ are even} \end{cases}$$

We will mostly use this to inform us about differentials and hidden extensions in the v_1-Bockstein spectral sequence. For example, we obtain the following results:

Lemma 12.1. *In the v_1-Bockstein spectral sequence, there are d_1-differentials*

$$d(R^{a_1}R^{a_2}R^{a_3}\dots R^{a_m}y_1) = v_1R^{a_1+2}R^{a_2+1}R^{a_3}\dots R^{a_m}y_1$$

when a_1 is odd and a_2 is even.

Lemma 12.2. *In the v_1-Bockstein spectral sequence, when x is a cycle there are hidden extensions*

$$v_0R^{a_1}R^{a_2}x = v_1R^{a_1+2}R^{a_2}x$$

when $R^{a_1}R^{a_2}x$ is admissible and a_2 is odd.

Proof. The differential

$$d(R^{a_1-1}R^{a_2}x) = v_0R^{a_1}R^{a_2}x + v_1R^{a_1+2}R^{a_2}x,$$

for x a cycle in the filtered Koszul complex, establishes this relation. □

Just as in the previous section, we can now determine the result of the v_1-Bockstein spectral sequence by a careful examination of edge cases.

Proposition 12.3. *The* Ext*-groups*

$$\mathrm{Ext}_{E(1)R\text{-mod}}(QH_*BP, \mathbb{F}_2)$$

are a direct sum of copies of three types of terms:

1. *a free copy of $\mathbb{F}_2[v_0, v_1]$ generated by y_1,*
2. *(v_0, v_1)-torsion copies of $\mathbb{F}_2$ indexed by those admissible monomials $R^{a_1}R^{a_2}\dots R^{a_m}y_1$ with a_1 and a_2 odd which do not end in any of the sequences R^5y_1, $R^9R^4y_1$, $R^{17}R^8R^4y_1$, and so on, and*
3. *"(v_0, v_1) sawtooth" patterns*

$$\mathbb{F}_2[v_0, v_1]\{R^{2k+1}x\}/(v_0R^{2k+1}x = v_1R^{2k+3}x)$$

as $R^{2k+1}x$ range over admissible monomials with x in the set $y_1, R^4y_1, R^8R^4y_1, \dots,$ mod the relations that $R^5y_1 = 0$, $R^9R^4y_1 = 0$, and so on.

The first sawtooth is pictured in Figure 3. It has generators R^7y_1, R^9y_1, $R^{11}y_1$, and so on, with $v_0R^7y_1 = v_1R^9y_1$, etc, mod the relation $v_1R^7y_1 = 0$. The second sawtooth has generators $R^{11}R^4y_1$, $R^{13}R^4y_1$, and so on.

Proof sketch. The first observation is that if $R^{a_1}R^{a_2}\dots R^{a_m}$ is an admissible monomial and a_1 and a_2 are odd, then $R^{a_1-2}R^{a_2-1}\dots R^{a_m}$ is also an admissible monomial. Therefore, every admissible monomial starting with two *odd* terms has a canonical lower admissible monomial with a differential that makes it v_1-torsion.

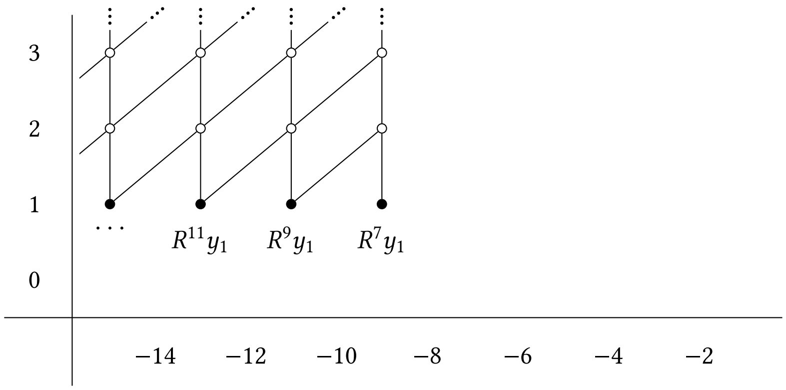

Figure 3: Part of the first sawtooth pattern in Ext for $n = 1$

Almost all of the admissible monomials $R^{a_1}R^{a_2}\dots R^{a_m}$ where a_1 is odd and a_2 is even, by contrast, support differentials and do not survive the spectral sequence. The only exception is when the admissible monomial $R^{a_1+2}R^{a_2+1}\dots R^{a_m}$ is already zero, and this occurs only when it ends with one of the sequences R^5, R^9R^4, $R^{17}R^8R^4$, and so on. In this case, there are two possibilities: either $R^{a_1}R^{a_2}\dots R^{a_m}$ *also* ends with this sequence and it was already the zero monomial, or it is one of the monomials R^a, R^aR^4, $R^aR^8R^4$, $R^{16}R^8R^4$, and so on, with a odd.

The differentials in the Koszul complex imply $d(R^4y_1) = v_1R^7y_1$, $d(R^8R^4y_1) = v_1R^{11}R^4y_1$, and so on. Therefore, in the Bockstein spectral sequence the elements $v_0^kR^4y_1$ support differentials to the hidden extensions $v_0^kv_1R^7y_1 = v_1^{k+1}R^{7+2k}y_1$, the elements $v_0^kR^8R^4y_1$ support differentials to the hidden extensions $v_1^{k+1}R^{11+2k}R^4y_1$, and so on. These impose the "right-hand edge" of the sawtooth patterns. □

13 Further calculations: $n = 2$

The calculation for $n = 2$ is carried out in a very similar fashion to the calculation for $n = 1$. Again, there are two spectral sequences calculating $\mathrm{Ext}_{E(2)R\text{-mod}}(QH_*BP, \mathbb{F}_2)$. The first is the Bockstein spectral sequence

$$\mathrm{Ext}_{E(1)R\text{-mod}}(QH_*BP, \mathbb{F}_2) \otimes \mathbb{F}_2[v_2] \Rightarrow \mathrm{Ext}_{E(2)R\text{-mod}}(QH_*BP, \mathbb{F}_2),$$

and the second is the spectral sequence

$$\mathrm{Ext}_{E(-1)R\text{-mod}}(QH_*BP, \mathbb{F}_2) \otimes \mathbb{F}_2[v_0, v_1, v_2] \Rightarrow \mathrm{Ext}_{E(2)R\text{-mod}}(QH_*BP, \mathbb{F}_2)$$

arising from filtering the Koszul complex. The E_1-term of the second has a basis of admissible monomials

$$v_0^jv_1^kv_2^lR^{a_1}R^{a_2}\dots R^{a_m}y_1$$

(except for those which are right multiples of R^5) with $\mathbb{F}_2[v_0, v_1, v_2]$-linear differential satisfying the following relations:

$$\begin{aligned}
dR^a(x) &= (a+1)(v_0 R^{a+1}x + v_1 R^{a+3}x + v_2 R^{a+7}x) + R^a(dx) \\
R^a(v_2 x) &= 0 \\
R^a(v_1 x) &= v_2 R^{a+4} x \\
R^a(v_0 x) &= v_2 R^{a+6} x + v_1 R^{a+2} x
\end{aligned}$$

We can again use these differentials to determine hidden extensions and Bockstein differentials on the basis calculated in the previous section. We begin with some hidden extensions.

Lemma 13.1. *In the v_2-Bockstein spectral sequence, when x is a cycle there are hidden extensions*

$$v_0 R^{a_1} R^{a_2} x = v_2 R^{a_1+4} R^{a_2+2} x$$

whenever $R^{a_1} R^{a_2} x$ is admissible and a_1 and a_2 are odd.

Proof. The differential

$$d(R^{a_1-1} R^{a_2} x + R^{a_1} R^{a_2-1} x) = v_0 R^{a_1} R^{a_2} x + v_2 R^{a_1+4} R^{a_2+2} x,$$

for x a cycle in the filtered Koszul complex, establishes this relation. □

We now calculate differentials. The first type are differentials on admissible monomials that start with two odd operations:

$$d(R^{a_1} R^{a_2} R^{a_3} x) = \begin{cases} 0 & \text{if } a_1,\ a_2, \text{ and } a_3 \text{ are odd} \\ v_2 R^{a_1+4} R^{a_2+2} R^{a_3+1} x & \text{if } a_1,\ a_2 \text{ are odd and } a_3 \text{ is even} \end{cases}$$

As before, this makes all admissible monomials of the form $R^{a_1} R^{a_2} R^{a_3} \dots R^{a_m} x$, where a_1, a_2, and a_3 are odd, into v_2-torsion elements. These are systematically eliminated by the admissible monomials of the form $R^{a_1} R^{a_2} R^{a_3} \dots R^{a_m} x$, where a_1 and a_2 are odd but a_3 is even, *except* for those among the following list:

$$\begin{aligned}
&R^{a_1} R^{a_2} R^4 y_1, \\
&R^{a_1} R^{a_2} R^8 R^4 y_1, \\
&R^{a_1} R^{a_2} R^{16} R^8 R^4 y_1, \dots
\end{aligned}$$

The second type are differentials on classes that are part of the "sawtooth" patterns. For a odd, these differentials take the following form:

$$\begin{aligned}
d(R^a y_1) &= 0, \\
d(R^a R^4 y_1) &= v_2 R^{a+4} R^7 y_1, \\
d(R^a R^8 R^4 y_1) &= v_2 R^{a+4} R^{11} R^4 y_1, \\
d(R^a R^{16} R^8 R^4 y_1) &= v_2 R^{a+4} R^{19} R^8 R^4 y_1, \dots
\end{aligned}$$

These differentials in the Bockstein spectral sequence hit v_0-torsion elements. However, the hidden extension

$$v_0 R^{a_1} R^{a_2} x = v_2 R^{a_1+4} R^{a_2+2} x$$

from Lemma 13.1 forces higher differentials:

$$\begin{aligned}
d(v_0^k R^a y_1) &= 0, \\
d(v_0^k R^a R^4 y_1) &= v_2^{k+1} R^{a+4k+4} R^{7+2k} y_1, \\
d(v_0^k R^a R^8 R^4 y_1) &= v_2^{k+1} R^{a+4k+4} R^{11+2k} R^4 y_1, \\
d(v_0^k R^a R^{16} R^8 R^4 y_1) &= v_2 R^{a+4k+4} R^{19+2k} R^8 R^4 y_1, \ldots
\end{aligned}$$

and so on. The targets span *all* of the admissible monomials which we previously described as being in the kernel of the differential.

Putting this calculation together, we are led to the following conclusion.

Proposition 13.2. *The* Ext*-groups*

$$\mathrm{Ext}_{E(2)R\text{-mod}}(QH_* BP, \mathbb{F}_2)$$

are a direct sum of copies of four types of terms:

1. *a free copy of $\mathbb{F}_2[v_0, v_1, v_2]$ generated by y_1,*
2. *a (v_0, v_1) sawtooth pattern generated by the classes $R^a y_1$ with a odd, $a \geq 7$,*
3. *(v_0, v_1, v_2)-torsion copies of $\mathbb{F}_2$ indexed by those admissible monomials $R^{a_1} R^{a_2} \ldots R^{a_m} y_1$ with a_1, a_2, and a_3 odd which do not end in any of the sequences $R^5 y_1$, $R^9 R^4 y_1$, $R^{17} R^8 R^4 y_1$, and so on, and*
4. *(v_0, v_2) sawtooth patterns*

$$\mathbb{F}_2[v_0, v_2]\{R^a R^b x\}/(v_0 R^a R^b x = v_2 R^{a+4} R^{b+2} x)$$

   as $R^a R^b x$ range over admissible monomials with a and b odd and x in the set $y_1, R^4 y_1, R^8 R^4 y_1, \ldots$, mod the relations that $R^5 y_1 = 0$, $R^9 R^4 y_1 = 0$, and so on.

The first (v_0, v_2) sawtooth pattern appears in Figure 4.

Remark 13.3. The v_1-multiplication on the (v_0, v_2)-sawtooth patterns is more complicated and we will not describe it here.

14 Final calculations in weight 2

The calculations we have been doing can be carried out for larger and larger values of n and obey systematic patterns, but require more and more bookkeeping with respect to the edge cases. However, we will ultimately be interested in weight 2. Here, we have already isolated everything: we have determined all the permanent cycles and their quotient by the differentials on classes in lower weight.

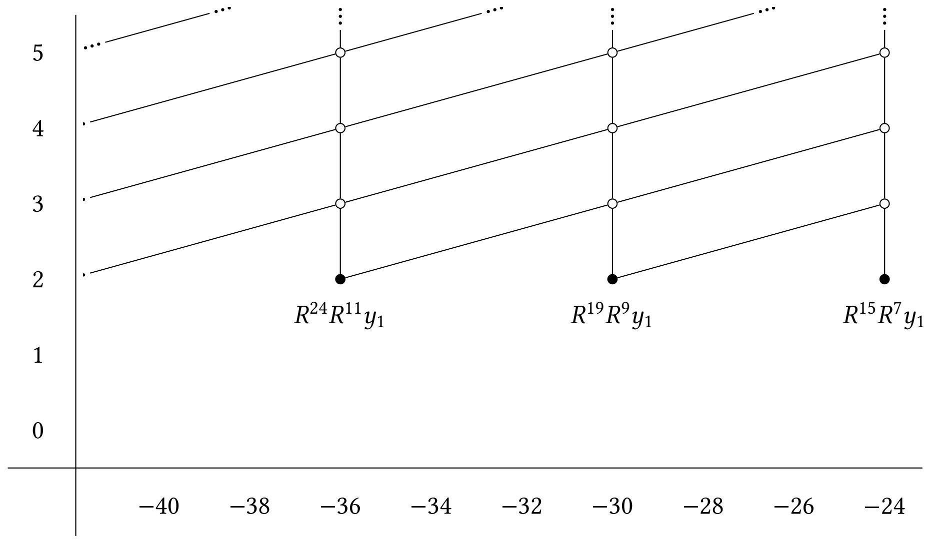

Figure 4: Part of the first (v_0, v_2) sawtooth pattern in Ext for $n = 2$

Proposition 14.1. *The part of*

$$Ext_{E(2)R\text{-mod}}(QH_*BP, \mathbb{F}_2)$$

on classes in weight less than or equal to 2 *is a sum of three terms:*

1. *in weight* 0*, there is a copy of* $\mathbb{F}_2[v_0, v_1, v_2]$ *generated by* y_1*;*
2. *in weight* 1*, there is a* (v_0, v_1)*-sawtooth pattern generated by the classes* $R^a y_1$ *for* $a \geq 7$*, a odd, with* $v_0 R^a y_1 = v_1 R^{a+2} y_1$*;*
3. *in weight* 2*, there are* (v_0, v_2)*-sawtooth patterns generated by the classes* $R^a R^b y_1$ *for* $b \geq 7$*,* $a > 2b$*, a and b odd, with* $v_0 R^a R^b y_1 = v_2 R^{a+4} R^{b+2} y_1$*.*[11]

In particular, these all lift to permanent cycles in the Koszul complex for $\mathrm{Ext}_{E(n)R\text{-mod}}(QH_*BP, \mathbb{F}_2)$*.*

In particular, the fact that we have permanent cycles implies that all the higher Bockstein spectral sequences degenerate in weights less than or equal to two.

Corollary 14.2. *For any* $n \geq 2$*, the Bockstein spectral sequences*

$$Ext_{E(2)R\text{-mod}}(QH_*BP, \mathbb{F}_2) \otimes \mathbb{F}_2[v_3, \ldots, v_n] \Rightarrow Ext_{E(n)R\text{-mod}}(QH_*BP, \mathbb{F}_2)$$

degenerate in weights less than or equal to two.

Since filtration is greater than or equal to weight, this gives a complete list of classes in Ext-filtration less than or equal to three except for classes of the form $R^a R^b R^c x$, all of which are in large negative degree.

[11]There is one sawtooth pattern for each odd positive value of $a - 2b$.

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