Adjoining roots in homotopy theory

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Abstract

We use a "twisted group algebra" method to constructively adjoin formal radicals $\sqrt[n]{\alpha}$, for α a unit in a commutative ring spectrum or an invertible object in a symmetric monoidal ∞-category. We show that this construction is classified by maps from Eilenberg–Mac Lane objects to the unit spectrum $\mathfrak{gl}_1$, the Picard spectrum $\mathfrak{pic}$, and the Brauer spectrum $\mathfrak{br}$.

Given a commutative ring R and an element $\alpha \in R$, we can adjoin a formal radical $\sqrt[n]{\alpha}$ to R by embedding R into the extension ring $R[x]/(x^n - \alpha)$. These ring extensions come equipped with a ready-made basis $\{1, x, \ldots, x^{n-1}\}$ over R and are fundamental constructions in algebra. In the derived setting, however, it is less clear when these types of constructions are possible. Given a commutative ring spectrum R and an element in $\alpha \in \pi_0 R$, one can construct a commutative R-algebra with an n'th root of α using the same type of presentation in terms of generators and relations. However, away from characteristic zero the universal property enjoyed by this construction is not as strong and its coefficient ring can be unpredictable.

We can try instead to lift the algebra directly by constructing a commutative R-algebra S with a map $R \to S$ such that, on coefficient rings, we have the algebraic extension: $\pi_* S \cong (\pi_* R)[x]/(x^n - \alpha)$. Depending on α, such an extension may not be possible or may not be unique. It is always possible to adjoin roots of 1, because those algebras are realized by the group algebras $R[C_n]$ of finite cyclic groups. If both α and n are units in $\pi_0 R$ then the resulting extension on coefficient rings is *étale*, and the obstruction theory of Robinson [Rob03] or Goerss–Hopkins [GH04] can be used to show that the extension ring S exists, is essentially unique, and has a universal property among R-algebras with such a root adjoined. If $1/2$ is inverted, this means that we can adjoint $\sqrt{-1}$. However, Schwänzl–Vogt–Waldhausen showed in [SVW99], using topological Hochschild homology, that it is impossible to adjoint a square root of -1 to the sphere spectrum. A different argument of Hopkins with $K(1)$-local power operations shows that the p-complete K-theory spectrum cannot admit p'th roots of unity [LN14, A.6.iii], and this was further generalized by Devalapurkar to $K(n)$-local theory [Dev17]. Further difficulties appear when attempting to adjoin a root that appears in a nonzero degree.

Our starting observation is that these formal radicals are a special case of twisted group algebras. Given an abelian group extension $0 \to G \to E \to A \to 0$ and a group homomorphism $G \to R^\times$, we can form the relative tensor product $\mathbb{Z}[E] \otimes_{\mathbb{Z}[G]} R$. The elements of A lift to a basis over R, and the resulting algebra differs from the group algebra $R[A]$ by this central extension. Moveover, all such extensions can be constructed in a universal case by pushing forward the extension from G to $R^\times$.

We will begin by showing that, when it is possible to construct similar extensions of the spectrum of units $\mathfrak{gl}_1(R)$, we can give systematic constructions of formal radicals and other twisted group algebras, and obstructions are detectable with sufficient knowledge of $\mathfrak{gl}_1(R)$. By then reinterpreting these constructions as Thom spectra for maps to $\mathfrak{pic}(R)$, we can use recent work of Antolín-Camarena–Barthel [AB14] to both generalize this and allow us to identify these algebras as having a universal property. This recovers several constructions: adjoining n'th roots of elements to a ring spectrum where n is invertible, usually carried out using obstruction thery, and adjoining similar roots to elements in gradings outside zero. There are also new constructions: we find that we can extend the first Postnikov stage $\tau_{\leq 1}\mathbb{S}_{(2)}$ of the 2-local sphere by adjoining $\sqrt{D}$ for $D \equiv 1 \bmod 4$.

Once we have accomplished this, our second goal will be to dig one categorical level down.

The same methods can be applied to adjoin formal radicals of elements in the Picard group. This allows us to take an extension of the Picard spectrum $\mathfrak{pic}(R)$ and use it to embed the category of R-modules into a graded category with a larger group of invertible objects. This formalism recovers algebraic examples, such as Rezk's ω-twisted tensor product for $\mathbb{Z}/2$-graded modules [Rez09]. There are also new topological examples: if R is an MU-algebra we can embed the category LMod_R of R-modules, where integer suspensions are possible, into a larger category $\prod_{\mathbb{Q}/Z} \mathrm{LMod}_R$ with a symmetric monoidal structure that allows suspensions by elements of $\mathbb{Q} \times \mathbb{Z}/2$. This is also possible for modules over the topological K-theory spectrum ku and the algebraic K-theory spectrum $K\mathbb{C}$. Although our focus is on ring spectra, many of the results are proved in the generality of presentable symmetric monoidal ∞-categories.

Further directions

A first issue is that our discussion of adjoining roots is less satisfying for units outside degree zero. In particular, the identification of such units is somewhat roundabout. Ideally a solution to this problem would make use of a spectrum of graded units similar to those developed by Sagave [Sag16], and in particular his construction of $\mathrm{bgl}_1^* R$.

Second, we restrict our attention to strictly commutative objects (meaning E_∞-ring objects). The constructions in this paper should have interesting and useful E_n-variants, making use of the iterated classifying spaces $B^{(n)}GL_1 R$.

Finally, we study unit groups because they are more easily analyzed via their associated spectrum. This means that we lose any ability to extract formal radicals of nonunit elements. We are hopeful that the future will bring a better understanding of the structure theory of E_∞-spaces, allowing us to move beyond unit groups to effectively study multiplicative monoids.

Conventions and background

Our paper is written homotopically, and in particular we use the phrase "commutative ring spectrum" to mean an E_∞-ring spectrum.

We will use the same name for both an abelian group A and the associated Eilenberg–Mac Lane spectrum, regarding the category of abelian groups as embedded fully faithfully into the category of spectra. In particular, a commutative ring k is equivalent to a commutative ring spectrum.

For a ring spectrum R, the unit group $\mathrm{GL}_1(R) \subset \Omega^\infty R$ is the space of units under the multiplicative monoidal product, or equivalently the space of self-equivalences of R as a left R-module. If R has a commutative ring structure we write $\mathfrak{gl}_1(R)$ for the associated spectrum of units [May77, ABG+14]. There is a unit map $\mathbb{S}[\mathrm{GL}_1(R)] \to R$ for the adjunction between unit groups and spherical group algebras.

For a monoidal ∞-category C, the Picard space $\mathrm{Pic}(C) \subset C^{\simeq}$ is the space of invertible objects and equivalences between them [Cla11, MS16].[1] If C has a symmetric monoidal structure then $\mathrm{Pic}(C)$ has an E_∞-structure and we write $\mathfrak{pic}(C)$ for the associated Picard spectrum. If $C = \mathrm{LMod}_R$ is the category of modules over R, we simply write $\mathrm{Pic}(R)$ and $\mathfrak{pic}(R)$ instead.

We will require known identifications of the groups $[A, \Sigma^k B]$ of homotopy classes of maps between Eilenberg–Mac Lane spectra, which we will simply state.[2]

1. The group $[A, B]$ is isomorphic to the group $\mathrm{Hom}(A, B)$.

2. The group $[A, \Sigma B]$ is isomorphic to the group $\mathrm{Ext}(A, B)$: this extension is identified with the fiber of a map $A \to \Sigma B$.

3. The group $[A, \Sigma^2 B]$ is isomorphic to the group $\mathrm{Hom}(A, B[2])$ of 2-torsion homomorphisms $A \to B$.

4. The group $[A, \Sigma^3 B]$ is part of a short exact sequence
$$0 \to \mathrm{Ext}(A, B[2]) \to [A, \Sigma^3 B] \to \mathrm{Hom}(A, B/2) \to 0.$$

5. These identifications respect composition. In particular, the composition map $[B, \Sigma^2 C] \times [A, \Sigma B] \to [A, \Sigma^3 C]$ is the Yoneda pairing
$$\mathrm{Hom}(B, C[2]) \times \mathrm{Ext}(A, B) \to \mathrm{Ext}(A, C[2]).$$
As a result, a pair (g, Γ) representing a homomorphism $B \to C[2]$ and an extension $0 \to B \to \Gamma \to A \to 0$ maps to zero if and only if the homomorphism g extends from B to all of Γ.

Given an ordinary symmetric monoidal category $\mathcal{D}$, the Picard space $\mathrm{Pic}(\mathcal{D})$ is the nerve of an ordinary groupoid, consisting of the invertible objects and isomorphisms between them. This means there is a fiber sequence
$$\mathfrak{pic}(\mathcal{D}) \to \pi_0\, \mathfrak{pic}(\mathcal{D}) \xrightarrow{k} \Sigma^2 \pi_1\, \mathfrak{pic}(\mathcal{D}).$$

[1](cf. [Mat14, §2.2]) If the unit of C is κ-compact for some cardinal κ, then the objects of $\mathrm{Pic}(C)$ are also κ-compact, and if C is presentable then the full subcategory of κ-compact objects is essentially small. Therefore, for presentable monoidal ∞-categories it is possible to identify $\mathrm{Pic}(C)$ with a small space even though C is large.

[2]All of these are determined by first calculating that the groups of maps $\mathbb{Z} \to \Sigma^k \mathbb{Z}$ are $\mathbb{Z}$, 0, 0, $\mathbb{Z}/2$, 0 for $k = 0 \dots 5$, and then using free resolutions of the source and target. The generator in degree 3 is the composite of mod-2 reduction, the Steenrod square Sq^2, and the Bockstein.

Here $\pi_0 \,\mathfrak{pic}(\mathcal{D})$ is the classical Picard group of the category $\mathcal{D}$, and $\pi_1 \,\mathfrak{pic}(\mathcal{D})$ is the automorphism group $\mathrm{Aut}_{\mathcal{D}}(\mathbb{I})$ of the monoidal unit. This first k-invariant is always expressed in terms of the twist map. Namely, given an invertible module γ, the twist-self-isomorphism $\gamma \otimes \gamma \to \gamma \otimes \gamma$ is multiplication by a 2-torsion automorphism $\tau(\gamma)$ of the monoidal unit, an element of $\pi_1 \,\mathfrak{pic}(\mathcal{D})$ that satisfies $\tau(\gamma) \circ \tau(\gamma) = \mathrm{id}$. The k-invariant $\pi_0 \,\mathfrak{pic}(\mathcal{D}) \xrightarrow{k} \Sigma^2 \pi_1 \,\mathfrak{pic}(\mathcal{D})$ is identified with τ.

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1 Formal radicals

Let $\alpha \in \pi_0(R)$ be a unit; α is then also represented by an element of $\pi_0(\mathfrak{gl}_1(R))$. Fix a positive integer n. From this, we can construct an extension of abelian groups

$$0 \to \pi_0(\mathfrak{gl}_1 R) \to E \to \mathbb{Z}/n \to 0$$

such that the generator 1 of $\mathbb{Z}/n$ has a chosen lift to an element $x \in E$ with $x^n = \alpha$. As a set with an action of $\pi_0 \,\mathfrak{gl}_1(R)$, $E \cong \coprod_{i=0}^{n-1} \pi_0 \,\mathfrak{gl}_1(R) \cdot \{x^i\}$.

This extension is determined by an extension class $\mathrm{Ext}^1(\mathbb{Z}/n, \pi_0 \,\mathfrak{gl}_1(R))$, or equivalently by a map

$$\bar{\rho}\colon \mathbb{Z}/n \to \Sigma\pi_0 \,\mathfrak{gl}_1(R)$$

between Eilenberg–Mac Lane spectra.

Definition 1. A *formal n'th root of* α is a lift ρ of $\bar{\rho}$ to $\mathfrak{gl}_1(R)$:

$$\begin{tikzcd} & \Sigma\,\mathfrak{gl}_1(R) \arrow[d] \\ \mathbb{Z}/n \arrow[ur, "\rho"] \arrow[r, "\bar{\rho}"'] & \Sigma\pi_0 \,\mathfrak{gl}_1(R) \end{tikzcd}$$

We refer to the fiber of ρ as the *extended unit spectrum* $\mathfrak{gl}_1(R, \rho)$ associated to ρ, and the associated infinite loop space as *extended unit group* $GL_1(R, \rho)$.

The extended unit spectrum is part of a fiber sequence

$$\mathfrak{gl}_1(R) \to \mathfrak{gl}_1(R, \rho) \to \mathbb{Z}/n,$$

and hence we get a decomposition $\mathrm{GL}_1(R, \rho) \cong \coprod_{i=0}^{n-1} \mathrm{GL}_1(R) \cdot \{x^i\}$ as spaces with an action of $\mathrm{GL}_1(R)$.

Remark 2. The map ρ determines $\bar{\rho}$: the map $\mathfrak{gl}_1(R) \to \mathfrak{gl}_1(R, \rho)$ is an isomorphism on π_k except when $k = 0$, when it is the inclusion $\pi_0 \mathfrak{gl}_1(R) \to E$. Therefore, ρ determines the extension E and hence determines α up to n'th powers.

Definition 3. Suppose that ρ is a formal n'th root of α. Then the algebra obtained by *adjoining this root* is the relative smash product

$$R[\rho] = \mathbb{S}[\mathrm{GL}_1(R, \rho)] \underset{\mathbb{S}[\mathrm{GL}_1(R)]}{\otimes} R.$$

Proposition 4. *The coefficient ring of* $R[\rho]$ *is*

$$\pi_* R[\rho] \cong \pi_* R[x]/(x^n - \alpha).$$

Proof. The decomposition $\mathrm{GL}_1(R, \rho) \cong \coprod_{i=0}^{n-1} \mathrm{GL}_1(R) \cdot \{x^i\}$ means that the spherical group algebra $\mathbb{S}[\mathrm{GL}_1(R, \rho)]$ decomposes as $\oplus_{i=0}^{n-1} \mathbb{S}[\mathrm{GL}_1(R)] \cdot \{x^i\}$, a free left $\mathbb{S}[\mathrm{GL}_1(R)]$-module. Therefore, there is a simple Künneth formula that gives us an isomorphism of modules:

$$\begin{aligned} \pi_* R[\rho] &\cong \pi_* \mathbb{S}[\mathrm{GL}_1(R, \rho)] \underset{\pi_* \mathbb{S}[\mathrm{GL}_1(R)]}{\otimes} \pi_* R \\ &\cong \bigoplus_{i=0}^{n-1} \pi_* R \cdot \{x^i\}. \end{aligned}$$

Moreover, the identity $x^n = \alpha$ for the element $x \in E = \pi_0 \mathrm{GL}_1(R, \rho)$ completely determines the multiplication in $\pi_* R[\rho]$. □

Remark 5. It is clear that, other than the calculation of the structure of the coefficient ring, there is nothing special about the group $\mathbb{Z}/n$ in the above discussion. Given a map $\rho\colon A \to \Sigma \mathfrak{gl}_1(R)$ for some abelian group A, lifting an extension

$$0 \to \pi_0 \mathfrak{gl}_1(R) \to E \to A \to 0,$$

there is an associated algebra $R[\rho]$ whose coefficient ring is a twisted central extension:

$$\begin{aligned} \pi_* R[\rho] &\cong \mathbb{Z}[E] \underset{\mathbb{Z}[\pi_0 \mathfrak{gl}_1(R)]}{\otimes} \pi_* R \\ &\cong \bigoplus_{a \in A} \pi_* R \cdot \{a\}. \end{aligned}$$

We will see similar algebras more extensively in later sections.

Example 6. Suppose that n is a unit in $\pi_0 R$. Then n acts invertibly on the homotopy groups $\pi_k \mathfrak{gl}_1(R) \cong \pi_k R$ for $k > 0$: therefore the spectrum of maps from $\mathbb{Z}/n$ to the 0-connected cover $\tau_{\geq 1} \mathfrak{gl}_1(R)$ is trivial. Using the fiber sequence

$$\Sigma \mathfrak{gl}_1(R) \to \Sigma \pi_0 \mathfrak{gl}_1(R) \to \Sigma^2 \tau_{\geq 1} \mathfrak{gl}_1(R),$$

we find that, for any unit α, the map $\mathbb{Z}/n \to \Sigma \pi_0 \mathfrak{gl}_1(R)$ lifts essentially uniquely to a formal n'th root $\mathbb{Z}/n \to \Sigma \mathfrak{gl}_1(R)$.

Example 7. Let K be the p-complete K-theory spectrum. Then it is possible to show that the group $[\mathbb{Z}/p, \Sigma\, \mathfrak{gl}_1 K]$ is trivial, and thus this method does not allow us to adjoin $x = \sqrt[p]{\alpha}$ for any nontrivial element α of $\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^p$. To give a proof, however, we need to know structural properties of the multiplication on K. The straight-line proof we know uses Rezk's $K(1)$-local logarithm $\ell_1 \colon \mathfrak{gl}_1 K \to K$ [Rez06], together with nontrivial knowledge of low-degree k-invariants for $\mathfrak{gl}_1 K$. In rough, the fact that Rezk's logarithm gives us an equivalence in degrees greater than 2 implies that there is a diagram of fiber sequences:

$$\begin{tikzcd}
\Sigma^4 (ku)^\wedge_p \arrow[r] & \tau_{\geq 2}\, \mathfrak{gl}_1 K \arrow[r] \arrow[d] & \mathfrak{gl}_1 K \arrow[d] \\
 & \Sigma^2 \mathbb{Z}_p \arrow[ul, dotted] & \mathbb{Z}_p^\times \arrow[ul, dotted]
\end{tikzcd}$$

Applying $[\mathbb{Z}/p, -]$ gives us a spectral sequence that computes $[\mathbb{Z}/p, \Sigma\, \mathfrak{gl}_1 K]$; in the critical group the k-invariants of $\mathfrak{gl}_1 K$ give this spectral sequence one nontrivial differential for $p > 2$ and two nontrivial differentials for $p = 2$.

However, the impossibility of adjoining these radicals can be shown directly using $K(1)$-local power operations, in a manner exactly analogous to the proof that one cannot adjoin roots of unity; in this form it generalizes. Let us sketch this argument.

If we had such an E_∞-ring K-algebra L, it would be p-complete and thus $K(1)$-local. Its coefficient ring $\mathbb{Z}_p[x]/(x^p - \alpha)$ would then have a $K(1)$-local power operation ψ^p, a ring endomorphism that agrees with the Frobenius mod p [Hop14]. The element $\zeta = \psi^p(x)/x$ would then be a p'th root of unity. If ζ is not in $\mathbb{Z}_p^\times$, then L is a $K(1)$-local E_∞-ring containing a nontrivial p'th root of unity and Devalapurkar has shown this to be impossible [Dev17]. If ζ is in $\mathbb{Z}_p^\times$, then $\alpha \equiv \psi^p x = \zeta^{-1} x \bmod p$, which contradicts the fact that $1, x, \ldots, x^{p-1}$ are a basis of this ring mod p.

2 Strict units

One source of formal roots is the theory of strictly commutative elements.

Definition 8. The space $\mathbb{G}_m(R)$ of *strictly commutative units* of R is the space of maps $\mathbb{Z} \to \mathfrak{gl}_1(R)$, or equivalently the space of E_∞-maps $\mathbb{Z} \to \mathrm{GL}_1(R)$. The generator $1 \in \mathbb{Z}$ induces forgetful maps $\mathbb{G}_m(R) \to \mathrm{GL}_1(R) \to \pi_0\, \mathfrak{gl}_1(R)$.

In particular, a strictly commutative unit of R has an underlying unit in $\pi_0(R)$.

Proposition 9. *Suppose α is a strictly commutative unit of R. Then, for any $n > 0$, α has a canonical lift to a formal n'th root.*

Proof. The canonical Bockstein map $\mathbb{Z}/n \to \Sigma\mathbb{Z}$ can be composed with the map $\mathbb{Z} \to \mathfrak{gl}_1(R)$ classifying α. □

Example 10. Let $R = S^{-1}(\tau_{\leq 1}\mathbb{S})$ be the localization of the first Postnikov truncation of the sphere spectrum with respect to some set S of primes (not containing 2). Then there is a fiber sequence

$$\mathfrak{gl}_1(R) \to (S^{-1}\mathbb{Z})^\times \xrightarrow{k} \Sigma^2\mathbb{Z}/2.$$

This k-invariant corresponds to a (2-torsion) homomorphism $(S^{-1}\mathbb{Z})^\times \to \mathbb{Z}/2$. This homomorphism is a classical calculation of orientation theory: it is the map

$$n \mapsto \begin{cases} 1 & \text{if } n \equiv +1 \mod 4, \\ -1 & \text{if } n \equiv -1 \mod 4. \end{cases}$$

As a result, one can determine the homotopy groups of the space of strictly commuting elements, and in particular there is an exact sequence

$$0 \to [\mathbb{Z}, \mathfrak{gl}_1(R)] \to [\mathbb{Z}, (S^{-1}\mathbb{Z})^\times] \to [\mathbb{Z}, \Sigma^2\mathbb{Z}/2].$$

We find that any unit in $S^{-1}\mathbb{Z}$ which is congruent to 1 mod 4 lifts, essentially uniquely, to a strictly commutative unit of R. This allows us to construct commutative algebras such as $R[\sqrt{5}]$ and $R[\sqrt{-3}]$, even though these are ramified extensions on the level of coefficient rings.

Remark 11. In the case of strictly commutative units, we obtain a second description of the algebra obtained by adjoining this root ρ. A strictly commutative element determines a composite map

$$\mathbb{S}[\mathbb{Z}] \to \mathbb{S}[\mathrm{GL}_1(R)] \to R,$$

and so we can construct the algebra $R[\rho]$ as $R \otimes_{\mathbb{S}[\mathbb{Z}]} \mathbb{S}[\frac{1}{n}\mathbb{Z}]$.

This has the benefit that it readily lifts to a *nonunit* version. If we define the *strictly commutative multiplicative monoid* $\mathbb{M}_m(R)$ to be the space of E_∞-maps

$$\mathbb{N} \to M_1(R) = \Omega^\infty_\otimes R$$

to the multiplicative monoid of R, then a strictly commutative element α can have an n'th root adjoined via the construction

$$\mathbb{S}[\tfrac{1}{n}\mathbb{N}] \otimes_{\mathbb{S}[\mathbb{N}]} R.$$

3 Strict gradings

The shift-by-1 in our definition of formal roots is strongly suggestive: the suspended unit spectrum $\Sigma\,\mathfrak{gl}_1(R)$ is a connective cover of the Picard spectrum $\mathfrak{pic}(R)$. In this section we will begin exploring Picard-graded analogues of our constructions.

Definition 12. Suppose that A is an abelian group and C is a symmetric monoidal ∞-category. The space of *strict A-gradings for C* is the space of maps $A \to \mathfrak{pic}(C)$, or equivalently the space of E_∞-maps $A \to \mathrm{Pic}(C)$. There is a composite $A \to \pi_0\,\mathfrak{pic}(C)$, which we refer to as the *underlying A-grading*.

Remark 13. Suppose that we have any symmetric monoidal functor $A \to C$. The space A is a groupoid, so its image lies in $C^{\simeq}$; the objects in A have inverses under the monoidal product, so monoidality of ρ implies that its image lies inside $\mathrm{Pic}(C)$. We will not distinguish between symmetric monoidal functors $A \to C$ and symmetric monoidal functors $A \to \mathrm{Pic}(C)$.

Example 14. Let C be a symmetric monoidal ∞-category. The *strict Picard space* of C, denoted by $\mathbb{P}\mathrm{ic}(C)$, is the space of strict $\mathbb{Z}$-gradings: maps $\mathbb{Z} \to \mathfrak{pic}(C)$, or equivalently E_∞-maps $\mathbb{Z} \to \mathrm{Pic}(C)$. The generator $1 \in \mathbb{Z}$ induces forgetful maps $\mathbb{P}\mathrm{ic}(C) \to \mathrm{Pic}(C) \to \pi_0 \mathrm{Pic}(C)$, and we refer to the image as the *underlying object.*

Example 15. The space of *strict n-torsion objects* of C, denoted by $\mathbb{P}\mathrm{ic}^{[n]}(C)$, is the space of strict $\mathbb{Z}/n$-gradings: maps $\rho\colon \mathbb{Z}/n \to \mathfrak{pic}(C)$, or equivalently maps $\mathbb{Z}/n \to \mathrm{Pic}(C)$ of E_∞-spaces. The cofiber sequence $\mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n$ of spectra gives rise to the following maps, where each double composite is a fiber sequence:

$$\mu_n(R) \to \mathbb{G}_m(R) \xrightarrow{n} \mathbb{G}_m(R) \xrightarrow{\partial} \mathbb{P}\mathrm{ic}^{[n]}(R) \to \mathbb{P}\mathrm{ic}(R) \xrightarrow{n} \mathbb{P}\mathrm{ic}(R)$$

In particular, the map ∂ sends a strictly commutative element $\alpha\colon \mathbb{Z} \to \mathfrak{gl}_1(R)$ to the image

$$\mathbb{Z}/n \to \Sigma\mathbb{Z} \xrightarrow{\alpha} \Sigma\, \mathfrak{gl}_1(R) \to \mathfrak{pic}(R)$$

of the formal n'th root associated to α.

Remark 16. For a commutative ring spectrum R, a strict n-torsion R-module with underlying left R-module L has a choice of equivalence $L^{\otimes_R n} \to R$. If the module L is equivalent to R, then the map $\mathbb{Z}/n \to \pi_0\, \mathfrak{pic}(R)$ is trivial and so the map lifts to a map $\mathbb{Z}/n \to \Sigma\, \mathfrak{gl}_1(R)$: a formal n'th root. We can detect which root (up to n'th powers) by making a choice of an equivalence $R \to L$; this determines a composite equivalence $R \simeq R^{\otimes_R n} \to I^{\otimes_R n} \to R$, and hence a unit in $\pi_0(R)$.

Example 17. Suppose that C is a symmetric monoidal stable ∞-category such that n is a unit in the ring $\pi_0 \mathrm{End}_C(\mathbb{I})$. Then for $k > 0$ there are isomorphisms

$$\pi_k\, \mathfrak{pic}(C) \cong \pi_{k+1} \mathrm{Aut}_C(\mathbb{I}) \cong \pi_{k+1} \mathrm{End}_C(\mathbb{I}),$$

and the latter are acted on invertibly by n. Therefore, the fiber sequence

$$\tau_{\geq 2}\, \mathfrak{pic}(C) \to \mathfrak{pic}(C) \to \mathfrak{pic}(hC)$$

induces an equivalence $\mathrm{Map}(\mathbb{Z}/n, \mathfrak{pic}(C)) \to \mathrm{Map}(\mathbb{Z}/n, \mathfrak{pic}(hC))$. In this case, strict n-torsion objects in C are equivalent to strict n-torsion objects in the homotopy category hC.

Example 18. For any ordinary ring k, there is a fiber sequence

$$\mathfrak{pic}(k) \to \mathbb{Z} \to \Sigma^2 k^\times,$$

where Σk is a generator of $\pi_0\, \mathfrak{pic}(k)$. The k-invariant is the twist permutation of Σk, and is represented by the homomorphism $\mathbb{Z} \twoheadrightarrow \{\pm 1\} \to k^\times$. This k-invariant becomes trivial on $2\mathbb{Z}$, and so the category of k-modules has a strict $2\mathbb{Z}$-grading. The spaces $\mathbb{G}_m(k)$ are connected but not contractible, so the $2\mathbb{Z}$-gradings are unique up to isomorphism but not canonical. We can make them canonical by choosing a $2\mathbb{Z}$-grading of $\mathbb{Z}$.[3]

Example 19 ([Law18, 1.3.7]). For any commutative ring spectrum R, the element ΣR in $\mathrm{Pic}(R)$ determines a map $\mathbb{Z} = \pi_0 \mathfrak{pic}(\mathbb{S}) \to \pi_0 \mathfrak{pic}(R)$. For any $d > 0$, we get a composite $d\mathbb{Z} \to \pi_0 \mathfrak{pic}(R)$. If this lifts to a strict $d\mathbb{Z}$-grading, we could give this a name: an E_∞^d-structure on R,[4] which should be a strengthening of the notion of an H_∞^d-structure from [BMMS86].

The universal property of the real bordism spectrum MO is that it is initial among commutative ring spectra with a nullhomotopy of the map $BO \to \mathrm{Pic}(\mathbb{S}) \to \mathrm{Pic}(MO)$ of E_∞-spaces. Equivalently, it is initial among commutative rings with a commutative diagram of spectra

$$\begin{tikzcd} ko \arrow[r] \arrow[d] & \mathfrak{pic}(\mathbb{S}) \arrow[d] \\ \mathbb{Z} \arrow[r] & \mathfrak{pic}(MO). \end{tikzcd}$$

In particular, this gives the real bordism spectrum MO a strict $\mathbb{Z}$-grading and any commutative MO-algebra inherits it. Similar considerations with complex or spin structures structures give the complex bordism spectrum MU a strict $2\mathbb{Z}$-grading and the spin bordism spectrum $MSpin$ a strict $4\mathbb{Z}$-grading.

Example 20. The Atiyah–Bott–Shapiro orientation lifts to give the complex K-theory spectrum ku a commutative MU-algebra structure, and the real K-theory spectrum ko a commutative $MSpin$-algebra structure, due to work of Joachim [Joa04]. Therefore, ku admits a strict $2\mathbb{Z}$-grading and ko admits a strict $4\mathbb{Z}$-grading.

Example 21. Let $K\mathbb{C}$ be the algebraic K-theory spectrum of the complex numbers, which comes equipped with a map $f\colon K\mathbb{C} \to ku$ to the complex topological K-theory spectrum. Work of Suslin showed that the map f is an equivalence after profinite completion, and hence the fiber of f is rational. We would like to show that $K\mathbb{C}$ has a strict $2\mathbb{Z}$-grading. (A similar argument applies to show that the strict $4\mathbb{Z}$-grading of the real K-theory spectrum ko lifts to the algebraic K-theory $K\mathbb{R}$.)

Consider the arithmetic square:

$$\begin{tikzcd} K\mathbb{C} \arrow[r] \arrow[d] & K\mathbb{C}_{\mathbb{Q}} \arrow[d] \\ K\mathbb{C}^\wedge \arrow[r] & (K\mathbb{C}^\wedge)_{\mathbb{Q}} \end{tikzcd}$$

The functor GL_1 preserves this pullback. When we apply $\mathfrak{pic}$ we get a diagram of

[3] If $2 = 0$ in k, this k-invariant is trivial and so the category of k-modules has a strict $\mathbb{Z}$-grading. Moreover, $\mathfrak{pic}(\mathbb{F}_2) \simeq \mathbb{Z}$, and so this $\mathbb{Z}$-grading is canonical.

[4] We are not very enthusiastic about extending this naming convention.

connective spectra:

$$\begin{tikzcd}
\mathfrak{pic}(K\mathbb{C}) \arrow[r] \arrow[d] & \mathfrak{pic}(K\mathbb{C}_{\mathbb{Q}}) \arrow[d] \\
\mathfrak{pic}(K\mathbb{C}^{\wedge}) \arrow[r] & \mathfrak{pic}((K\mathbb{C}^{\wedge})_{\mathbb{Q}})
\end{tikzcd}$$

On π_0 this is the constant square $\mathbb{Z}$, and on π_1 we get a bicartesian square

$$\begin{tikzcd}
\mathbb{Z}^{\times} \arrow[r] \arrow[d] & \mathbb{Q}^{\times} \arrow[d] \\
\widehat{\mathbb{Z}}^{\times} \arrow[r] & (\widehat{\mathbb{Z}}_{\mathbb{Q}})^{\times}.
\end{tikzcd}$$

Together these show that this diagram of Picard spectra is a homotopy pullback diagram. Let C be the cofiber of $\mathfrak{pic}(K\mathbb{C}) \to \mathfrak{pic}(K\mathbb{C}^{\wedge})$; its homotopy groups are then rational above degree 2, and equal to the torsion-free group $\widehat{\mathbb{Z}}^{\times}/\mathbb{Z}^{\times}$ in degree one.

The obstruction to lifting the strict $2\mathbb{Z}$-grading

$$2\mathbb{Z} \to \mathfrak{pic}(ku) \to \mathfrak{pic}(ku^{\wedge}) \simeq \mathfrak{pic}(K\mathbb{C}^{\wedge})$$

to a $2\mathbb{Z}$-grading of $\mathfrak{pic}(K\mathbb{C})$ is the map $2\mathbb{Z} \to C$. However, consider the fiber sequence

$$\tau_{\geq 2}C \to C \to \Sigma\widehat{\mathbb{Z}}^{\times}/\mathbb{Z}^{\times}.$$

Since the left term is 1-connected and rational and the right term is torsion-free, there are no nontrivial homotopy classes of maps from $2\mathbb{Z}$ into either term, and hence $[\mathbb{Z}, C] = 0$. Therefore, the strict $2\mathbb{Z}$-grading of ku can be extended to $K\mathbb{C}$.

4 Trivializing algebras

The ring spectrum constructed by adjoining formal radicals turns out to be a special case of a more general construction associated to strict gradings. From this point forward we will need to make use of [Lur09, Lur17].

Definition 22. Suppose that C is a presentable symmetric monoidal ∞-category, A is an abelian group regarded as a discrete symmetric monoidal category, and that $\rho\colon A \to C$ is a strict A-grading. We define the *trivializing algebra* T_ρ to be the homotopy colimit of ρ.

Remark 23. Since A is a discrete category, there is an equivalence in C of the form

$$T_\rho \simeq \coprod_{a \in A} \rho(a).$$

This homotopy colimit is a very special case of the Thom spectrum construction. As such, work of Antolín-Camarena–Barthel gives the trivializing algebra a universal property.

Proposition 24. *Suppose that A is an abelian group and that $\rho\colon A \to \mathcal{C}$ is a symmetric monoidal functor with trivializing algebra T_ρ.*

1. *T_ρ has a natural lift to a commutative algebra object: $T_\rho \in \mathrm{CAlg}(\mathcal{C})$.*

2. *The algebra T_ρ is universal among commutative algebras in $\mathrm{CAlg}(\mathcal{C})$ with a chosen commuting diagram*

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d] & \mathfrak{pic}(\mathcal{C}) \arrow[d] \\ * \arrow[r] & \mathfrak{pic}(\mathrm{LMod}_{T_\rho}). \end{tikzcd}$$

In particular, for any $a \in A$ the algebra T_ρ has a chosen equivalence of left T_ρ-modules $\phi_a\colon T_\rho \to T_\rho \otimes \rho(a)$, and there are chosen coherences $\phi_a \otimes 1 \circ \phi_b \simeq \phi_{ab}$.

Proof. The colimit has a commutative algebra structure by [AB14, Theorem 2.8].

We will now prove the universal property essentially, following the same line of argument in [AB14, Lemma 3.15]. Applying [AB14, Theorem 2.13] to the functor $A \to \mathcal{C}$ of symmetric monoidal ∞-categories, we find the following: maps $T_\rho \to R$ in $\mathrm{CAlg}(\mathcal{C})$ are equivalent to lax symmetric monoidal lifts in the diagram

$$\begin{tikzcd} & & \mathcal{C}_{/R} \arrow[d] \\ A \arrow[r] \arrow[urr, dotted] & \mathrm{Pic}(\mathcal{C}) \arrow[r] & \mathcal{C}. \end{tikzcd}$$

The objects of $\mathcal{C}_{/R}$ are maps $N \to R$, with symmetric monoidal product given by

$$(N \to R) \otimes (M \to R) \simeq (N \otimes M \to R \otimes R \to R).$$

The monoidal unit is the map $\mathbb{I} \to R$. There is a symmetric monoidal functor $\mathcal{C}_{/R} \to \mathcal{C}$, given by forgetting the structure map, and a symmetric monoidal functor $\mathcal{C}_{/R} \to (\mathrm{LMod}_R)_{/R}$, given by $(L \to R) \mapsto (R \otimes L \to R)$.

However, since A is grouplike the image $L \to R$ of any object must be contained in the invertible objects of $\mathcal{C}_{/R}$. This implies first that L is an invertible object of $\mathcal{C}$. This implies second that $R \otimes L \to R$ is an invertible object of $(\mathrm{LMod}_R)_{/R}$; this happens only when this adjoint structure map is an equivalence. Conversely, if $L \to R$ is a map whose adjoint $R \otimes L \to R$ is an equivalence, tensoring with L^{-1} gives an equivalence of R-modules $R \to R \otimes L^{-1}$ of left R-modules, whose inverse is adjoint to a map $L^{-1} \to R$. □

Example 25. The trivializing algebra for the map $2\mathbb{Z} \to \mathfrak{pic}(MU)$ is the periodic MU-spectrum MUP, whose coefficient ring is $\pi_* MU[u^{\pm 1}]$ for a generator u in degree 2. It is universal among commutative algebras R with a nullhomotopy of the composite $ku \to \mathfrak{pic}(\mathbb{S}) \to \mathfrak{pic}(R)$. The algebra MUP is often useful for translating between even-periodic and $\mathbb{Z}/2$-graded interpretations in chromatic theory.

5 Root obstructions

The symmetric monoidal functor from C to its homotopy category hC induces a map of Picard spectra $\mathfrak{pic}(C) \to \mathfrak{pic}(hC)$, identifying $\mathfrak{pic}(hC)$ with the first nontrivial stage $\tau_{\leq 1} \mathfrak{pic}(C)$ in the Postnikov tower for $\mathfrak{pic}(C)$. For us to lift a map $\bar{\rho}\colon A \to \pi_0 \mathfrak{pic}(C)$ to the first Postnikov stage $\mathfrak{pic}(hC)$, it is necessary and sufficient that the composite map

$$A \xrightarrow{\bar{\rho}} \pi_0 \mathfrak{pic}(C) \xrightarrow{k} \Sigma^2 \pi_1 \mathfrak{pic}(C)$$

is trivial. The result is an obstruction class: an element of $[A, \Sigma^2 \pi_1 \mathfrak{pic}(C))]$, and a lift exists if and only if this obstruction vanishes. Two different choices of lift to a map $A \to \mathfrak{pic}(hC)$ are represented by homotopy classes of maps $A \to \Sigma\pi_1 \mathfrak{pic}(C)$.

The identification of the first k-invariant with the twist homomorphism τ leads us to the following definition [Rez09].

Definition 26. An element $\gamma \in \pi_0 \mathfrak{pic}(C)$ is *symmetric* if $\tau(\gamma) = \mathrm{id}$ in $\pi_1 \mathfrak{pic}(C)$.

This allows us to conclude the following.

Proposition 27. *A map $\bar{\rho}\colon A \to \pi_0 \mathfrak{pic}(C)$ lifts to a map $A \to \mathfrak{pic}(hC)$ if and only if the elements $\bar{\rho}(a)$ are symmetric for all $a \in A$. Two such lifts determine a difference class in* $\mathrm{Ext}(A, \pi_1 \mathfrak{pic}(hC))$.

Example 28. Suppose that L is an invertible object such that there is an equivalence $v\colon \mathbb{I} \to L^{\otimes n}$. This equivalence v gives rise to a commutative diagram of symmetric monoidal ∞-categories

$$\begin{tikzcd} F(L^{\otimes n}) \arrow[r] \arrow[d] & F(L) \arrow[d] \\ * \arrow[r] & \mathrm{Pic}(C), \end{tikzcd}$$

where $F(x)$ is the free E_∞-space on an object named x. Taking associated spectra gives a diagram

$$\begin{tikzcd} \mathbb{S} \arrow[r, "n"] \arrow[d] & \mathbb{S} \arrow[d] \\ * \arrow[r] & \mathfrak{pic}(C), \end{tikzcd}$$

or equivalently a map $\mathbb{S}/n \to \mathfrak{pic}(C)$. Conversely, there is a short exact sequence

$$0 \to \pi_1 \mathfrak{pic}(C)/[\pi_1 \mathfrak{pic}(C)]^n \to [\mathbb{S}/n, \mathfrak{pic}(C)] \to \pi_0 \mathfrak{pic}(C)[n] \to 0.$$

The quotient expresses that a map from $\mathbb{S}/n$ determines an underlying n-torsion object L in $\pi_0 \mathfrak{pic}(C)$. The kernel expresses that a map from $\mathbb{S}/n$ expresses that two different maps representing the same object L may differ in their choice of equivalence $v\colon \mathbb{I} \to L^{\otimes n}$ (modulo n'th powers).

If there is a symmetric monoidal functor $f\colon C \to \mathcal{D}$ such that there is an equivalence $u\colon \mathbb{I} \to f(L)$ in $\mathcal{D}$, then the map $\mathbb{S} \to \mathfrak{pic}(\mathcal{D})$ determining $f(L)$ becomes trivial.

However, the extended map $\mathbb{S}/n \to \mathfrak{pic}(\mathcal{D})$ does not always become trivial: it becomes trivial precisely when there exists a choice of $u\colon \mathbb{I} \to f(L)$ in $\mathcal{D}$ such that $u^n = f(v)$.

The bottom homotopy group of $\mathbb{S}/n$ is $\mathbb{Z}/n$. The map $\mathbb{S}/n \to \mathfrak{pic}(hC)$ extends to a map $\mathbb{Z}/n \to \mathfrak{pic}(hC)$ if and only if L is symmetric, and a strict $\mathbb{Z}/n$-grading would be an extension to a map $\mathbb{Z}/n \to \mathfrak{pic}(C)$. In rough, we can think of this in the following way. If we have a strict $\mathbb{Z}/n$-grading, then it is a rigid version of choosing an object L with an equivalence $v\colon \mathbb{I} \to \mathbb{L}^{\otimes n}$; the trivializing algebra T then extracts an n'th root of this *chosen* equivalence v.

Example 29. Suppose that R is a commutative ring spectrum which is $2n$-periodic: there is a unit $v \in \pi_{2n}R$. This determines a symmetric element $\Sigma^2 R$, and a lift of $\Sigma^2 R$ to a strict n-torsion object allows us to construct an R-algebra whose $\mathbb{Z}$-graded coefficient ring is

$$\pi_* R[x]/(x^n - av)$$

for some well-defined $a \in (\pi_0 R)^\times/[(\pi_0 R)^\times]^n$. If n is a unit in $\pi_0 R$ then we also find that such algebras can be constructed, essentially uniquely, for any value of a. These extensions appear, for example, when relating completed Johnson–Wilson spectra to Lubin–Tate spectra [LN12, §4].

6 Grading extensions

In this section, we will begin the process of extending gradings by adjoining formal radicals to elements in the Picard group. We fix a symmetric monoidal presentable ∞-category C, and let $\mathrm{Pic}(C)$ be the Picard space of invertible elements in C.

Definition 30. Let

$$0 \to \pi_0 \mathfrak{pic}(C) \to \Gamma \to B \to 0$$

be an extension of abelian groups, represented by a map

$$\bar{\rho}\colon\ B \to \Sigma\pi_0 \mathfrak{pic}(C)$$

between Eilenberg–Mac Lane spectra. A *extension to* Γ*-grading* is a lift of $\bar{\rho}$ to $\mathfrak{pic}(C)$:

$$\begin{tikzcd} & \Sigma\,\mathfrak{pic}(C) \arrow[d] \\ B \arrow[ur, "\rho"] \arrow[r, "\bar{\rho}"'] & \Sigma\pi_0\,\mathfrak{pic}(C) \end{tikzcd}$$

We refer to the fiber of ρ as the *extended Picard spectrum* $\mathfrak{pic}(C, \rho)$ associated to ρ, and the associated infinite loop space as the *extended Picard group* $\mathrm{Pic}(C, \rho)$.

The extended Picard spectrum is part of a fiber sequence

$$\mathfrak{pic}(C) \to \mathfrak{pic}(C, \rho) \to B,$$

and on π_0 this realizes the extension $0 \to \pi_0 \mathfrak{pic}(C) \to \Gamma \to B \to 0$. We get a decomposition $\mathrm{Pic}(C, \rho) \cong \coprod_{b \in B} \mathrm{Pic}(C) \cdot \{b\}$ as spaces with an action of $\mathrm{Pic}(C)$.

Remark 31. Again, the map ρ determines $\bar{\rho}$ and the extension Γ.

By definition, there is an inclusion $i\colon \operatorname{Pic}(C) \subset C$ of symmetric monoidal ∞-categories. The latter is presentable, whereas the former is (essentially) small. By formally adjoining colimits to C, we obtain a factorization

$$\operatorname{Pic}(C) \to \mathcal{P}(\operatorname{Pic}(C)) \to C$$

through the presheaf ∞-category, where the second functor preserves colimits.

Proposition 32. *For a symmetric monoidal ∞-category C, there is a diagram*

$$\mathcal{P}(\operatorname{Pic}(C,\rho)) \leftarrow \mathcal{P}(\operatorname{Pic}(C)) \to C$$

of symmetric monoidal presentable ∞-categories.

Proof. Fix a regular cardinal κ such that the unit of C is κ-compact. Then all of the objects of $\operatorname{Pic}(C)$ are contained inside the essentially small subcategory C^κ of κ-compact objects.

The functor $\mathcal{P}$ is the left adjoint in an adjunction between κ-small ∞-categories and κ-presentable ∞-categories; $(-)^\kappa$ is the right adjoint. Moreover, the tensor product of presentable ∞-categories is universal with respect to functors that are colimit-preserving in each variable separately; in particular, this gives us canonical identifications

$$\operatorname{Fun}^{PrL}(\mathcal{P}(\Pi S_i), C) \simeq \operatorname{Fun}^{PrL}(\otimes \mathcal{P}(S_i), C)$$

natural in C. This makes the functor $\mathcal{P}$ strong symmetric monoidal, and lifts it to a left adjoint to the functor taking a κ-presentable symmetric monoidal ∞-category to the symmetric monoidal subcategory of κ-compact objects. For a small symmetric monoidal ∞-category S, the induced symmetric monoidal structure on the category $\mathcal{P}(S)$ is given by left Kan extension: this is the Day convolution monoidal structure [Gla16, Lur17]. It is colimit-preserving in each variable, and for objects s and t of S with associated presheaves j_s and j_t there is a natural isomorphism $j_s \otimes j_t \cong j_{s\otimes t}$.

The Day convolution makes the functor $\mathcal{P}(\operatorname{Pic}(C)) \to \mathcal{P}(\operatorname{Pic}(C,\rho))$ symmetric monoidal, and the adjunction gives us a composite symmetric monoidal functor

$$\mathcal{P}(\operatorname{Pic}(C)) \to \mathcal{P}(C^\kappa) \to C$$

as desired. □

Definition 33. Suppose that ρ is an extension to Γ-grading. We define the category obtained by *extending gradings to* Γ to be the symmetric presentable ∞-category

$$C[\rho] = \mathcal{P}(\operatorname{Pic}(C,\rho)) \otimes_{\mathcal{P}(\operatorname{Pic}(R))} C.$$

Proposition 34. *As a presentable category left-tensored over C, we have*

$$C[\rho] \cong \prod_{b\in B} C.$$

In particular, $C[\rho]$ is isomorphic to the category of B-graded objects of C.

Proof. Since $\mathrm{Pic}(C,\rho) \cong \coprod_{b\in B} \mathrm{Pic}(C)$ as categories left-tensored over $\mathrm{Pic}(C)$,

$$\mathcal{P}(\mathrm{Pic}(C,\rho)) \cong \coprod_{b\in B}^{pres} \mathcal{P}(\mathrm{Pic}(C))$$

as presentable ∞-categories left-tensored over $\mathcal{P}(C)$—here the coproduct taking place within presentable ∞-categories. The relative tensor product preserves colimits in each variable, and thus we have

$$C[\rho] \simeq \coprod_{b\in B}^{pres} C$$

as categories left-tensored over C. However, within presentable ∞-categories, coproducts and products over a small index set coincide. □

One source of grading extensions is the theory of strict gradings.

Proposition 35. *Suppose that $0 \to G \to \Gamma \to B \to 0$ is an extension of abelian groups. Then every strict G-grading of C has a naturally associated extension to a Γ-grading.*

Proof. The extension Γ is classified by a map $B \to \Sigma G$, which can be composed with the strict G-grading $G \to \mathfrak{pic}(C)$. □

Example 36. Since MO has a strict $\mathbb{Z}$-grading, for any MO-algebra R we can then adjoin invertible objects L to the category of MO-modules such that $L^{\otimes n} \simeq \Sigma MO$, or extend to any grading $\mathbb{Z} \subset \Gamma$. For example, we can embed the category of MO-modules into the category of $\mathbb{Q}/\mathbb{Z}$-graded MO-modules, giving the latter a symmetric monoidal structure where shifts by integers are extended to shifts by rational numbers.

Example 37. Similarly, the strict $2\mathbb{Z}$-grading on MU allow us to adjoin invertible objects L such that $L^{\otimes n} \cong \Sigma^2 MU$. (Note that if we take $n = 2$ in this construction, we find that the object $\Sigma^{-1}L$ is a nontrivial object with $(\Sigma^{-1}L)^{\otimes 2} \cong MU$.) This allows us to extend from a $\mathbb{Z}$-grading on MU-modules to a grading over $\mathbb{Z} \oplus_{2\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q} \times \mathbb{Z}/2$. Similar constructions are possible with MU-algebras like ku or with the algebraic K-theory spectrum $K\mathbb{C}$.

7 Grading obstructions

As in §5, we can identify $\mathfrak{pic}(hC)$ with the first nontrivial stage $\tau_{\le 1}\mathfrak{pic}(C)$ in the Postnikov tower for $\mathfrak{pic}(C)$. For us to lift a map $\bar{\rho}\colon B \to \Sigma\pi_0\,\mathfrak{pic}(C)$ to the first Postnikov stage, it is necessary and sufficient that the associated obstruction

$$B \xrightarrow{\bar{\rho}} \Sigma\pi_0\,\mathfrak{pic}(C) \xrightarrow{k} \Sigma^3\pi_1\,\mathfrak{pic}(C)$$

is trivial. Two different choices of lift to a map $B \to \Sigma\,\mathfrak{pic}(hC)$ differ by an element of $[B \to \Sigma^2\pi_1\,\mathfrak{pic}(C)]$.

This can be concisely packaged into the following result.

Proposition 38. *Given an extension $0 \to \pi_0 \mathfrak{pic}(C) \to \Gamma \to B \to 0$, the lifts of the map $\bar{\rho}\colon B \to \Sigma\pi_0 \mathfrak{pic}(C)$ to a map $\rho_{\leq 1}\colon B \to \Sigma \mathfrak{pic}(hC)$ are in bijective correspondence with extensions of the twist homomorphism $\tau\colon \pi_0 \mathfrak{pic}(C) \to \pi_1 \mathfrak{pic}(C)[2]$ to all of Γ.*

Example 39. This construction can recover the twisted $\mathbb{Z}/2$-graded categories of Rezk [Rez09, §2].[5] Let C be an ordinary presentable symmetric monoidal category and ω an invertible object of C. We would like to construct a larger category $C[\sqrt{\omega}]$ where ω is the square of another invertible module. This is would be a symmetric monoidal category of $\mathbb{Z}/2$-graded objects (A_0, A_1) of C, representing $A_0 \oplus (\sqrt{\omega} \otimes A_1)$, with a tensor product satisfying

$$(A_0, A_1) \otimes (B_0, B_1) \cong \Big((A_0 \otimes B_0) \oplus (\omega \otimes A_1 \otimes B_1), (A_0 \otimes B_1) \oplus (A_1 \otimes B_0)\Big).$$

In this case, the underlying category is ordinary and so $\mathfrak{pic}(C) = \mathfrak{pic}(hC)$. For us to extend gradings, it is necessary and sufficient that ω be symmetric. If this is the case, the possible choices of extension represent choices of 2-torsion element $\tau(\sqrt{\omega}) \in \mathrm{Aut}_C(\mathbb{I})$, a Koszul sign rule for the twist isomorphism on the square root of ω. If C is additive, choosing $\tau(\sqrt{\omega}) = -1$ recovers Rezk's *ω-twisted tensor product*.

8 Brauer roots

Just as the unit spectrum is extended by the Picard spectrum, the Picard spectrum is extended by the Brauer spectrum [GL16, Hau17]. Fix a symmetric monoidal presentable ∞-category C and let Cat_C be the category of presentable ∞-categories left-tensored over C. This has a symmetric monoidal under $\otimes_C$. We write $\mathrm{Br}(C) = \mathrm{Pic}(\mathrm{Cat}_C)$ for the Brauer space parametrizing invertible objects of Cat_C that admit a compact generator, and $\mathfrak{br}(C)$ for the associated spectrum.

The unit of Cat_C is C itself, and all C-linear functors are of the form $X \mapsto X \otimes B$ for some $B \in C$; in particular, this identifies the C-linear functors $C \to C$ with $C^{\simeq}$ itself, with composition given by the tensor in C. As a result, the space of self-equivalences of the unit is $\mathrm{Pic}(C)$, and so there is a fiber sequence

$$\Sigma \mathfrak{pic}(C) \to \mathfrak{br}(C) \to \pi_0 \mathfrak{br}(C).$$

Definition 40. Suppose that C is a presentable symmetric monoidal ∞-category, B is an abelian group regarded as a discrete symmetric monoidal category, and that $\rho\colon B \to \mathrm{Cat}(C)$ is a symmetric monoidal functor. We define the *trivializing category* $\mathcal{T}_\rho$ to be the homotopy colimit of ρ, calculated in Cat_C.

Remark 41. Since B is a discrete category, there is an equivalence in Cat_C of the form

$$\mathcal{T}_\rho \simeq \coprod_{b \in B}^{\mathrm{Cat}_C} \rho(b) \simeq \prod_{b \in B} \rho(b).$$

[5]Rezk gives these constructions in the case where C is merely additive, which is not covered by our assumption that C is presentable.

Proposition 42. *Suppose that B is an abelian group and that $\rho\colon B \to C$ is a symmetric monoidal functor with trivializing category $\mathcal{T}_\rho$.*

1. *The map ρ has a natural lift to a map $B \to \mathrm{Br}(C)$.*
2. *$\mathcal{T}_\rho$ has a natural lift to a symmetric monoidal ∞-category under C.*
3. *The symmetric monoidal ∞-category $\mathcal{T}_\rho$ is universal among symmetric monoidal presentable ∞-categories under C with a chosen commutative diagram*

$$\begin{tikzcd} B \arrow[r] \arrow[d] & \mathfrak{br}(C) \arrow[d] \\ * \arrow[r] & \mathfrak{br}(\mathcal{T}_\rho). \end{tikzcd}$$

In particular, for any $b \in B$ there is an equivalence $\phi_b\colon \mathcal{T}_\rho \to \mathcal{T}_\rho \otimes_C \rho(b)$ of presentable ∞-categories left-tensored over $\mathcal{T}_\rho$, and there are chosen equivalences of functors $\phi_a \otimes 1 \circ \phi_b \xrightarrow{\sim} \phi_{ab}$.

Proof. This is Proposition 24, applied to the (large) ∞-category of presentable ∞-categories. □

Example 43. Let $C = \mathrm{LMod}_R$ where R is a commutative ring spectrum. Then each object of $\mathrm{Br}(R)$ is represented by the category of right modules over an Azumaya R-algebra Q which is well-defined up to Morita equivalence [BRS], [GL16, 5.13]. Given a symmetric monoidal functor $\rho\colon B \to \mathrm{Br}(R)$, we can therefore choose algebras $Q(b)$ so that there is an equivalence

$$\mathcal{T}_\rho \simeq \prod_{b\in B} \mathrm{RMod}_{Q(b)}\,.$$

The symmetric monoidal structure on this category takes more work to describe. The symmetric monoidal structure on ρ gives Morita equivalences between $Q(p) \otimes_R Q(q)$ and $Q(p+q)$, which are expressed by $Q(p) \otimes_R Q(q) \otimes_R Q(p+q)^{op}$-modules $M_{p,q}$. The symmetric monoidal structure is given by a formula of the form

$$(X_b)_{b\in B} \otimes (Y_b)_{b\in B} \cong \left(\bigoplus_{p+q=a} (X_p \otimes Y_q) \underset{Q(p)\otimes Q(q)}{\otimes} M_{p,q}\right)_{b\in B}.$$

However, describing the full symmetric monoidal structure in this fashion would require us to carefully express coherence relations between tensor products of the bimodules $M_{p,q}$. We can think of these objects as “coefficients” for the multiplication on this product category.

Remark 44. The first k-invariant in the Brauer spectrum is a map $\pi_0\,\mathfrak{br}(C) \to \Sigma^2\pi_0\,\mathfrak{pic}(C)$, and this is still expressed by the twist self-equivalence. However, now this twist self-equivalence occurs on the level of module categories. For a commutative ring spectrum R with an Azumaya R-algebra Q, the twist equivalence of $\mathrm{RMod}_{Q\otimes_R Q}$ is expressed by tensoring with the a $Q \otimes_R Q$-bimodule $Q \otimes_R Q^\tau$, where the action on the left is the standard one and on the right factors through the twist automorphism. The fact that $Q \otimes_R Q$ is Azumaya means that this bimodule must be of the form $Q \otimes_R Q \otimes_R \tau(Q)$ for some $\tau(Q) \in \mathrm{Pic}(R)$.[6]

More concretely, we can identify $\tau(Q)$ with the R-module

$$F_{Q\otimes_R Q\text{-bimod}}(Q \otimes_R Q, (Q \otimes_R Q)^\tau).$$

This assignment remains relatively mysterious to the author. In particular, it is not clear whether there are algebraic examples where it is nontrivial.

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$$1 \to k^\times \to T^\times \to \mathrm{Aut}_{\mathrm{Alg}(k)}(T) \to \pi_0\, \mathfrak{pic}(k)$$

that expresses the potential failure of algebra automorphisms to be inner. If k is a field this recovers the Noether–Skolem theorem.

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