KHOVANOV SPECTRA FOR TANGLES

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Abstract. We define stable homotopy refinements of Khovanov's arc algebras and tangle invariants.

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1. Introduction

1.1. **Context.** Quantum topology began in the 1980s with the Jones polynomial [Jon85], and Witten's reinterpretation of it via Yang-Mills theory [Wit89]. Witten's work was at a physical level of rigor, but Atiyah [Ati90], Reshetikhin-Turaev [RT91], and others introduced mathematically rigorous definitions of topological field theories and related them to both the Jones polynomial and deep questions in representation theory.

Around the same time, topological field theories also began to appear in dimension 4, in the work of Donaldson [Don90], Floer [Flo88], and others. Unlike the Jones polynomial, these 4-dimensional invariants all required partial differential equations to define. (Curiously, while Donaldson's and Floer's invariants were archetypal examples for what Witten called topological field theories [Wit88], they do not satisfy the axioms mathematicians came to insist on for topological field theories.) The connection between these invariants and representation theory was also less apparent.

In the 1990s, Crane-Frenkel proposed that the Jones polynomial and its siblings might be extended to 4-dimensional topological field theories via "a categorical version of a Hopf algebra" [CF94]. Inspired by this suggestion Khovanov categorified the Jones polynomial [Kho00], and Rasmussen showed that this categorification could be used to study smooth knot concordance and deduce the existence of exotic smooth structures on $\mathbb{R}^4$ without recourse to gauge theory [Ras10].

Answering a question of Khovanov's, Jacobsson proved that Khovanov homology extends to a (3+1)-dimensional topological field theory [Jac04]. His proof, which involved explicitly checking the myriad *movie moves* relating different movie presentations of a surface, was long and intricate. Khovanov [Kho02, Kho06] and, independently, Bar-Natan [Bar05] gave simpler proofs of functoriality of Khovanov homology, by extending it downwards, to tangles (as Reshetikhin-Turaev had done for the Jones polynomial). (Their tangle invariants are different, and since then several more Khovanov homology invariants of tangles have also been given [APS06, CK14, BS11, Rob16].) This tangle invariants also led to categorifications of quantum groups [Lau10, KL09, Rou], and tensor products of representations [Web16], and many other interesting advances.

Returning to gauge theory and related invariants, in the 1990s, Cohen-Jones-Segal proposed a program to give stable homotopy refinements of Floer homology groups, in certain cases [CJS95], though this program has yet to be carried out rigorously. Using other techniques, stable homotopy refinements have been given for certain Floer homologies [Man03, KM, Coh10, Kra]. The Cohen-Jones-Segal program is in two steps: first they use the Floer data to build a *framed flow category*, and then they use the framed flow category to build a space; it is the first step for which technical difficulties have not yet been resolved.

In a previous paper, we built a framed flow category combinatorially and then used the second step of the Cohen-Jones-Segal program to define a Khovanov stable homotopy type [LS14a]. Hu-Kriz-Kriz gave another construction of a Khovanov stable homotopy type, using the Elmendorf-Mandell infinite loop space machine [HKK16]. In another previous paper we were able to show that these two constructions give equivalent invariants [LLS17]. Hu-Kriz-Kriz's construction factors through the embedded cobordism category of $(\mathbb{R}^2, [0,1] \times \mathbb{R}^2)$, a point that will be important in our construction of tangle invariants.

Computations show that this lift is strictly stronger than Khovanov homology [LS14c, See] and can be used to give additional concordance information [LS14b, LLS17]. (A space-level lift of Khovanov homology which does not have more information than Khovanov homology was given by Everitt-Turner [ET14, ELST16].)

We would like to use the Khovanov homotopy type to study smoothly embedded surfaces in $\mathbb{R}^4$. Following Khovanov and Bar-Natan, as a step towards this goal, in this paper we construct an extension of the Khovanov stable homotopy type to tangles.

Remark 1.1. Hu-Kriz-Somberg have outlined a construction of a stable homotopy type refining $\mathfrak{sl}_n$ Khovanov-Rozansky homology [HKS]. Their construction passes through *oriented tangles*, i.e., tangles in $[0,1]\times\mathbb{D}^2$ every strand of which runs from $\{0\}\times\mathbb{D}^2$ to $\{1\}\times\mathbb{D}^2$. At the time of writing, their construction is restricted to a homotopy type localized at a "large" prime p (depending on n).

1.2. **Statement of results.** In this paper, we give two extensions of two extensions of the Khovanov homotopy type to tangles. The first is combinatorial, and has the form of a multifunctor $\underline{\mathsf{MB}}_T$ from a particular multicategory to the Burnside category. The functor $\underline{\mathsf{MB}}_T$ is well-defined up to a notion of stable equivalence (Theorem 3). (For the special case of knots, this essentially reduces to the combinatorial invariant described in a previous paper [LLS].) To summarize:

Theorem 1. *Given a $(2m,2n)$-tangle T with N crossings, there is an associated multifunctor*

$$\underline{\mathsf{MB}}_T\colon \underline{2}^N \tilde{\times}_m\mathcal{T}_n \to \underline{\mathscr{B}}.$$

Up to stable equivalence, $\underline{\mathsf{MB}}_T$ is an invariant of the isotopy class of T. The composition of $\underline{\mathsf{MB}}_T$ with the forgetful map $\underline{\mathscr{B}}\to\underline{\mathsf{Ab}}$ is identified with Khovanov's tangle invariant [Kho02].

(This is restated and proved as Lemma 3.19 and Theorem 3, below.)

Next, we use the Elmendorf-Mandell machine to define a spectral category (category enriched over spectra) $\mathscr{H}^m$ so that the homology of $\mathscr{H}^m$ is the Khovanov arc algebra H^m. We then turn $\underline{\mathsf{MB}}_T$ into a (spectral) bimodule $\mathscr{X}(T)$ over $\mathscr{H}^m$ and $\mathscr{H}^n$, so that the singular chain complex of $\mathscr{X}(T)$ is quasi-isomorphic, as a complex of (H^m,H^n)-bimodules, to the Khovanov tangle invariant of T. We then prove:

Theorem 2. *Up to equivalence of $(\mathscr{H}^m,\mathscr{H}^n)$-bimodules, $\mathscr{X}(T)$ is an invariant of the isotopy class of T. Further, given an $(2n,2p)$-tangle T',*

$$\mathscr{X}(T'\circ T)\simeq \mathscr{X}(T)\otimes^{\mathbb{L}}_{\mathscr{H}^n}\mathscr{X}(T')$$

(where tensor product denotes the tensor product of module spectra).

(This is restated and proved as Theorems 4 and 5, below.)

The outline of the construction is as follows:

(1) We construct a multicategory $\underline{\widetilde{\mathsf{Cob}_d}}$ enriched in groupoids of *divided cobordisms* so that:
   (a) there is at most one 2-morphism between any pair of morphisms in $\underline{\widetilde{\mathsf{Cob}_d}}$;
   (b) the Khovanov-Burnside functor V_{HKK} from the embedded cobordism category to the Burnside category induces a functor $\underline{V}_{HKK}$ from $\underline{\widetilde{\mathsf{Cob}_d}}$ to the Burnside category; and
   (c) the cobordisms involved in the Khovanov arc algebras and tangle invariants have (essentially canonical) representatives in $\underline{\widetilde{\mathsf{Cob}_d}}$.

   (Sections 3.1 and 3.2.3.)
(2) We define an *arc-algebra shape multicategory* $\mathcal{H}_n^0$ and *tangle shape multicategory* ${}_m\mathcal{T}_n^0$ so that the Khovanov arc algebras and tangle invariants are equivalent to multifunctors $\mathcal{H}_n^0\to\underline{\mathsf{Ab}}$ and ${}_m\mathcal{T}_n^0\to\underline{\mathsf{Kom}}$. There are also groupoid-enriched versions of $\mathcal{H}_n$ and ${}_m\mathcal{T}_n$, and projection maps $\mathcal{H}_n\to\mathcal{H}_n^0$, ${}_m\mathcal{T}_n\to{}_m\mathcal{T}_n^0$. (Sections 2.3 and 3.2.2.)
(3) The functor $\mathcal{H}_n^0\to\underline{\mathsf{Ab}}$ factors through a functor $\mathcal{H}_n\to\underline{\widetilde{\mathsf{Cob}_d}}$. Similarly, the tangle invariant ${}_m\mathcal{T}_n^0\to\underline{\mathsf{Kom}}$ factors through a functor $\underline{2}^N\tilde{\times}_m\mathcal{T}_n\to\underline{\widetilde{\mathsf{Cob}_d}}$ from (an appropriate kind of) product of ${}_m\mathcal{T}_n$ and a cube. So, we can compose with $\underline{V}_{HKK}$ to get functors $\underline{\mathsf{MB}}_n\colon\mathcal{H}_n\to\underline{\mathscr{B}}$ and $\underline{\mathsf{MB}}_T\colon\underline{2}^N\tilde{\times}_m\mathcal{T}_n\to\underline{\mathscr{B}}$. We also digress to note that we can view $\underline{\mathsf{MB}}_T$ as a tangle invariant in an appropriate derived category. (Section 3.5.)

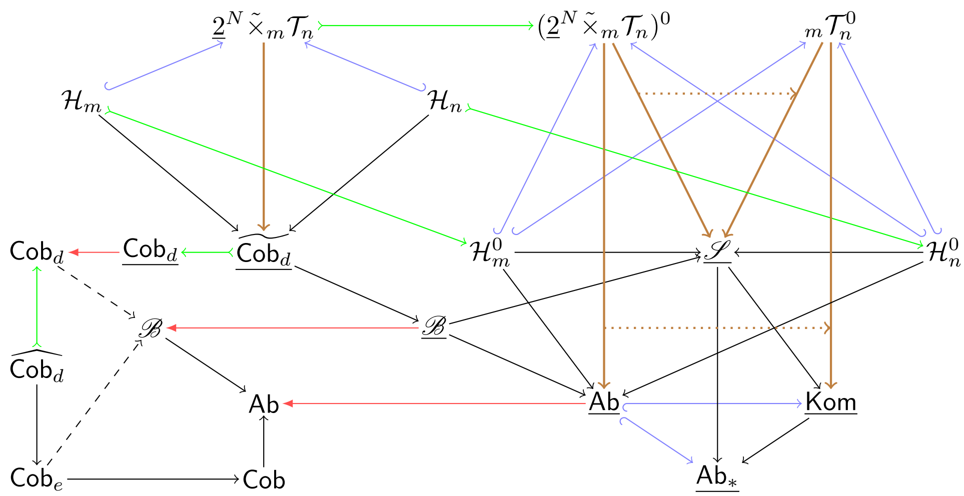


FIGURE 1.1. **The outline of the construction.** We construct the above diagram starting with a $(2m, 2n)$-tangle diagram T. Hook-tailed arrows are subcategory inclusions, split-tailed arrows are strictifications from groupoid enriched multicategories to ordinary multicategories, and solid-headed arrows convert a multicategory to an ordinary category by forgetting multimorphisms. Only the thick arrows depend on the tangle T. Solid arrows are strict, while the dashed arrows are lax. The two dotted arrows are functors between functor categories, $\underline{\mathscr{S}}^{(\underline{2}^N \tilde{\times}_m \mathcal{T}_n)^0} \to \underline{\mathscr{S}}^{{}_m\mathcal{T}_n^0}$ and $\underline{\mathsf{Ab}}^{(\underline{2}^N \tilde{\times}_m \mathcal{T}_n)^0} \to \underline{\mathsf{Kom}}^{{}_m\mathcal{T}_n^0}$ (their only dependence on the tangle is in an overall grading shift). The diagram commutes, with the understanding that anything involving the strictification arrows only commute up to (zigzags of) natural equivalences, and arrows to $\underline{\mathsf{Kom}}$ only commute up to quasi-isomorphisms. The picture does not encompass the quantum gradings.

(4) Using the Elmendorf-Mandell K-theory machine and rectification results, we can turn $\underline{\mathsf{MB}}_n$ and $\underline{\mathsf{MB}}_T$ into functors $\mathcal{H}_n^0 \to \underline{\mathscr{S}}$ and ${}_m\mathcal{T}_n^0 \to \underline{\mathscr{S}}$. We reinterpret these functors as a spectral category and spectral bimodule, respectively. Whitehead's theorem combined with familiar invariance arguments implies that the functor ${}_m\mathcal{T}_n^0 \to \underline{\mathscr{S}}$ is a tangle invariant. (Section 4.)

(5) The gluing theorem for tangles follows by considering a map from a larger multicategory to $\widetilde{\underline{\mathsf{Cob}_d}}$; the corresponding result for the Khovanov bimodules; projectivity (sweetness) of the Khovanov bimodules; and, again, a version of Whitehead's theorem. (Section 5.)

We precede these constructions with a review of Khovanov's tangle invariants and some algebraic topology background (Section 2), and follow it with some modest structural applications (Section 7). We concentrate the discussion of quantum gradings in Section 6.

The outline of the construction is summarized by Figure 1.1. The partial diagrams at the bottom of the pages, starting on page 6 track the progress of our construction.

Remark 1.2. To construct both the combinatorial and topological tangle invariants, we use the language of multicategories. There is another construction of a combinatorial invariant with at least as much information, using the language of enriched bicategories (cf. [GS16]); we may return to this point in a future paper.

Acknowledgments. We thank Finn Lawler for pointing us to [GS16] and Andrew Blumberg and Aaron Royer for helpful conversations.

2. Background

2.1. **Homological grading conventions.** In this paper, we will work with chain complexes. We view cochain complexes as chain complexes by negating the grading. In particular, the Khovanov complex was originally defined as a cochain complex [Kho00], but we will view it as a chain complex. So, our homological gradings differ from Khovanov's by a sign.

2.2. **Multicategories.**

Definition 2.1. A *multicategory* (or *colored operad*) $\mathscr{C}$ consists of:

(M-1) A set (or, more generally, class) $\mathrm{Ob}(\mathscr{C})$ of *objects*,
(M-2) For each $n \geq 0$ and objects $x_1, \dots, x_n, y \in \mathrm{Ob}(\mathscr{C})$ a set $\mathrm{Hom}(x_1, \dots, x_n; y)$ of *multimorphisms* from $(x_1, \dots, x_n)$ to y,
(M-3) a composition map

$$\mathrm{Hom}(y_1,\dots,y_n;z)\times\mathrm{Hom}(x_{1,1},\dots,x_{1,m_1};y_1)\times\cdots\times\mathrm{Hom}(x_{n,1},\dots,x_{n,m_n};y_n)\to\mathrm{Hom}(x_{1,1},x_{1,2},\dots,x_{n,m_n};z),$$

and
(M-4) A distinguished element $\mathrm{Id}_x \in \mathrm{Hom}(x; x)$, called the *identity* or *unit*,

satisfying the following conditions:

(M-5) Composition is associative, in the sense that the following diagram commutes:

$$\begin{array}{ccc}
\begin{array}{l}\mathrm{Hom}(y_1,\dots,y_n;z)\\ \times\prod_{i=1}^n\mathrm{Hom}(x_{i,1},\dots,x_{i,m_i};y_i)\\ \times\prod_{i=1}^n\prod_{j=1}^{m_i}\mathrm{Hom}(w_{i,j,1},\dots,w_{i,j,k_{i,j}};x_{i,j})\end{array} & \longrightarrow & \begin{array}{l}\mathrm{Hom}(x_{1,1},\dots,x_{n,m_n};z)\\ \times\prod_{i=1}^n\prod_{j=1}^{m_i}\mathrm{Hom}(w_{i,j,1},\dots,w_{i,j,k_{i,j};x_{i,j}})\end{array}\\
\downarrow & & \downarrow\\
\begin{array}{l}\mathrm{Hom}(y_1,\dots,y_n;z)\\ \times\prod_{i=1}^n\mathrm{Hom}(w_{i,1,1},\dots,w_{i,m_i,k_{i,m_i}};y_i)\end{array} & \longrightarrow & \mathrm{Hom}(w_{1,1,1},\dots,w_{n,m_n,k_{n,m_n}};z).
\end{array}$$

(Here, all of the maps are composition maps.)

(M-6) The identity elements are right identities for composition, in the sense that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Hom}(x_1,\dots,x_n;y) & \xrightarrow{=} & \mathrm{Hom}(x_1,\dots,x_n;y)\\
\Big\downarrow{\scriptstyle \mathrm{Id}\times\prod\mathrm{Id}_{x_i}} & \nearrow{\scriptstyle\circ} & \\
\mathrm{Hom}(x_1,\dots,x_n;y)\times\prod_{i=1}^n\mathrm{Hom}(x_i,x_i). & &
\end{array}$$

(M-7) The identity elements are left identities for composition, in the sense that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Hom}(x_1,\dots,x_n;y) & \xrightarrow{=} & \mathrm{Hom}(x_1,\dots,x_n;y)\\
\Big\downarrow{\scriptstyle \mathrm{Id}_y\times\mathrm{Id}} & \nearrow{\scriptstyle\circ} & \\
\mathrm{Hom}(y,y)\times\mathrm{Hom}(x_1,\dots,x_n;y). & &
\end{array}$$

Given multicategories $\mathscr{C}$ and $\mathscr{D}$, a *multifunctor* $F\colon \mathscr{C} \to \mathscr{D}$ is a map $F\colon \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{D})$ and, for each $x_1, \dots, x_n, y \in \mathrm{Ob}(\mathscr{C})$, a map $\mathrm{Hom}_{\mathscr{C}}(x_1, \dots, x_n; y) \to \mathrm{Hom}_{\mathscr{D}}(F(x_1), \dots, F(x_n); F(y))$ which respects multi-composition and identity elements.

Multicategories, which model the notion of multilinear maps, are a common generalization of a category (a multicategory in which only multimorphism sets with one input are nonempty) and an operad (a multicategory with one object). Multicategories were introduced by Lambek [Lam69] and Boardman-Vogt [BV73]. In Boardman-Vogt's work and most modern algebraic topology, the multimorphism sets in multicategories are equipped with actions of the symmetric group; the definition we have given would be called a non-symmetric multicategory. Some of our multicategories (notably $\underline{\mathscr{B}}$, Sets/X, and $\mathscr{S}$) are, in fact, symmetric multicategories. In particular, the multicategories Sets/X to which we apply Boardman-Vogt's K-theory are symmetric multicategories.

A monoidal category $(\mathscr{C}, \otimes)$ produces a multicategory, which we will denote $\underline{\mathscr{C}}$, as follows. The objects of $\underline{\mathscr{C}}$ are the same as the objects of $\mathscr{C}$, and the multimorphism sets are given by

$$\mathrm{Hom}_{\underline{\mathscr{C}}}(x_1, \dots, x_n; y) = \mathrm{Hom}_{\mathscr{C}}(x_1 \otimes \cdots \otimes x_n; y)$$

(for any choice of how to parenthesize the tensor product). If the monoidal category happened to be a symmetric monoidal category, as in the case of abelian groups Ab, graded abelian groups Ab_*, or chain complexes Kom, then the corresponding multicategory is a symmetric multicategory. (These are examples of Hu-Kriz-Kriz's $\star$-*categories* [HKK16].)

Many of our multicategories will be enriched in groupoids. That is, the multimorphism sets will be groupoids (i.e., categories in which all the morphisms are invertible) and the composition maps are maps of groupoids (i.e., functors).

Most of our non-enriched multicategories will be rather simple, in a sense we make precise:

Definition 2.2. Given a finite set X, the *shape multicategory of* X has objects $X \times X$, and the multimorphism set $\mathrm{Hom}((a_1, b_1), (a_2, b_2), \dots, (a_n, b_n); (b_0, a_{n+1}))$ consists of a single element if $b_i = a_{i+1}$ for all $0 \le i \le n$, and all other multimorphism sets empty. We allow the special case $n = 0$ which produces a unique zero-input multimorphism in $\mathrm{Hom}(\varnothing; (a, a))$ for each $a \in X$.

Generalizing Definition 2.2, we have the following variant.

Definition 2.3. Given a finite sequence of finite sets $X^1, \dots, X^k$, the *shape multicategory of* $(X^1, \dots, X^k)$ has objects $\coprod_{i \le j} X^i \times X^j$ and $\mathrm{Hom}((a_1, b_1), (a_2, b_2), \dots, (a_n, b_n); (b_0, a_{n+1}))$ consists of a single element if $b_i = a_{i+1}$ for all $0 \le i \le n$, and all other multimorphism sets empty. Once again, we allow the special case $n = 0$ which produces a unique zero-input multimorphism in $\mathrm{Hom}(\varnothing; (a, a))$ for each $a \in \coprod_i X^i$.

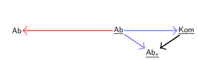
2.3. **Linear categories and multifunctors to abelian groups.** Many of the algebras that we will encounter in this paper will come equipped with an extra structure, which we abstract below.

Definition 2.4. *An algebra equipped with an orthogonal set of idempotents* is an algebra A and a finite subset $I \subset A$, so that

- $\iota^2 = \iota$ for all $\iota \in I$,
- $\iota\iota' = \iota'\iota = 0$ for all distinct $\iota, \iota' \in I$, and
- $\sum_{\iota \in I} \iota = 1$.

The following three notions are equivalent.

(1) A ring A (algebra over $\mathbb{Z}$) equipped with an orthogonal set of idempotents X.
(2) A linear category (category enriched over abelian groups Ab) with objects a finite set X.



(3) A multifunctor from the shape multicategory M of a finite set X to the multicategory $\underline{\mathsf{Ab}}$ of abelian groups.

(A similar statement holds for algebras over any ring R; the corresponding linear category has to be enriched over R-modules, and the corresponding multifunctor should map to the multicategory of R-modules.)

To see the equivalence, given a multifunctor $F\colon M \to \underline{\mathsf{Ab}}$ there is a corresponding linear category with objects X, $\mathrm{Hom}(x,y) = F((x,y))$, composition $\mathrm{Hom}(y,z)\otimes\mathrm{Hom}(x,y) \to \mathrm{Hom}(x,z)$ is the image of the unique morphism $(x,y),(y,z) \to (x,z)$, and the identity $\mathrm{Id}_x \in \mathrm{Hom}(x,x)$ is the image of 1 under the maps $\mathbb{Z} \to \mathrm{Hom}(x,x)$, which is the image under F of the unique morphism $\varnothing \to (x,x)$. Given a linear category $\mathscr{C}$ with finitely many objects, we can form a ring $A_{\mathscr{C}} = \bigoplus_{x,y\in\mathrm{Ob}(\mathscr{C})} \mathrm{Hom}_{\mathscr{C}}(x,y)$ with multiplication given by composition (i.e., $a\cdot b := b\circ a$) when defined and 0 otherwise; the ring $A_{\mathscr{C}}$ is equipped with the orthogonal set of idempotents $\{\mathrm{Id}_x \mid x \in \mathrm{Ob}(\mathscr{C})\}$.

In a similar fashion, given linear categories $\mathscr{C}$ and $\mathscr{D}$ with finitely many objects, the following are equivalent notions for bimodules.

(1) A left-$A_{\mathscr{C}}$ right-$A_{\mathscr{D}}$ bimodule B.
(2) An enriched functor $F_A\colon \mathscr{C}^{\mathrm{op}}\times\mathscr{D} \to \mathsf{Ab}$; an enriched functor between linear categories is one for which the map on morphisms $\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}\times\mathscr{D}}((c,d),(c',d')) \to \mathrm{Hom}_{\mathsf{Ab}}(F_A(c,d),F_A(c',d'))$ is linear, or equivalently, $\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}\times\mathscr{D}}((c,d),(c',d'))\times F_A(c,d) \to F_A(c',d')$ is bilinear.
(3) A multifunctor from the shape multicategory $M(\mathscr{C},\mathscr{D})$ of $(\mathrm{Ob}(\mathscr{C}),\mathrm{Ob}(\mathscr{D}))$ to $\underline{\mathsf{Ab}}$, which restricts to the multifunctors corresponding to $\mathscr{C}$, respectively $\mathscr{D}$, (as defined above) on the subcategory of $M(\mathscr{C},\mathscr{D})$ which is the shape multicategory of $\mathrm{Ob}(\mathscr{C})$, respectively $\mathrm{Ob}(\mathscr{D})$.

Recall from Definition 2.3 that $M(\mathscr{C},\mathscr{D})$ consists of the following data.

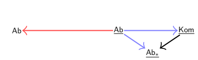
- Three kinds of objects:
  - Pairs $(x_1,x_2) \in \mathrm{Ob}(\mathscr{C})\times\mathrm{Ob}(\mathscr{C})$.
  - Pairs $(y_1,y_2) \in \mathrm{Ob}(\mathscr{D})\times\mathrm{Ob}(\mathscr{D})$.
  - Pairs (x,y) where $x\in\mathrm{Ob}(\mathscr{C})$ and $y \in \mathrm{Ob}(\mathscr{D})$. For notational clarity, we will write (x,y) instead as $(x,[B],y)$.
- A single multimorphism in each of the following cases:
  - $(x_1,x_2),(x_2,x_3),\dots,(x_{m-1},x_m) \to (x_1,x_m)$ where $x_1,\dots,x_m \in \mathrm{Ob}(\mathscr{C})$.
  - $(y_1,y_2),(y_2,y_3),\dots,(y_{n-1},y_n) \to (y_1,y_n)$ where $y_1,\dots,y_n \in \mathrm{Ob}(\mathscr{D})$.
  - $(x_1,x_2),\dots,(x_{m-1},x_m),(x_m,[B],y_1),(y_1,y_2),\dots,(y_{n-1},y_n) \to (x_1,[B],y_n)$ where $x_1,\dots,x_m \in \mathrm{Ob}(\mathscr{C})$ and $y_1,\dots,y_n\in\mathrm{Ob}(\mathscr{D})$.

The bimodule B defines a multifunctor $F_B\colon M(\mathscr{C},\mathscr{D}) \to \underline{\mathsf{Ab}}$ by:

- On objects, for $x_1,x_2 \in \mathrm{Ob}(\mathscr{C})$ and $y_1,y_2\in\mathrm{Ob}(\mathscr{D})$, $F_B(x_1,x_2) = \mathrm{Hom}_{\mathscr{C}}(x_1,x_2) = \mathrm{Id}_{x_1} A_{\mathscr{C}}\,\mathrm{Id}_{x_2}$, $F_B(y_1,y_2) = \mathrm{Hom}_{\mathscr{D}}(y_1,y_2) = \mathrm{Id}_{y_1} A_{\mathscr{D}}\,\mathrm{Id}_{y_2}$, and $F_B(x_1,[B],y_1) = \mathrm{Id}_{x_1} B\,\mathrm{Id}_{y_1}$.
- On the first and second types of multimorphisms, F_B is simply composition. For the third type, the map F_B sends the multimorphism
$$(x_1,x_2),\dots,(x_{m-1},x_m),(x_m,[B],y_1),(y_1,y_2),\dots,(y_{n-1},y_n) \to (x_1,[B],y_n)$$
to the product
$$\mathrm{Id}_{x_1} R_{\mathscr{C}}\,\mathrm{Id}_{x_2}\otimes\cdots\otimes\mathrm{Id}_{x_{m-1}} R_{\mathscr{C}}\,\mathrm{Id}_{x_m}\otimes\mathrm{Id}_{x_m} B\,\mathrm{Id}_{y_1}\otimes\mathrm{Id}_{y_1} R_{\mathscr{D}}\,\mathrm{Id}_{y_2}\otimes\cdots\otimes\mathrm{Id}_{y_{n-1}} R_{\mathscr{D}}\,\mathrm{Id}_{y_n} \to \mathrm{Id}_{x_1} B\,\mathrm{Id}_{y_n}.$$

Conversely, every multifunctor $M(\mathscr{C},\mathscr{D}) \to \underline{\mathsf{Ab}}$ of the given form arises as F_B for the bimodule $B = \bigoplus_{x\in\mathrm{Ob}(\mathscr{C}),y\in\mathrm{Ob}(\mathscr{D})} F_B(x,[B],y)$.

Similarly, given a multifunctor $F_B\colon M(\mathscr{C},\mathscr{D}) \to \underline{\mathsf{Ab}}$, we can construct an enriched functor $F_A\colon \mathscr{C}^{\mathrm{op}}\times\mathscr{D} \to \mathsf{Ab}$ by defining:



- On objects, $F_A(c,d) = F_B(c,[B],d)$.
- On morphisms, $\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}\times\mathscr{D}}((c,d),(c',d'))\otimes F_A(c,d)\to F_A(c',d')$ as the composition

$$\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}\times\mathscr{D}}((c,d),(c',d'))\otimes F_A(c,d) = F_B(c',c)\otimes F_B(c,[B],d)\otimes F_B(d,d')\to F_B(c',[B],d').$$

There are similar equivalences for the notions of differential $(A_{\mathscr{C}},A_{\mathscr{D}})$-bimodules, enriched functors $\mathscr{C}^{\mathrm{op}}\times\mathscr{D}\to\mathsf{Kom}$, and multifunctors $M(\mathscr{C},\mathscr{D})\to\underline{\mathsf{Kom}}$.

2.4. **Trees and canonical groupoid enrichments.** To define some enriched multicategories, we will first need some terminology about trees.

A *planar, rooted tree* is a tree $\Yup$ with some number $n\geq 1$ of leaves, which has been embedded in $\mathbb{R}\times[0,1]$ so that $k\leq n-1$ of the leaves are embedded in $\mathbb{R}\times\{0\}$, one leaf is embedded in $\mathbb{R}\times\{1\}$, and no other edges or vertices are mapped to $\mathbb{R}\times\{0,1\}$. The vertices mapped to $\mathbb{R}\times\{0\}$ are called *inputs* of $\Yup$ and the vertex mapped to $\mathbb{R}\times\{1\}$ is the *output* or *root* of $\Yup$. We call the remaining vertices of $\Yup$ *internal*. We view planar, rooted trees as directed graphs, in which edges point away from the inputs and towards the output. In particular, given a valence m internal vertex p of $\Yup$, $(m-1)$ of the edges adjacent to p are *input edges* to p and one edge is the *output edge* of p, and the inputs of p are ordered. We allow the case $m=1$, and call such 0-input 1-output internal vertices *stump leaves*. We view two planar, rooted trees as equivalent if there is an orientation-preserving self-homeomorphism of $\mathbb{R}\times[0,1]$ which preserves $\mathbb{R}\times\{0\}$ and $\mathbb{R}\times\{1\}$ and takes one tree to the other.

Given a tree $\Yup$, the *collapse* of $\Yup$ is the result of collapsing all internal edges of $\Yup$, to obtain a tree with one internal vertex (i.e., a *corolla*).

2.4.1. *Canonical groupoid enrichments.* First, given a non-enriched multicategory $\mathscr{C}$ we can enrich $\mathscr{C}$ over groupoids *trivially* as follows. Given elements $f,g\in\mathrm{Hom}_{\mathscr{C}}(x_1,\dots,x_n;y)$ define $\mathrm{Hom}(f,g)$ to be empty if $f\neq g$ and to consist of a single element, the identity map, if $f=g$.

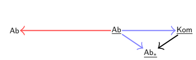
Next we give a different way of enriching multicategories over groupoids, which provides a tool for turning lax multifunctors into strict ones (from a different source), though we will avoid ever actually defining or using the notion of a lax multifunctor or multicategory. Suppose $\mathscr{C}$ is an unenriched multicategory. The *canonical thickening* $\widetilde{\mathscr{C}}$ is the multicategory enriched in groupoids, defined as follows. The objects of $\widetilde{\mathscr{C}}$ are the same as the objects of $\mathscr{C}$. Informally, an object in $\mathrm{Hom}_{\widetilde{\mathscr{C}}}(x_1,\dots,x_n;y)$ is a sequence of composable multimorphisms starting at $x_1,\dots,x_n$ and ending at y. The 2-morphisms record whether two sequences compose to the same multimorphism.

More precisely, an object of $\mathrm{Hom}_{\widetilde{\mathscr{C}}}(x_1,\dots,x_n;y)$ is a tree $\Yup$ with n inputs, together with a labeling of each edge of $\Yup$ by an object of $\mathscr{C}$ and each internal vertex of $\Yup$ by a multimorphism of $\mathscr{C}$, subject to the following conditions:

(1) The input edges of $\Yup$ are labeled by $x_1,\dots,x_n$ (in that order).
(2) The output edge of $\Yup$ is labeled by y.
(3) At a vertex v, if the input edges to v are labeled $w_1,\dots,w_k$ and the output edge is labeled z then the vertex v is labeled by an element of $\mathrm{Hom}_{\mathscr{C}}(w_1,\dots,w_k;z)$. In particular, stump leaves of $\Yup$ are labeled by multimorphisms in $\mathrm{Hom}_{\mathscr{C}}(\varnothing;z)$, i.e., by 0-input multimorphisms.

Given a morphism $f\in\mathrm{Hom}_{\widetilde{\mathscr{C}}}(x_1,\dots,x_n;y)$, we can compose the multimorphisms labeling the vertices according to the tree to obtain a morphism $f^\circ\in\mathrm{Hom}_{\mathscr{C}}(x_1,\dots,x_n;y)$. Given morphisms $f,g\in\mathrm{Hom}_{\widetilde{\mathscr{C}}}(x_1,\dots,x_n;y)$, define $\mathrm{Hom}_{\widetilde{\mathscr{C}}}(f,g)$ to have one element if $f^\circ=g^\circ$ and to be empty otherwise. The unit in $\mathrm{Hom}_{\widetilde{\mathscr{C}}}(x;x)$ is the tree with one input, one output, no internal vertices, and edge labeled x. This completes the definition of the multimorphism groupoids in $\widetilde{\mathscr{C}}$.

Composition of multimorphisms is simply concatenation of trees.



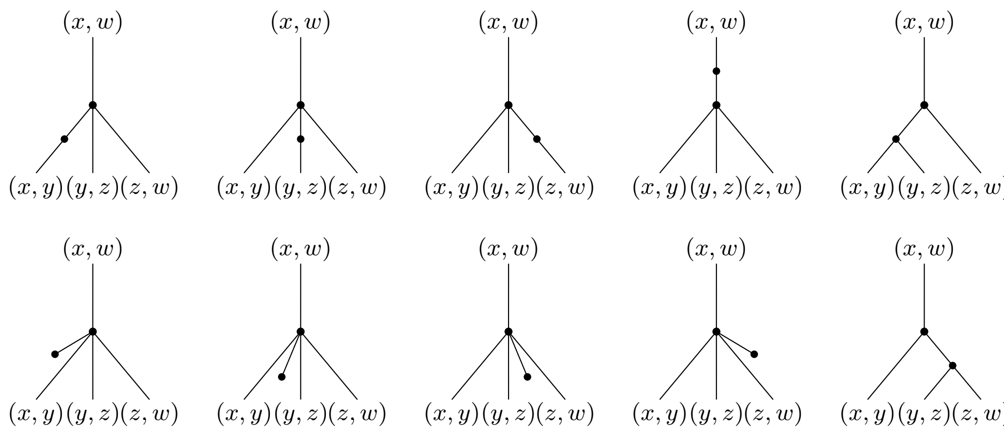


FIGURE 2.1. **Some of the multimorphisms in** $\mathrm{Hom}((x,y),(y,z),(z,w);(x,w))$ **from Example 2.7.** The edges are labeled by the objects and the internal vertices are labeled by multimorphisms in the original multicategory (i.e., basic multimorphisms). Edges ending in a node are stumps. The original multicategory being the shape multicategory of a set, the vertex labels and the internal edge labels are forced, and are not shown.

Lemma 2.5. *This definition of composition extends uniquely to morphisms in the multimorphism groupoids.*

Proof. This is immediate from the definitions. ☐

Lemma 2.6. *These definitions make $\widetilde{\mathscr{C}}$ into a multicategory enriched in groupoids.*

Proof. At the level of objects of the multimorphism groupoids, associativity follows from associativity of composition of trees. At the level of morphisms of the multimorphism groupoids, associativity trivially holds. The unit axioms follow from the fact that gluing on a tree with no internal vertices has no effect. ☐

We will call a multimorphism in $\widetilde{\mathscr{C}}$ *basic* if the underlying tree has only one internal vertex. Every object in the multimorphism groupoid $\mathrm{Hom}_{\widetilde{\mathscr{C}}}(x_1,\dots,x_n;y)$ is a composition of basic multimorphisms.

Example 2.7. Consider the canonical groupoid enrichment of the shape multicategory of some set X (cf. Definition 2.2). For any $x,y,z,w \in X$, the multimorphism set $\mathrm{Hom}((x,y),(y,z),(z,w);(x,w))$ consists of infinitely many elements since the underlying trees could contain arbitrary number of internal vertices. However, there is exactly one multimorphism when the underlying tree has exactly one internal vertex, exactly ten when the underlying tree has exactly two interval vertices (shown in Figure 2.1), exactly sixty-two when the underlying tree has exactly three interval vertices, and so on.

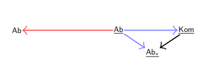
There is a canonical projection multifunctor $\widetilde{\mathscr{C}} \to \mathscr{C}$ which is the identity on objects and composes the multimorphisms associated to the vertices of a tree according to the edges. (Here, we view $\mathscr{C}$ as trivially enriched in groupoids.)

Lemma 2.8. *The projection map $\widetilde{\mathscr{C}} \to \mathscr{C}$ is a weak equivalence.*

(See [EM06, Definition 12.1] for the definition of a weak equivalence.)

Proof. We must check that projection induces an equivalence on the categories of components and that for each $x_1,\dots,x_n,y$ the projection map gives a weak equivalence of simplicial nerves

$$\mathscr{N}\,\mathrm{Hom}_{\widetilde{\mathscr{C}}}(x_1,\dots,x_n;y) \to \mathscr{N}\,\mathrm{Hom}_{\mathscr{C}}(x_1,\dots,x_n;y). \tag{2.1}$$



The first statement follows from the fact that the components of the groupoid $\mathrm{Hom}_{\widetilde{\mathscr{C}}}(x_1,\dots,x_n;y)$ correspond (under the projection) to the elements of $\mathrm{Hom}_{\mathscr{C}}(x_1,\dots,x_n;y)$. The second statement follows from the fact that in each component of the multimorphism groupoid $\mathrm{Hom}_{\widetilde{\mathscr{C}}}(x_1,\dots,x_n;y)$, every object is initial (and terminal), so $\mathscr{N}\,\mathrm{Hom}_{\widetilde{\mathscr{C}}}(x_1,\dots,x_n;y)$ is contractible. □

A related construction is strictification:

Definition 2.9. Given a multicategory enriched in groupoids $\mathscr{C}$ there is a *strictification* $\mathscr{C}^0$ of $\mathscr{C}$, which is an ordinary multicategory, with objects $\mathrm{Ob}(\mathscr{C}^0)=\mathrm{Ob}(\mathscr{C})$ and multimorphism sets $\mathrm{Hom}_{\mathscr{C}^0}(x_1,\dots,x_n;y)$ the set of isomorphism classes (path components) in the groupoid $\mathrm{Hom}_{\mathscr{C}}(x_1,\dots,x_n;y)$. If we view $\mathscr{C}^0$ as trivially enriched in groupoids then there is a projection multifunctor $\mathscr{C}\to\mathscr{C}^0$.

Strictification is a left inverse to thickening, i.e., for any non-enriched multicategory $\mathscr{C}$,

$$(\widetilde{\mathscr{C}})^0\cong\mathscr{C}.$$

A more general notion than a multicategory enriched in groupoids is a *simplicial multicategory*, i.e., a multicategory enriched in simplicial sets. Given a multicategory enriched in groupoids $\mathscr{C}$, replacing each Hom groupoid $\mathrm{Hom}_{\mathscr{C}}(x,y)$ by its nerve gives a simplicial multicategory. One can also *strictify* a simplicial multicategory $\mathscr{D}$ by replacing each Hom simplicial set by its set of path components. If $\mathscr{D}$ came from a multicategory $\mathscr{C}$ enriched in groupoids by taking nerves then the strictification $\mathscr{C}^0$ of $\mathscr{C}$ and the strictification $\mathscr{D}^0$ of $\mathscr{D}$ are are naturally equivalent. Our main reason for introducing simplicial multicategories is that some of the background results we use are stated in that more general language. For instance, spectra form a simplicial multicategory.

2.5. **Homotopy colimits.** In this section we will discuss homotopy colimits in the categories of simplicial sets and chain complexes.

Given an index category I and a functor F from I to the category SSet_* of based simplicial sets, there is a based homotopy colimit denoted by $\mathrm{hocolim}_I F$: it is a quotient of the space

$$\coprod_{p\geq 0}\ \coprod_{i_0\to i_1\to\cdots\to i_p} F(i_0)\wedge(\Delta^p)_+$$

by an equivalence relation induced by simplicial face and degeneracy operations [BK72, XII.2]. Similarly, if instead we are given a functor F from I to the category Kom of complexes, there is a homotopy colimit $\mathrm{hocolim}_I F$ (denoted $\coprod_* F$ in [BK72]): it is a quotient of the complex

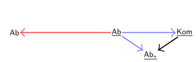
$$\bigoplus_{p\geq 0}\ \bigoplus_{i_0\to i_1\to\cdots\to i_p} F(i_0)\otimes C_*(\Delta^p),$$

where C_* is the normalized chain functor on simplicial sets. (More explicit chain-level descriptions can be given.) In particular, the natural commutative and associative Eilenberg-Zilber shuffle pairing $\widetilde{C}_*(X)\otimes\widetilde{C}_*(Y)\to\widetilde{C}_*(X\wedge Y)$, applied to the above constructions, gives rise to a natural transformation $\mathrm{hocolim}(\widetilde{C}_*\circ F)\to\widetilde{C}_*(\mathrm{hocolim}\,F)$.

In the following, we use the shorthand *equivalence* to denote both a weak equivalence of simplicial sets and a quasi-isomorphism of chain complexes.

Proposition 2.10. *Homotopy colimits satisfy the following properties.*

- *Homotopy colimits are functorial: a natural transformation $F\to F'$ induces a map $\mathrm{hocolim}\,F\to\mathrm{hocolim}\,F'$ that makes hocolim functorial in F, and a map of diagrams $j\colon I\to J$ induces a natural transformation $\mathrm{hocolim}(F\circ j)\to\mathrm{hocolim}\,F$ that makes hocolim functorial in I.*



- *Homotopy colimits preserve equivalences: any natural transformation $F \to F'$ of functors such that $F(i) \to F'(i)$ is an equivalence for all i induces an equivalence* $\operatorname{hocolim} F \to \operatorname{hocolim} F'$*.*
- *For a diagram F indexed by $I \times J$, there is a natural transformation*

$$\operatorname{hocolim}_{i \in I}(\operatorname{hocolim}_{j \in J} F(i \times j)) \to \operatorname{hocolim}_{I \times J} F$$

coming from the (non-commutative) Alexander-Whitney pairing (not the commutative Eilenberg-Zilber shuffle pairing). This is an isomorphism for a homotopy colimit in simplicial sets, and a quasi-isomorphism for a homotopy colimit in complexes. This is associative in I and J, but not commutative.
- *The reduced chain functor $\widetilde{C}_*$ preserves homotopy colimits: given a functor $F\colon I \to \mathsf{SSet}_*$, the natural map* $\operatorname{hocolim}(\widetilde{C}_* \circ F) \to \widetilde{C}_*(\operatorname{hocolim} F)$ *is a quasi-isomorphism.*
- *The smash product $\wedge$ and tensor product $\otimes$ preserve homotopy colimits in each variable, and this is compatible with the Eilenberg-Zilber shuffle pairing.*

In particular, these combine to give a natural quasi-isomorphism

$$(\operatorname{hocolim}_I F) \otimes (\operatorname{hocolim}_J G) \to \operatorname{hocolim}_{I \times J}(F \otimes G)$$

which is compatible with associativity (but not commutativity) of the tensor product.

Homotopy colimits in the category Kom are closely related to left derived functors. In the following, we view Ab as a subcategory of Kom, given by the chain complexes concentrated in degree zero.

Proposition 2.11. *Homotopy colimits of complexes satisfy the following properties.*

- *Write Ab^I for the category of functors $I \to \mathsf{Ab}$ and colim_I for the colimit functor $\mathsf{Ab}^I \to \mathsf{Ab}$. Then there is a natural isomorphism between the left derived functor $\mathbb{L}_p \operatorname{colim}_I(F)$ and the homology group $H_p(\operatorname{hocolim} F)$, for each $p \geq 0$* [BK72, XII.5].
- *For a functor $F\colon I \to \mathsf{Kom}$, there is a convergent spectral sequence*

$$\mathbb{L}_p \operatorname{colim}_I(H_q \circ F) \Rightarrow H_{p+q}(\operatorname{hocolim}_I F).$$

- *For a functor $F\colon I \to \mathsf{SSet}_*$, there is a convergent spectral sequence*

$$\mathbb{L}_p \operatorname{colim}_I(\widetilde{H}_q \circ F) \Rightarrow \widetilde{H}_{p+q}(\operatorname{hocolim}_I F)$$

for the homology groups of a homotopy colimit [BK72, XII.5.7].

Proposition 2.12 ([BK72, XII.5.6])**.** *Suppose Δ denotes the category of finite ordinals and order-preserving maps, and $A\colon \Delta^{\mathrm{op}} \to \mathsf{Kom}$ represents a simplicial chain complex $A_\bullet$. Then the chain complex* $\operatorname{hocolim}_{\Delta^{\mathrm{op}}} A$ *is quasi-isomorphic to the total complex of the double complex*

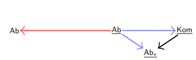
$$\cdots \to A_2 \to A_1 \to A_0 \to 0,$$

where the "horizontal" boundary maps are given by the standard alternating sum of the face maps of $A_\bullet$.

Proposition 2.13. *If A is an abelian group, represented by a functor $F\colon I \to \mathsf{Ab}$ from the trivial category with one object, then the complex* $\operatorname{hocolim}_I F$ *is the complex with A in degree 0 and 0 in all other degrees.*

Proposition 2.14. *Suppose I is a category and we have a natural transformation $\phi\colon F \to G$ of functors $I \to \mathsf{Kom}$. Let P denote the category $\{* \leftarrow 0 \to 1\}$, and define a functor $C\phi\colon P \times I \to \mathsf{Kom}$ on objects by*

$$C\phi(x,y) = \begin{cases} 0 & \text{if } x = *, \\ F(y) & \text{if } x = 0, \\ G(y) & \text{if } x = 1 \end{cases}$$



with morphisms determined by F, G, and ϕ. Then the chain complex $\mathrm{hocolim}_{P\times I}(C\phi)$ *is quasi-isomorphic to the standard mapping cone of the map of chain complexes* $\mathrm{hocolim}_I F \to \mathrm{hocolim}_I G$.

Using the previous two propositions to iterate a mapping cone construction gives the following result for cube-shaped diagrams.

Corollary 2.15. *Let P denote the category $\{* \leftarrow 0 \to 1\}$ and $\underline{2}$ denote the subcategory $\{0 \to 1\}$. Given a functor $F\colon \underline{2}^n \to \mathsf{Ab}$, its totalization is defined to be the chain complex*

$$\bigoplus_{\substack{v\in\underline{2}^n\\|v|=0}} F(v) \to \bigoplus_{\substack{v\in\underline{2}^n\\|v|=1}} F(v) \to \cdots \to \bigoplus_{\substack{v\in\underline{2}^n\\|v|=n}} F(v), \tag{2.2}$$

graded so that $\bigoplus_{\substack{v\in\underline{2}^n\\|v|=i}} F(v)$ is in grading $n-i$ (where $|v|$ denotes the number of 1's in v), and the differential counts the sum of the edge maps of F with standard signs. Let $\widetilde{F}\colon P^n \to \mathsf{Ab}$ be the extended functor given by

$$\widetilde{F}(v) = \begin{cases} F(v) & \text{if } v \in \underline{2}^n, \\ 0 & \text{otherwise.} \end{cases}$$

Then the complex $\mathrm{hocolim}_{P^n} \tilde{F}$ *is quasi-isomorphic to the totalization of F.*

2.6. **Classical spectra.** In this section we will review some of the models for the category of spectra and some of the properties we will need.

For us, a *classical spectrum* X (sometimes called a *sequential spectrum*) is a sequence of based simplicial sets X_n, together with structure maps $\sigma_n\colon X_n \wedge S^1 \to X_{n+1}$. A map $X \to Y$ is a sequence of based maps $f_n\colon X_n \to Y_n$ such that the diagrams

$$\begin{array}{ccc} X_n \wedge S^1 & \xrightarrow{\sigma_n^X} & X_{n+1} \\ \Big\downarrow{\scriptstyle f_n\wedge \mathrm{Id}} & & \Big\downarrow{\scriptstyle f_{n+1}} \\ Y_n \wedge S^1 & \xrightarrow[\sigma_n^Y]{} & Y_{n+1} \end{array}$$

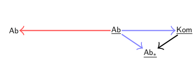
all commute. The structure maps produce natural homomorphisms on homotopy groups $\pi_k(X_n) \to \pi_{k+1}(X_{n+1})$ and (reduced) homology groups $\widetilde{H}_k(X_n) \to \widetilde{H}_{k+1}(X_{n+1})$, allowing us to define homotopy and homology groups

$$\pi_k(X) = \mathrm{colim}_n\, \pi_{k+n} X_n \qquad\qquad H_k(X) = \mathrm{colim}_n\, \widetilde{H}_{k+n} X_n$$

for all $k \in \mathbb{Z}$ that are functorial in X. A map of classical spectra $X \to Y$ is defined to be a *weak equivalence* if it induces an isomorphism $\pi_* X \to \pi_* Y$, and the *stable homotopy category* is obtained from the category of classical spectra by inverting the weak equivalences. The functors π_* and H_* both factor through the stable homotopy category. (This description is due to Bousfield and Friedlander [BF78], and they show that it gives a stable homotopy category equivalent to the one defined by Adams [Ada74]. It has the advantage that maps of spectra are easier to describe, but the disadvantage that maps $X \to Y$ in the stable homotopy category are not defined as homotopy classes of maps $X \to Y$.)

Classical spectra X and Y have a *handicrafted smash product* given by

$$(X \wedge Y)_n = \begin{cases} X_k \wedge Y_k & \text{if } n = 2k, \\ (X_k \wedge Y_k) \wedge S^1 & \text{if } n = 2k+1. \end{cases}$$



The structure map $(X \wedge Y)_n \wedge S^1 \to (X \wedge Y)_{n+1}$ is the canonical isomorphism when n is even and is obtained from the structure maps of X and Y when n is odd. This smash product is not associative or unital, but it induces a smash product functor that makes the stable homotopy category symmetric monoidal. There is a Künneth formula for homology: there is a multiplication pairing $H_p(X) \otimes H_q(Y) \to H_{p+q}(X \wedge Y)$ that is part of a natural exact sequence

$$0 \to \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \to H_n(X \wedge Y) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_p(X), H_q(Y)) \to 0$$

that can be obtained by applying colimits to the ordinary Künneth formula. In particular, this multiplication pairing is an isomorphism if the groups $H_*(X)$ or $H_*(Y)$ are all flat over $\mathbb{Z}$.

Given a functor F from I to the category of classical spectra, there is a homotopy colimit $\mathrm{hocolim}_I F$ obtained by applying homotopy colimits levelwise. Homotopy colimits preserve weak equivalences, and the handicrafted smash product preserves homotopy colimits in each variable. There is also a derived functor spectral sequence

$$\mathbb{L}_p \operatorname{colim}_I (H_q \circ F) \Rightarrow H_{p+q}(\mathrm{hocolim}_I F)$$

for calculating the homology of a homotopy colimit. (In fact, this spectral sequence exists for stable homotopy groups π_* as well.)

The Hurewicz theorem for spaces translates into a Hurewicz theorem for spectra:

Definition 2.16. For an integer n, an object X in the stable homotopy category is *n-connected* if $\pi_k X = 0$ for $k \leq n$. If $n = -1$, we simply say that X is *connective*.

Theorem 2.17. *There is a natural Hurewicz map $\pi_n(X) \to H_n(X)$, which is an isomorphism if X is $(n-1)$-connected.*

This induces a homology Whitehead theorem:

Theorem 2.18. *If $f \colon X \to Y$ is a map of spectra that induces an isomorphism $H_*(X) \to H_*(Y)$ and both X and Y are n-connected for some n, then f is an equivalence.*

Spectra have suspensions and desuspensions:

Definition 2.19. For a spectrum X, there are *suspension* and *loop functors*, as well as formal *shift functors*, as follows:

$$(S^1 \wedge X)_n = S^1 \wedge (X_n) \qquad (\Omega X)_n = \Omega(X_n)$$

$$\mathrm{sh}(X)_n = X_{n+1} \qquad \mathrm{sh}^{-1}(X)_n = \begin{cases} X_{n-1} & \text{if } n > 0 \\ * & \text{if } n = 0 \end{cases}$$

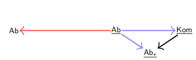
Proposition 2.20. *The pairs $(S^1 \wedge (-), \Omega)$ and (sh^{-1}, sh) are adjoint pairs, and all unit and counit maps are weak equivalences.*

In the stable homotopy category, there are isomorphisms

$$S^1 \wedge X \cong \mathrm{sh}(X) \qquad \Omega X \cong \mathrm{sh}^{-1}(X)$$

In particular, suspension and desuspension are inverse to each other.

Although it looks like there are natural maps $S^1 \wedge X \to \mathrm{sh}(X)$ and $\Omega X \to \mathrm{sh}^{-1}(X)$ that implement these equivalences, there are not.



2.7. **Symmetric spectra.** Many of our constructions make use of Elmendorf-Mandell's paper [EM06], which uses Hovey-Shipley-Smith's more structured category of symmetric spectra [HSS00]. In this section we review some details about symmetric spectra and their relationship to classical spectra.

A *symmetric spectrum* (which, in this paper, we may simply call a *spectrum*) is a sequence of based simplicial sets X_n, together with actions of the symmetric group $\mathfrak{S}_n$ on X_n, and structure maps $\sigma_n \colon X_n \wedge S^1 \to X_{n+1}$. These are required to satisfy the following additional constraint. For any n and m, the iterated structure map

$$X_n \wedge S^m \cong X_n \wedge (S^1 \wedge S^1 \wedge \ldots \wedge S^1) \to X_{n+m}$$

has actions of $\mathfrak{S}_n \times \mathfrak{S}_m$ on the source and target: via the actions on the two factors for the source, and via the standard inclusion $\mathfrak{S}_n \times \mathfrak{S}_m \to \mathfrak{S}_{n+m}$ in the target. The structure maps are required to intertwine these two actions. A map of symmetric spectra consists of a sequence of based, $\mathfrak{S}_n$-equivariant maps $f_n \colon X_n \to Y_n$ commuting with the structure maps. We write $\mathscr{S}$ for the category of symmetric spectra.

A symmetric spectrum can also be described as the following equivalent data. To a finite set S, a symmetric spectrum assigns a simplicial set $X(S)$, and this is functorial in isomorphisms of finite sets. To a pair of finite sets S and T, there is a structure map $X(S) \wedge \left(\bigwedge_{t \in T} S^1\right) \to X(S \coprod T)$, and this is compatible with isomorphisms in S and T as well as satisfying an associativity axiom in T. We recover the original definition by setting $X_n = X(\{1, 2, \ldots, n\})$.

Symmetric spectra also have a more rigid monoidal structure $\wedge$, characterized by the property that a map $X \wedge Y \to Z$ is equivalent to a natural family of maps $X(S) \wedge Y(T) \to Z(S \coprod T)$ compatible with the structure maps in both variables. This makes the category of symmetric spectra symmetric monoidal closed.

Again, the constructions of homotopy colimits are compatible enough that they extend to symmetric spectra. Given a functor F from I to the category of symmetric spectra, there is a homotopy colimit $\operatorname{hocolim}_I F$ obtained by applying homotopy colimits levelwise. Homotopy colimits preserve weak equivalences. The smash product also behaves well with respect to homotopy colimits, as follows.

Proposition 2.21. *The smash product of symmetric spectra preserves homotopy colimits in each variable.*

The category of symmetric spectra has an internal notion of weak equivalence, and a homotopy category of symmetric spectra. Both symmetric spectra and classical spectra have model structures [HSS00, BF78], and we have the following results.

Theorem 2.22 ([HSS00, 4.2.5])**.** *The forgetful functor U from symmetric spectra to classical spectra has a left adjoint V, and this pair of adjoint functors is a Quillen equivalence between these model categories.*

Corollary 2.23. *The homotopy category of symmetric spectra is equivalent to the stable homotopy category.*

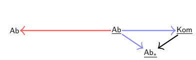
Corollary 2.24. *The equivalence between symmetric spectra and classical spectra preserves homotopy colimits.*

Note that the forgetful functor U does not preserve weak equivalences except between certain symmetric spectra, the so-called *semistable* ones [HSS00, Section 5.6]. Any fibrant symmetric spectrum is semistable, and any symmetric spectrum is weakly equivalent to a semistable one.

Theorem 2.25 ([MMSS01, 0.3])**.** *The equivalence between the homotopy category of symmetric spectra and the stable homotopy category preserves smash products.*

Remark 2.26. In order for $X \wedge Y$ to have the correct homotopy type, X and Y should both be cofibrant symmetric spectra.

These results allow us to define homotopy and homology groups for a symmetric spectrum X as a composite: take the image of X in the homotopy category of symmetric spectra; apply the (right) derived



functor of U to get an element in the homotopy category of classical spectra; and then apply homotopy or homology groups. The homology groups of symmetric spectra therefore inherit the following properties from classical spectra.

Proposition 2.27. *For symmetric spectra X and Y, there is a natural Künneth exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \to H_n(X \wedge Y) \to \bigoplus_{p+q=n-1} \operatorname{Tor}_1^{\mathbb{Z}}(H_p(X), H_q(Y)) \to 0.$$

Proposition 2.28. *For a diagram $F\colon I \to \mathscr{S}$ of symmetric spectra, there is a convergent derived functor spectral sequence*

$$\mathbb{L}_p \operatorname{colim}_I (H_q \circ F) \Rightarrow H_{p+q}(\operatorname{hocolim}_I F).$$

It will be convenient for us to have a lift of these homology groups to a chain functor. Let L denote the reduced chain complex $\widetilde{C}_*(S^1)$ of the simplicial set S^1. This is a complex with value $\mathbb{Z}$ in degree 1 and zero elsewhere. For complexes C and D, let $\underline{\operatorname{Hom}}(C, D)$ be the function complex.

Definition 2.29. Fix a symmetric spectrum X. For an inclusion of finite sets $T \subset U$, there is a natural map

$$\underline{\operatorname{Hom}}(L^{\otimes T}, \widetilde{C}_* X(T)) \xrightarrow{\sim} \underline{\operatorname{Hom}}(L^{\otimes T} \otimes L^{\otimes U \setminus T}, \widetilde{C}_* X(T) \otimes L^{\otimes U \setminus T}) \to \underline{\operatorname{Hom}}(L^{\otimes U}, \widetilde{C}_* X(U)).$$

Now, given any set S (infinite or not), these maps make the complexes $\underline{\operatorname{Hom}}(L^{\otimes T}, \widetilde{C}_* X(T))$ into a directed system indexed by finite subsets $T \subset S$. Define the chain complex

$$\widehat{C}_k(X)_S = \operatorname{colim}_{T \subset S \text{ finite}} \underline{\operatorname{Hom}}(L^{\otimes T}, \widetilde{C}_* X(T)).$$

If S is finite of size n, $\widehat{C}_k(X)_S$ is isomorphic to the shift $\widetilde{C}_* X_n[-n]$. More generally, these structure maps naturally make the system of chain groups and homology groups $\{H_{n+k}(X_n)\}$ into a functor from the category of finite sets and injections to the category of abelian groups (i.e., an FI-module in the language of [CEF15]).

There is a natural pairing

$$\widehat{C}_*(X)_S \otimes \widehat{C}_*(Y)_T \to \widehat{C}_*(X \wedge Y)_{S \coprod T}.$$

The construction of $\widehat{C}_*$ is also natural in injections $S \to S'$.

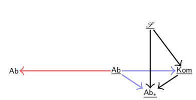
Definition 2.30. Let $\mathcal{M}$ be the category whose objects are the countable sets of the form $\coprod^k \mathbb{N}$ for $k > 1$, and whose morphisms are monomorphisms of sets. For a symmetric spectrum X, we define

$$C_*(X) = \operatorname{hocolim}_{S \in \mathcal{M}} (\widehat{C}_*(X)_S).$$

Let M be the monoid of monomorphisms $\mathbb{N} \to \mathbb{N}$. Since all objects in the category $\mathcal{M}$ are isomorphic to $\mathbb{N}$, this homotopy colimit is quasi-isomorphic to the homotopy colimit over this one-object subcategory, which can be re-expressed as the derived tensor product $\mathbb{Z} \otimes^{\mathbb{L}}_{\mathbb{Z}[M]} \widehat{C}_*(X)_{\mathbb{N}}$. See [Sch08] and [Sch, Exercise E.II.13] for a discussion of this functor.

Proposition 2.31. *The chain functor $C_*\colon \mathscr{S} \to \mathsf{Kom}$ satisfies the following properties.*

- *The homology groups of $C_* X$ are the classical homology groups of the image of X in the stable homotopy category.*
- *The associative disjoint union operation $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ gives rise to a natural quasi-isomorphism $\bigotimes C_*(X_i) \to C_*(\bigwedge X_i)$, which respects the associativity isomorphisms for $\wedge$ and $\otimes$.*
- *The functor C_* preserves homotopy colimits: for a diagram $F\colon I \to \mathscr{S}$, there is a natural quasi-isomorphism* $\operatorname{hocolim}(C_* \circ F) \to C_*(\operatorname{hocolim} F)$.



Therefore, if $\underline{\mathscr{S}}$ denotes the associated multicategory of symmetric spectra, C_* induces a multifunctor $\underline{\mathscr{S}} \to \underline{\mathsf{Kom}}$. To a multimorphism in symmetric spectra realized by a map $X_1 \wedge \cdots \wedge X_n \to Y$, C_* associates the chain map $C_*(X_1) \otimes \cdots \otimes C_*(X_n) \to C_*(X_1 \wedge \cdots \wedge X_n) \to C_*(Y)$. This definition of C_* respects multi-composition. (The multifunctor C_* is not compatible with the symmetries interchanging factors, if we regard $\underline{\mathscr{S}}$ and $\underline{\mathsf{Kom}}$ as symmetric multicategories.)

If we defined homotopy and homology groups

$$\widehat{\pi}_k(X) = \operatorname{colim} \pi_{k+n}(X_n) \qquad \widehat{H}_k(X) = \operatorname{colim} H_{k+n}(X_n)$$

using the same formula as for classical spectra, we obtain "naïve" homotopy and homology groups of a symmetric spectrum X which are not preserved under weak equivalence. If we tensor with the sign representation of $\mathfrak{S}_n$ and take $\operatorname{colim} H_{n+k}(X_n) \otimes sgn$, the result is isomorphic to $\widehat{H}_k(X)_{\mathbb{N}}$ with its action of the monoid M of injections $\mathbb{N} \to \mathbb{N}$ [Sch08]. The natural map $\widehat{H}_k(X) \to H_k(X)$ to the true homology groups factors through the quotient by M. A similar action and factorization hold relating the naïve homotopy groups $\widehat{\pi}_k(X)$ to the true homotopy groups $\pi_k(X)$.

A similar warning holds for homotopy colimits. If F is a diagram of symmetric spectra, it is not the case that $U(\operatorname{hocolim} F) \simeq \operatorname{hocolim}(U \circ F)$ unless F is a diagram of semistable symmetric spectra. However, it is always possible to replace F with a weakly equivalent diagram F' of semistable symmetric spectra so that $\operatorname{hocolim} F \simeq \operatorname{hocolim} F'$, and then $U(\operatorname{hocolim} F') \simeq \operatorname{hocolim}(U \circ F')$.

Symmetric spectra have suspensions and desuspensions.

Definition 2.32. For a symmetric spectrum X, there are suspension and loop functors, as well as formal shift functors, as follows:

$$(S^1 \wedge X)_n = S^1 \wedge (X_n) \qquad (\Omega X)_n = \Omega(X_n)$$

$$\operatorname{sh}(X)_n = X_{1+n} \qquad \operatorname{sh}^{-1}(X)_n = \begin{cases} (\mathfrak{S}_{1+m})_+ \wedge_{\mathfrak{S}_m} X_m & \text{if } n = 1+m \\ * & \text{if } n = 0 \end{cases}$$

The notation $1 + n$ in the shift functor sh indicates that the $\mathfrak{S}_n$-action on X_{1+n} is via the inclusion $\mathfrak{S}_1 \times \mathfrak{S}_n \to \mathfrak{S}_{1+n}$.

Proposition 2.33. *The pairs $(S^1 \wedge (-), \Omega)$ and $(\operatorname{sh}^{-1}, sh)$ are adjoint pairs, and all unit and counit maps are weak equivalences.*

There are natural weak equivalences of symmetric spectra $S^1 \wedge X \to \operatorname{sh}(X)$ and $\operatorname{sh}^{-1} X \to \Omega X$. These become equivalent to the standard shift functors in the stable homotopy category.

For example, the map $S^1 \wedge X \to \operatorname{sh}(X)$ is the composite

$$S^1 \wedge X_n \cong X_n \wedge S^1 \to X_{n+1} \xrightarrow{\sigma} X_{1+n},$$

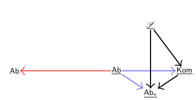
where the final map σ is a block permutation in $\mathfrak{S}_{n+1}$: this is necessary to ensure that this commutes with the structure maps.

Proposition 2.34. *The suspension functor $S^1 \wedge (-)$ and the formal shift functors preserve homotopy colimits. They also preserve smash products: there are natural isomorphisms*

$$\operatorname{sh}(X) \wedge Y \to \operatorname{sh}(X \wedge Y) \qquad X \wedge \operatorname{sh}(Y) \to X \wedge \operatorname{sh}(Y)$$

$$(S^1 \wedge X) \wedge Y \to S^1 \wedge (X \wedge Y) \qquad X \wedge (S^1 \wedge Y) \to S^1 \wedge (X \wedge Y).$$

As with chain complexes, order matters in these identities. For example, the two isomorphisms for $(S^1 \wedge X) \wedge (S^1 \wedge Y)$ do not commute with each other, but differ by a transposition of $(S^1 \wedge S^1)$; the two isomorphisms of $(\operatorname{sh} X) \wedge (\operatorname{sh} Y)$ with $\operatorname{sh}(\operatorname{sh}(X \wedge Y))$ differ by a transposition in $\mathfrak{S}_{2+n}$.



Proposition 2.35. *There are natural isomorphisms* $\underline{\mathrm{Hom}}(L, C_*(\mathrm{sh}(X)) \to C_*(X)$ *and* $C_*(\mathrm{sh}^{-1}(X)) \to \underline{\mathrm{Hom}}(L, C_*(X))$*, as well as natural quasi-isomorphisms* $C_*(S^1 \wedge X) \to C_*(\mathrm{sh}(X))$*.*

In more standard notation, this implies that $C_*(\mathrm{sh}(X)) \cong C_*(X)[1]$ and $C_*(\mathrm{sh}^{-1}(X)) \cong C_*(X)[-1]$. The isomorphism for $\mathrm{sh}(X)$ is true before taking homotopy colimits for $\mathcal{M}$, but the isomorphism for sh^{-1} is not.

2.8. **The Elmendorf-Mandell machine.** A permutative category is a category $\mathscr{C}$ together with a 0-object, an associative operation $\oplus\colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, and a natural isomorphism $\gamma\colon a \oplus b \to b \oplus a$ satisfying certain coherence conditions (see [EM06, Definition 3.1]). An example is the category Sets/X of finite sets over X, with:

- Objects pairs $(Y, f\colon Y \to X)$ of a finite set Y and a map from Y to X,
- Morphisms $\mathrm{Hom}((Y, f), (Z, g)) = \{h\colon Y \to Z \mid f = g \circ h\}$,
- Zero object the pair $(\varnothing, \iota)$ (where ι is the unique map $\varnothing \to X$), and
- Sum $\oplus$ given by disjoint union.

The category Sets/X can be made small by requiring that all sets Y are elements of some chosen, large set. For instance, by a finite set we could mean a pair (n, S) where $n \in \mathbb{N}$ and S is a finite subset of $\mathbb{R}^n$; we will elide this point. Given a finite correspondence $A\colon X \to Y$, i.e., a finite set A and a map $(\pi_X, \pi_Y)\colon A \to X \times Y$, there is a corresponding functor of permutative categories

$$F_A\colon \mathsf{Sets}/X \to \mathsf{Sets}/Y$$
$$F_A(Z, f) = (A \times_X Z, \pi_Y) = \big(\{(a, z) \in A \times Z \mid \pi_X(a) = f(z)\}, (a, z) \mapsto \pi_Y(a)\big).$$

The collection of all (small) permutative categories forms a simplicial multicategory Permu [EM06, Definition 3.2]. The category $\underline{\mathscr{S}}$ of symmetric spectra also forms a simplicial multicategory, and Elmendorf-Mandell construct an enriched multifunctor, *K-theory*,

$$K\colon \mathsf{Permu} \to \underline{\mathscr{S}}.$$

Their functor K takes the category Sets/X to $\bigvee_{x \in X} \mathbb{S}$, a wedge of copies of the sphere spectrum. Further, given a correspondence A from X to Y, the induced map $K(A)\colon K(X) \to K(Y)$ sends $\mathbb{S}_x$ to $\mathbb{S}_y$ (for $x \in X$, $y \in Y$) by a map of degree $\#\big(\pi_X^{-1}(x) \cap \pi_Y^{-1}(y)\big)$. (This special case can be understood concretely, using the Pontrjagin-Thom construction; see, for example, [LLS17, Section 5].)

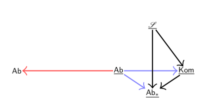
We note that that K is invariant under equivalence in the following sense. Because K respects the enrichments of Permu and $\underline{\mathscr{S}}$ in simplicial sets, it takes natural isomorphisms between functors of permutative categories to homotopies between maps of K-theory spectra. Therefore, equivalent permutative categories give homotopy equivalent answers.

This concludes our general introduction to Elmendorf-Mandell's K-theory machine. In the rest of this section, we discuss a precise sense in which multifunctors from different multicategories can be equivalent. This will be used in Section 4.1 to replace multifunctors from floppy multicategories (enriched in groupoids) with multifunctors from more rigid (unenriched) multicategories.

Definition 2.36 (cf. [Lur, 2.0.0.1])**.** Suppose I is a multicategory. The *associated monoidal category* $I^{\otimes}$ is the category defined as follows. An object of $I^{\otimes}$ is a (possibly empty) tuple $(i_1, \dots, i_n)$ of objects of I. The maps $(i_1, \dots, i_n) \to (j_1, \dots, j_m)$ are given by

$$\coprod_{f\colon \{1,\dots,n\} \to \{1,\dots,m\}} \prod_{k=1}^{m} \mathrm{Hom}_I(f^{-1}(j_k); j_k).$$

The monoidal structure on $I^{\otimes}$ is given by concatenation of tuples, with unit given by the empty tuple.



Definition 2.37. Given multicategories I and J and functors $f\colon I \to J$ and $G\colon I \to \underline{\mathscr{S}}$ there is a map $f_*G\colon J \to \underline{\mathscr{S}}$, the *left Kan extension of* G, defined on objects by

$$(f_*G)(j) = \operatorname{colim}_{(f(i^{\otimes})\to j)\in I^{\otimes}\downarrow j} G(i^{\otimes}). \tag{2.3}$$

(Here $I^{\otimes} \downarrow j$ denotes the overcategory of j.) Left Kan extension is functorial in G, i.e., gives a functor of diagram categories $f_*\colon \underline{\mathscr{S}}^I \to \underline{\mathscr{S}}^J$.

There is also a restriction map $f^*\colon \underline{\mathscr{S}}^J \to \underline{\mathscr{S}}^I$, and f_* is left adjoint to f^*.

Following Elmendorf-Mandell [EM06, Definition 12.1], a map $f\colon \mathcal{M} \to \mathcal{N}$ between simplicial multicategories is a *(weak) equivalence* if the induced map on the strictifications $f^0\colon \mathcal{M}^0 \to \mathcal{N}^0$ is an equivalence of (ordinary) categories and for any $x_1, \dots, x_n, y \in \operatorname{Ob}(\mathcal{M})$, the map $\operatorname{Hom}_{\mathcal{M}}(x_1, \dots, x_n; y) \to \operatorname{Hom}_{\mathcal{N}}(f(x_1), \dots, f(x_n); f(y))$ is a weak equivalence of simplicial sets.

A key technical result of Elmendorf-Mandell's is:

Theorem 2.38 ([EM06, Theorems 1.3 and 1.4])**.** *Let $\mathcal{M}$ be a simplicial multicategory. Then the functor categories $\underline{\mathscr{S}}^{\mathcal{M}}$ and $\underline{\mathscr{S}}^{\mathcal{M}^0}$ are simplicial model categories with weak equivalences (respectively fibrations) the maps which are objectwise weak equivalences (respectively fibrations).*

Further, suppose $\mathcal{N}$ is another simplicial multicategory and

$$f\colon \mathcal{M} \to \mathcal{N}$$

is an equivalence. Then there are Quillen equivalences

$$\underline{\mathscr{S}}^{\mathcal{M}} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \underline{\mathscr{S}}^{\mathcal{N}},$$

where f_* is left Kan extension and f^* is restriction.

For instance, in Theorem 2.38, $\mathcal{M}$ might be (the nerve of) a multicategory enriched in groupoids whose every component is contractible, and $\mathcal{N}$ might be (the nerve of) its strictification $\mathcal{M}^0$.

We will need some additional cofibrancy for the rectification results we apply (see Section 2.9). In particular, Elmendorf-Mandell also show that $\underline{\mathscr{S}}^{\mathcal{M}}$ is cofibrantly generated [EM06, Section 11] and it is combinatorial in the sense of [Lur09, Definition A.2.6.1]. Using a small object argument, Chorny [Cho06] constructs functorial cofibrant factorizations that apply, in particular, to combinatorial model categories such as $\underline{\mathscr{S}}$. So, his construction gives a cofibrant replacement functor

$$Q^{\mathcal{M}}\colon \underline{\mathscr{S}}^{\mathcal{M}} \to \underline{\mathscr{S}}^{\mathcal{M}}.$$

His construction satisfies the following property:

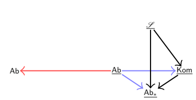
Proposition 2.39. *Suppose $j\colon \mathcal{N} \hookrightarrow \mathcal{M}$ is a full subcategory such that* $\operatorname{Hom}(m_1, \dots, m_k; n) = \varnothing$ *if $n \in \mathcal{N}$ and $m_i \notin \mathcal{N}$ for some i. (That is, there are no arrows into $\mathcal{N}$; we call such a full subcategory $\mathcal{N}$* blockaded.*) Then the small object argument is preserved by restriction: there is a natural isomorphism*

$$j^*Q^{\mathcal{M}} \xrightarrow{\simeq} Q^{\mathcal{N}} j^*.$$

We note that various operations preserve cofibrancy.

Lemma 2.40. *If X is a cofibrant symmetric spectrum then* $\operatorname{sh}(X)$ *and* $\operatorname{sh}^{-1}(X)$ *are also cofibrant. Further, if $F\colon I \to \mathscr{S}$ is a diagram of symmetric spectra which is pointwise cofibrant (i.e., $F(x)$ is cofibrant for all $x \in \operatorname{Ob}(I)$) then* $\operatorname{hocolim} F$ *is cofibrant.*

Proof. This is mechanical to verify from the definitions in [HSS00, Section 3.4], because shifts of the generating cofibrations are cofibrations. ☐



Lemma 2.41. *If $\mathcal{M}$ is a multicategory and $F\colon \mathcal{M} \to \underline{\mathscr{S}}$ is cofibrant then for each object $x \in \mathrm{Ob}(\mathcal{M})$, $F(x)$ is a cofibrant spectrum.*

Proof. The functor $ev_x\colon \underline{\mathscr{S}}^{\mathcal{M}} \to \underline{\mathscr{S}}$, given by $F \mapsto F(x)$, has a right adjoint given by right Kan extension. Given a symmetric spectrum X, the value of this right Kan extension on an object y is

$$\prod_{n\geq 0} X^{\mathrm{Hom}_{\mathcal{M}}(x,x,\dots,x;y)}.$$

In particular, any fibration $X \to Y$ becomes a fibration on applying right Kan extension. Therefore, ev_x is a left Quillen functor and so preserves cofibrations and cofibrant objects. ☐

2.9. **Rectification.** In the process of defining the arc algebras and tangle invariants, we will construct a number of cobordisms which are not equal but are canonically isotopic. The lax nature of the construction will be encoded by defining multifunctors from multicategories in which the Hom sets are groupoids in which each component is contractible: the objects in the groupoids are mapped to the cobordisms while the morphisms in the groupoids are mapped to the isotopies; and contractibility of the groupoids encodes the fact that these isotopies are canonical. We then use the Khovanov-Burnside functor and the Elmendorf-Mandell machine to produce functors from these multicategories to spectra. At that point, we want to collapse the enriched multicategories to ordinary multicategories, to obtain simpler invariants. This collapsing is called rectification, and is accomplished as follows.

Definition 2.42. Let $\mathcal{M}$ be a simplicial multicategory (e.g., the nerve of a multicategory enriched in groupoids), $\mathcal{M}^0$ the strictified discrete multicategory, and $f\colon \mathcal{M} \to \mathcal{M}^0$ the projection. Given a functor $G\colon \mathcal{M} \to \underline{\mathscr{S}}$, the *rectification of* G is the composite

$$f_* Q^{\mathcal{M}} G\colon \mathcal{M}^0 \to \underline{\mathscr{S}}.$$

Lemma 2.43. *If the projection map $\mathcal{M} \to \mathcal{M}^0$ is an equivalence then rectification is part of a Quillen equivalence. In particular, if the projection is an equivalence then for any $G\colon \mathcal{M} \to \underline{\mathscr{S}}$, the functors G and $f^* f_* Q^{\mathcal{M}} G\colon \mathcal{M} \to \underline{\mathscr{S}}$ are naturally equivalent.*

Proof. By definition of cofibrant replacement, the natural transformation $Q^{\mathcal{M}} G \to G$ is an equivalence of diagrams: for every object in $x \in \mathcal{M}$ the map $(Q^{\mathcal{M}} G)(x) \to G(x)$ is an equivalence. Thus it suffices to show that the unit map from $Q^{\mathcal{M}} G$ to $f^* f_* Q^{\mathcal{M}} G$ is an equivalence.

By Theorem 2.38, the adjoint pair f^* and f_* form a Quillen equivalence. This implies that for any fibrant replacement $f_* Q^{\mathcal{M}} G \to (f_* Q^{\mathcal{M}} G)_{\mathit{fib}}$ in $\underline{\mathscr{S}}^{\mathcal{M}^0}$, the composite

$$Q^{\mathcal{M}} G \to f^* f_* Q^{\mathcal{M}} G \to f^* (f_* Q^{\mathcal{M}} G)_{\mathit{fib}}$$

is an equivalence. For every object in $x \in \mathcal{M}$ the composite

$$(Q^{\mathcal{M}} G)(x) \to (f_* Q^{\mathcal{M}} G)(f(x)) \to (f_* Q^{\mathcal{M}} G)_{\mathit{fib}}(f(x))$$

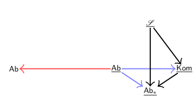
is therefore an equivalence. However, by definition of fibrant replacement the map

$$(f_* Q^{\mathcal{M}} G)(y) \to (f_* Q^{\mathcal{M}} G)_{\mathit{fib}}(y)$$

is an equivalence for any $y \in \mathcal{M}^0$, and hence $Q^{\mathcal{M}}(G) \to f^* f_* Q^{\mathcal{M}} G$ is also an equivalence by the 2-out-of-3 property. ☐

Lemma 2.44. *Suppose that $j\colon \mathcal{N} \hookrightarrow \mathcal{M}$ is a blockaded subcategory and let $j^0\colon \mathcal{N}^0 \to \mathcal{M}^0$ denote the strictification. For any functor $G\colon \mathcal{M} \to \underline{\mathscr{S}}$, there is a natural isomorphism of rectifications*

$$f_*^{\mathcal{N}} Q^{\mathcal{N}} j^* G \cong (j^0)^* f_*^{\mathcal{M}} Q^{\mathcal{M}} G.$$



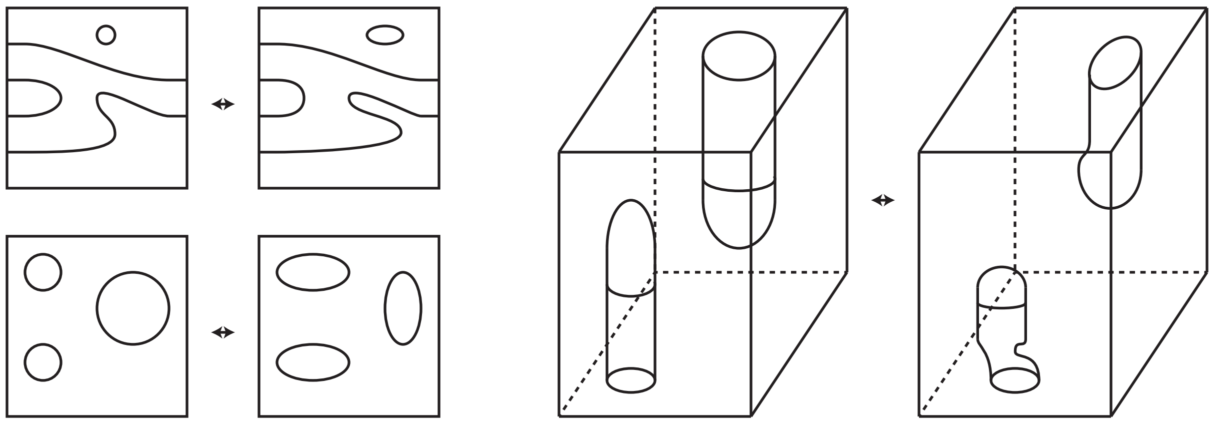


FIGURE 2.2. **The actions of** Diff^1 **and** Diff^2**.** Left: two flat $(4,2)$-tangles related by the action of $(\mathrm{Diff}^1, \mathrm{Id})$ and two flat $(0,0)$-tangles related by the action of $(\mathrm{Diff}^1, \mathrm{Id})$. Right: two flat $(0,0)$-tangle cobordisms related by the action of $(\mathrm{Diff}^2, \mathrm{Id})$.

Proof. There is a natural transformation $f_*^{\mathcal{N}} j^* G \to (j^0)^* f_*^{\mathcal{M}} G$, the *mate*. Note that if $K \subset I$ is blockaded and $j \in K$ then the colimit in Equation (2.3) only sees the objects of K. Thus, the mate is a natural isomorphism

$$f_*^{\mathcal{N}} j^* G \cong (j^0)^* f_*^{\mathcal{M}} G$$

(i.e., satisfies the Beck-Chevalley condition). So, the result follows from Proposition 2.39. ☐

2.10. **Khovanov invariants of tangles.**

Convention 2.45. All embedded cobordisms will be assumed to be the same as the product cobordism in some neighborhood of the boundary.

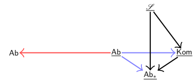
Definition 2.46. Let Diff^1 denote the group of orientation-preserving diffeomorphisms $\phi\colon [0,1] \to [0,1]$ so that there is some $\epsilon = \epsilon(\phi) > 0$ so that $\phi|_{[0,\epsilon)\cup(1-\epsilon,1]} = \mathrm{Id}$. This restriction that ϕ be the identity near the boundary is similar to Convention 2.45.

Definition 2.47. Let Diff^2 denote the group of orientation-preserving diffeomorphisms $\phi\colon [0,1]^2 \to [0,1]^2$ so that there is some $\epsilon = \epsilon(\phi) > 0$ and some $\psi_0, \psi_1 \in \mathrm{Diff}^1$ so that $\phi|_{[0,1]\times([0,\epsilon)\cup(1-\epsilon,1])} = \mathrm{Id}$, and $\phi(p,q) = (p, \psi_0(q))$ for all $p \in [0,\epsilon)$, and $\phi(p,q) = (p,\psi_1(q))$ for all $p \in (1-\epsilon, 1]$.

See Figure 2.2 for examples of the actions of elements in Diff^1 and Diff^2.

By the $2n$ *standard points in* $(0,1)$ we mean $[2n]_{\mathrm{std}} = \{1/(2n+1), \dots, 2n/(2n+1)\}$. A *flat* $(2m,2n)$-*tangle* is an embedded cobordism in $[0,1] \times (0,1)$ from $\{0\} \times [2m]_{\mathrm{std}}$ to $\{1\} \times [2n]_{\mathrm{std}}$. More generally, a $(2m,2n)$-*tangle* is an embedded cobordism in $\mathbb{R} \times [0,1] \times (0,1)$ from $\{0\} \times \{0\} \times [2m]_{\mathrm{std}}$ to $\{0\} \times \{1\} \times [2n]_{\mathrm{std}}$. We call flat tangles T and T' *equivalent* if there is a $\phi \in \mathrm{Diff}^1$ so that $T' = (\phi \times \mathrm{Id}_{(0,1)})(T)$. Similarly, tangles T and T' are *equivalent* if there is a $\phi \in \mathrm{Diff}^1$ so that $T' = (\mathrm{Id}_{\mathbb{R}} \times \phi \times \mathrm{Id}_{(0,1)})(T)$.

Convention 2.48. From now on, by *tangle* (respectively *flat tangle)* we mean an equivalence class of tangles (respectively flat tangles).



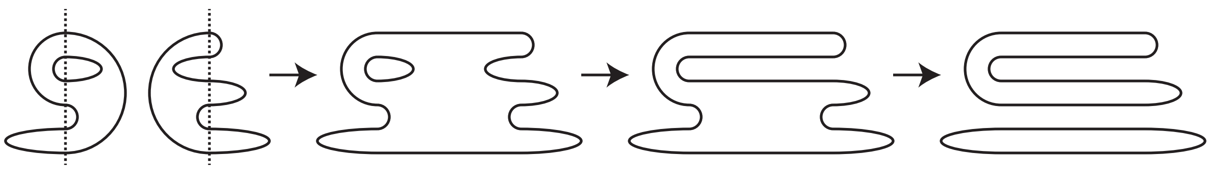


FIGURE 2.3. **Flat tangles and the multiplication on $\mathcal{H}^n$.**

Remark 2.49. We are writing tangles horizontally, while Khovanov [Kho02] (and many others) writes tangles vertically.

Khovanov [Kho02] associated an algebra H^n to each integer n; an (H^m, H^n)-bimodule $\mathcal{C}_{Kh}(T)$ to a flat $(2m, 2n)$-tangle T; and more generally a chain complex of (H^m, H^n)-bimodules to any $(2m, 2n)$-tangle. We will review Khovanov's construction briefly. Because we reserve H^n for singular cohomology, we will use the notation $\mathcal{H}^n$ for Khovanov's algebra H^n.

The constructions start from Khovanov's Frobenius algebra $V = H^*(S^2) = \mathbb{Z}[X]/(X^2)$ with comultiplication $1 \mapsto 1 \otimes X + X \otimes 1, X \mapsto X \otimes X$ and counit $1 \mapsto 0, X \mapsto 1$.

Let ${}_m\widehat{\mathsf{B}}_n$ denote the collection of flat $(2m, 2n)$-tangles. Composition of flat tangles, followed by scaling $[0, 2] \times (0, 1) \to [0, 1] \times (0, 1)$, is a map ${}_m\widehat{\mathsf{B}}_n \times {}_n\widehat{\mathsf{B}}_p \to {}_m\widehat{\mathsf{B}}_p$, which we will write $(a, b) \mapsto ab$. (This map is associative and has strict identities because we quotiented by Diff^1.) Reflection is a map ${}_m\widehat{\mathsf{B}}_n \to {}_n\widehat{\mathsf{B}}_m$, which we will write $a \mapsto \overline{a}$.

The isotopy classes of ${}_0\widehat{\mathsf{B}}_n$ with no closed components are called *crossingless matchings*. For each crossingless matching a, we choose a namesake representative $a \subset [0, 1] \times (0, 1)$ in ${}_0\widehat{\mathsf{B}}_n$ so that the projection $a \to [0, 1]$ to the x-coordinate is Morse with exactly n critical points with distinct critical values; therefore, we may view the set of crossingless matchings, B_n, as a subset of ${}_0\widehat{\mathsf{B}}_n$.

Given a collection of disjoint, embedded circles Z in the plane, let $V(Z) = \bigotimes_{C \in \pi_0(Z)} V$. As a $\mathbb{Z}$-module, the ring $\mathcal{H}^n$ is given by

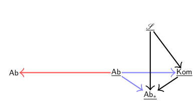
$$\mathcal{H}^n = \bigoplus_{a,b \in \mathsf{B}_n} V(a\bar{b}).$$

The product on $\mathcal{H}^n$ satisfies $xy = 0$ if $x \in V(a\bar{b})$ and $y \in V(c\bar{d})$ with $b \neq c$. To define the product $V(a\bar{b}) \otimes V(b\bar{c}) \to V(a\bar{c})$, consider the representative $b \subset [0, 1] \times (0, 1)$ and let $\mu_1, \dots, \mu_n$ be the critical points of the projection $b \to [0, 1]$, ordered according to the critical values. Define a sequence of $(2n, 2n)$-tangles γ_i, $i = 0, \dots, n$, inductively by setting $\gamma_0 = \bar{b}b$ and obtaining γ_{i+1} by performing embedded surgery on γ_i along an arc connecting $\overline{\mu_{i+1}}$ and μ_{i+1}. (See Figure 2.3.) Observe that γ_n is canonically isotopic to the identity tangle on $2n$ strands. The Frobenius structure on V induces a map $V(a\gamma_i\bar{c}) \to V(a\gamma_{i+1}\bar{c})$; define the product $V(a\bar{b}) \otimes V(b\bar{c}) \to V(a\bar{c})$ to be the composition

$$V(a\bar{b}) \otimes V(b\bar{c}) \cong V(a\gamma_0\bar{c}) \to V(a\gamma_1\bar{c}) \to \cdots \to V(a\gamma_n\bar{c}) \cong V(a\bar{c}).$$

Lemma 2.50 ([Kho02, Proposition 1])**.** *The multiplication just defined is associative and unital, and is independent of the choice of the representative in ${}_0\widehat{\mathsf{B}}_n$ of the $b \in \mathsf{B}_n$.*

Sketch of proof. The key point is that a Frobenius algebra is the same as a $(1 + 1)$-dimensional topological field theory. Multiplication is induced by certain merge cobordisms (see Section 3.3). Up to homeomorphism these cobordisms are independent of the choices of ordering of the saddles, and a composition of these merge cobordisms is another merge cobordism. (Units are also induced by canonical cup cobordisms.) □



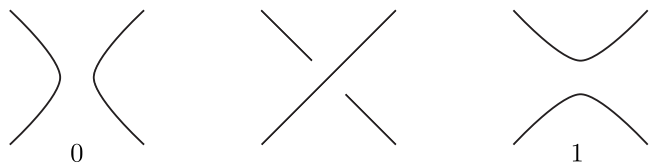


Figure 2.4. **Resolutions of a crossing.**

Given a flat $(2m, 2n)$-tangle $T \in {}_m\widehat{\mathsf{B}}_n$, the bimodule $\mathcal{C}_{Kh}(T)$ is given additively by

$$\mathcal{C}_{Kh}(T) = \bigoplus_{\substack{a \in \mathsf{B}_m \\ b \in \mathsf{B}_n}} V(aT\overline{b}).$$

The left action of $\mathcal{H}^m$ (respectively, the right action of $\mathcal{H}^n$) is defined similarly to the multiplication on $\mathcal{H}^n$: multiplication sends $V(a\overline{b}) \otimes V(cT\overline{d})$ to 0 unless $b = c$ (respectively, sends $V(cT\overline{d}) \otimes V(e\overline{f})$ to 0 unless $d = e$), and the product $V(a\overline{b}) \otimes V(bT\overline{c}) \to V(aT\overline{c})$ (respectively, $V(bT\overline{c}) \otimes V(c\overline{d}) \to V(bT\overline{d})$) is defined by a sequence of merge and split maps, turning the tangle $\overline{b}b$ (respectively, $\overline{c}c$) into the identity tangle.

Lemma 2.51 ([Kho02, Section 2.7])**.** *The bimodule structure on $\mathcal{C}_{Kh}(T)$ is independent of the choices in its construction and defines an associative, unital action.*

Sketch of proof. Like Lemma 2.50, this follows from the fact that these operations are induced by cobordisms which, up to homeomorphism, themselves satisfy the associativity and unitality axioms. ☐

Now let ${}_m\mathsf{C}_n$ denote the collection of all $(2m, 2n)$-tangles in $\mathbb{R} \times [0,1] \times (0,1)$, with each component oriented. Call such a tangle *generic* if its projection to $[0,1] \times (0,1)$ has no cusps, tangencies, or triple points. A *tangle diagram* is a generic tangle along with a total ordering of its crossings (double points of the projection to $[0,1] \times (0,1)$). Let ${}_m\mathsf{D}_n$ be the set of all $(2m, 2n)$-tangle diagrams. (Forgetting the ordering of the crossings, followed by an inclusion, gives a map ${}_m\mathsf{D}_n \to {}_m\mathsf{C}_n$.)

Given a $(2m, 2n)$-tangle diagram $T \in {}_m\mathsf{D}_n$ with N (totally ordered) crossings, and any crossingless matchings $a \in \mathsf{B}_m$ and $b \in \mathsf{B}_n$, there is a corresponding link $aT\overline{b}$, which has an associated Khovanov complex $\mathcal{C}_{Kh}(aT\overline{b})$. Additively, $\mathcal{C}_{Kh}(aT\overline{b})$ is a direct sum over the complete resolutions T_v, $v \in \{0,1\}^N$, of $V(aT_v\overline{b})$. (Our conventions for resolutions are shown in Figure 2.4.) Thus,

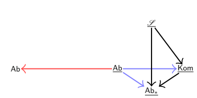
$$\mathcal{C}_{Kh}(T) \coloneqq \bigoplus_{\substack{a \in \mathsf{B}_m \\ b \in \mathsf{B}_n \\ v \in \{0,1\}^N}} V(aT_v\overline{b})$$

inherits the structure of a chain complex, as a direct sum over the a and b of $\mathcal{C}_{Kh}(aT\overline{b})$, and of a bimodule over $\mathcal{H}^m$ and $\mathcal{H}^n$, as a direct sum over v of $\mathcal{C}_{Kh}(T_v)$.

Lemma 2.52 ([Kho02, Section 3.4])**.** *The differential and bimodule structures on $\mathcal{C}_{Kh}(T)$ commute, making $\mathcal{C}_{Kh}(T)$ into a chain complex of bimodules.*

Sketch of proof. Again, this follows from the fact that both the differential and multiplication are induced by Khovanov's TQFT, and the cobordisms inducing the differential and the multiplication commute up to homeomorphism. Indeed, this is a kind of far-commutation: the non-identity portions of the cobordisms inducing multiplication and differentials are supported over different regions of the diagram. ☐

These chain complexes of bimodules have the following TQFT property:



Proposition 2.53 ([Kho02, Proposition 13])**.** *If $T_1 \in {}_m\mathsf{D}_n$ is a $(2m,2n)$-tangle diagram and $T_2 \in {}_n\mathsf{D}_p$ is a $(2n,2p)$-tangle diagram, then the complexes of $(\mathcal{H}^m,\mathcal{H}^p)$-bimodules $\mathcal{C}_{Kh}(T_1T_2)$ and $\mathcal{C}_{Kh}(T_1)\otimes_{\mathcal{H}^n}\mathcal{C}_{Kh}(T_2)$ are isomorphic.*

Sketch of proof. Suppose T_1 has N_1 crossings and T_2 has N_2 crossings. Then the isomorphism

$$\mathcal{C}_{Kh}(T_1)\otimes_{\mathcal{H}^n}\mathcal{C}_{Kh}(T_2)\xrightarrow{\cong}\mathcal{C}_{Kh}(T_1T_2)$$

identifies the summand of $\mathcal{C}_{Kh}(T_1)\otimes_{\mathcal{H}^n}\mathcal{C}_{Kh}(T_2)$ over the vertices $v\in\{0,1\}^{N_1}$ and $w\in\{0,1\}^{N_2}$ with the summand of $\mathcal{C}_{Kh}(T_1T_2)$ over $(v,w)\in\{0,1\}^{N_1+N_2}$. For these flat tangles $T_{1,v}$, $T_{2,w}$, and $(T_1T_2)_{(v,w)}$, the gluing map

$$\mathcal{C}_{Kh}(T_{1,v})\otimes_{\mathcal{H}^n}\mathcal{C}_{Kh}(T_{2,w})\to\mathcal{C}_{Kh}((T_1T_2)_{(v,w)})$$

is induced by the multi-saddle cobordism (cf. Section 3.3) map

$$\mathcal{C}_{Kh}(aT_{1,v}\overline{b})\otimes_{\mathbb{Z}}\mathcal{C}_{Kh}(bT_{2,w}\overline{c})\to\mathcal{C}_{Kh}(a(T_1T_2)_{(v,w)}\overline{c})$$

[Kho02, Theorem 1]. ☐

Proposition 2.54 ([Kho02, Theorem 2])**.** *For any tangle diagram $T\in{}_m\mathsf{D}_n$, the chain homotopy type of the chain complex of bimodules $\mathcal{C}_{Kh}(T)$ is an invariant of the isotopy class of T viewed as a tangle in ${}_m\mathsf{C}_n$.*

For comparison with our constructions later, note that each of the 1-manifolds $a\overline{b}$ in the construction of $\mathcal{H}^n$ lies in $(0,1)^2\subset[0,1]\times(0,1)$; and so does each of the 1-manifolds $aT\overline{b}$ in the construction of $\mathcal{C}_{Kh}(T)$ for a flat tangle T. There is a disjoint union operation on embedded 1-manifolds in $(0,1)^2$ induced by the map

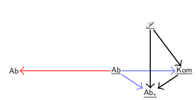
$$(0,1)^2\amalg(0,1)^2\to(0,1)^2$$

which identifies the first copy of $(0,1)^2$ with $(0,1/2)\times(0,1)$ and the second copy of $(0,1)^2$ with $(1/2,1)\times(0,1)$, by affine transformations. Since we have quotiented by the action of Diff^1 on the first $(0,1)$-factor, this disjoint union operation is strictly associative. Further, we can view the maps inducing the multiplication on $\mathcal{H}^n$, the actions on $\mathcal{C}_{Kh}(T)$, and the differential on $\mathcal{C}_{Kh}(T)$ when T is non-flat as induced by cobordisms embedded in $[0,1]\times(0,1)^2$. For instance, the multiplication $V(a\overline{b})\otimes V(b\overline{c})\to V(a\overline{c})$ is induced by a cobordism in $[0,1]\times(0,1)^2$ from $\{0\}\times(a\overline{b}\amalg b\overline{c})$ to $\{1\}\times(a\overline{c})$. For this section, only the abstract (not embedded) cobordisms are relevant; but for the space-level refinement we will need the embedded cobordisms.

2.10.1. *Gradings.* Khovanov homology has both a quantum (internal) and homological grading.

We start with the quantum grading. We grade V so that $\mathrm{gr}_q(1)=-1$ and $\mathrm{gr}_q(X)=1$. Then the grading of $\mathcal{H}^n$ is obtained by shifting the grading on each $V(a\overline{b})$ up by n. In particular, the elements of lowest degree in $\mathcal{H}^n$ are the idempotents in $V(a\overline{a})$, in which each of the n circles is labeled by 1, and these generators lie in quantum grading 0. All homogeneous, non-idempotent elements lie in positive quantum grading. Similarly, for the invariants of flat tangles, if $T\in{}_m\widehat{\mathsf{B}}_n$ then the quantum grading on $V(aT\overline{b})$ is shifted up by n. Given a tangle diagram T with N crossings and a vertex $v\in\{0,1\}^N$, we shift the grading of $\mathcal{C}_{Kh}(T_v)$, the part of $\mathcal{C}_{Kh}(T)$ lying over the vertex v, down by an additional $|v|$. (Here, $|v|$ denotes the number of 1's in v.) The grading on the whole cube is then shifted down by N_+-2N_-, where N_+, respectively N_-, is the number of positive, respectively negative, crossings in T; this is where the orientation of T is used. In other words, for T a $(2m,2n)$-tangle diagram, the quantum grading on $V(aT_v\overline{b})\subset\mathcal{C}_{Kh}(T)$ is shifted up by $n-|v|-N_++2N_-$.

For the homological gradings, all of $\mathcal{H}^n$ lies in grading 0. The homological grading on $\mathcal{C}_{Kh}(T_v)\subset\mathcal{C}_{Kh}(T)$ is given by $N_--|v|$. The differential on $\mathcal{C}_{Kh}(T)$ preserves the quantum grading and decreases the homological grading by 1. The isomorphism of Proposition 2.53 and the chain homotopy equivalences of Proposition 2.54 respect both gradings.



Remark 2.55. Khovanov's first paper on $\mathfrak{sl}_2$ knot homology [Kho00] and his paper on its extension to tangles [Kho02] use different conventions for the quantum grading: in the first paper, $\mathrm{gr}_q(X) = \mathrm{gr}_q(1) - 2$ while in the second $\mathrm{gr}_q(X) = \mathrm{gr}_q(1) + 2$. Our first papers on Khovanov homotopy type [LS14a, LLS17] follow Khovanov's original convention from [Kho00]. In this paper we switch to Khovanov's newer quantum grading convention of [Kho02].

Khovanov's homological grading conventions are the same in all of his papers, but our homological gradings also differ from his by a sign. This is because we treat the Khovanov complex as a chain complex, not a cochain complex; see our conventions from Section 2.1.

2.11. The Khovanov-Burnside 2-functor.

Definition 2.56. Informally, the *Burnside category* $\mathscr{B}$ is the bicategory with objects finite sets X, $\mathrm{Hom}(X,Y)$ the class of finite correspondences $A\colon X \to Y$, i.e., diagrams of sets

$$\begin{array}{ccccc} & & A & & \\ & \overset{s}{\swarrow} & & \overset{t}{\searrow} & \\ X & & & & Y, \end{array}$$

$2\mathrm{Hom}(A,B)$ the set of isomorphisms of correspondences from A to B, i.e., commutative diagrams

$$\begin{array}{ccccc} & A & \xrightarrow{\cong} & B & \\ \swarrow & & \times & & \searrow \\ X & & & & Y. \end{array}$$

Composition of correspondences is fiber product: given $A\colon X \to Y$ and $B\colon Y \to Z$, $B \circ A = A \times_Y B$. Note that one can think of a correspondence $A\colon X \to Y$ as an $(Y \times X)$-matrix of sets, i.e., for each $(y,x) \in Y \times X$ a set $A_{y,x} = s^{-1}(x) \cap t^{-1}(y)$. Composition of correspondences then corresponds to multiplication of matrices, using the Cartesian product and disjoint union to multiply and add sets.

Note that, with this definition, composition is not strictly associative since $(A \times_Y B) \times_Z C$ is in canonical bijection with, but not equal to, $A \times_Y (B \times_Z C)$. Composition also lacks strict identities since $A \times_X X$ is in canonical bijection with, but not equal to, A. There are many ways to rectify this; here is one.

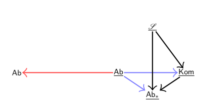
Instead of correspondences, let $\mathrm{Hom}(X,Y)$ denote the set of pairs of an integer n and a $(Y \times X)$-matrix $(A_{y,x})_{x\in X,\ y\in Y}$ of finite subsets $A_{y,x}$ of $\mathbb{R}^n$, with the following property:

(D) $A_{y,x} \cap A_{y',x} = \varnothing$ if $y \neq y'$ and $A_{y,x} \cap A_{y,x'} = \varnothing$ if $x \neq x'$.

(A $(Y \times X)$-matrix of subsets of $\mathbb{R}^n$ is a function $Y \times X \to 2^{\mathbb{R}^n}$.) Given subsets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$, $A \times B$ is a subset of $\mathbb{R}^{n+m}$. Composition is defined by

$$(A_{z,y})_{y\in Y,\ z\in Z} \circ (A_{y,x})_{x\in X,\ y\in Y} = \left(\bigcup_{y\in Y} A_{z,y} \times A_{y,x} \right)_{x\in X,\ z\in Z}.$$

The condition that $A_{y,x} \cap A_{y',x} = \varnothing$ whenever $y \neq y'$ implies that the sets in the union are disjoint. Given $x \neq x'$, $(A_{z,y} \times A_{y,x}) \cap (A_{z,y'} \times A_{y',x'})$ is empty unless $y = y'$ (by looking at the first factor), and thus is empty unless $x = x'$ (by looking at the second factor). Similarly, $(A_{z,y} \times A_{y,x}) \cap (A_{z',y'} \times A_{y',x}) = \varnothing$ if $z \neq z'$. Thus, the composition has Property (D). Composition is clearly strictly associative. The (strict) identity element of X is the $(X \times X)$-diagonal matrix with diagonal entries the 1-element subset of $\mathbb{R}^0$. A 2-morphism of correspondences $\phi\colon (A_{y,x})_{x\in X,\ y\in Y} \to (B_{y,x})_{x\in X,\ y\in Y}$ is a collection of bijections $(\phi_{y,x}\colon A_{y,x} \xrightarrow{\cong} B_{y,x})_{x\in X,\ y\in Y}$; note that 2-morphisms ignore the embedding information.



Throughout, when we talk about the Burnside category we mean this latter, strict version of the category. Typically, however, the embedding data can be chosen arbitrarily, and in those cases we will not specify it.

The free abelian group construction gives a functor $\mathscr{B} \to \mathsf{Ab}$, by

$$(\mathscr{B}) \ni X \mapsto \bigoplus_{x \in X} \mathbb{Z}$$
$$(A_{y,x})_{x \in X,\ y \in Y} \mapsto (|A_{y,x}|)_{x \in X,\ y \in Y}$$

where $|A_{y,x}|$ denotes the number of elements of $A_{y,x}$; the right-hand side is a $(Y \times X)$-matrix of non-negative integers, specifying a homomorphism $\mathbb{Z}\langle X \rangle \to \mathbb{Z}\langle Y \rangle$.

Definition 2.57. The *embedded cobordism category* of 1-manifolds in $(0,1)^2$, $\mathsf{Cob}_e = \mathsf{Cob}_e^{1+1}((0,1)^2)$, has:

- Objects equivalence classes of smooth, closed, one-dimensional submanifolds $Z \subset (0,1)^2$ (i.e., finite collections of disjoint, embedded circles in the open square). Here, we view Z and Z' as equivalent if there is a diffeomorphism $\phi \in \mathrm{Diff}^1$ so that $(\phi \times \mathrm{Id}_{(0,1)})(Z) = Z'$.
- Morphisms $\mathrm{Hom}(Z, W)$ equivalence classes of proper cobordisms embedded in $[0,1] \times (0,1)^2$ from $\{0\} \times Z$ to $\{1\} \times W$, which intersect $[0,\epsilon] \times (0,1)^2$ and $[1-\epsilon, 1] \times (0,1)^2$ as $[0,\epsilon] \times Z$ and $[1-\epsilon,1] \times W$, respectively, for some $\epsilon > 0$ (which may depend on the cobordism; compare Convention 2.45), and so that each component of the cobordism intersects $\{1\} \times (0,1)^2$. Here, we view two cobordisms Σ, Σ' as equivalent if there is a diffeomorphism $\phi \in \mathrm{Diff}^2$ so that $(\phi \times \mathrm{Id}_{(0,1)})(\Sigma) = \Sigma'$.
- Two-morphisms the set of isotopy classes of isotopies of cobordisms.

Note the morphisms are well-defined, because if an embedded one-manifold Z, respectively W, is equivalent (related by Diff^1) to Z', respectively W', and if Σ is any embedded cobordism from Z to W, then there is an embedded cobordism Σ' from Z' to W' which is equivalent (related by Diff^2) to Σ. Note that composition maps and identity maps are strict, because we quotiented by the action of diffeomorphisms of $[0,1]$ (the first factor in $[0,1] \times (0,1)^2$). There is also a disjoint union operation on objects and morphisms induced by $(0,1) \coprod (0,1) \to (0,1/2) \coprod (1/2,1) \hookrightarrow (0,1)$, where $(0,1)$ is the first factor in $(0,1)^2$. This operation is strictly associative because we quotiented by the action of diffeomorphisms on this factor. Finally note that we have explicitly disallowed closed surfaces in morphisms; see Remark 2.59.

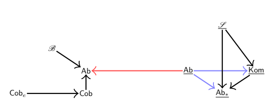
There is a forgetful map from the embedded cobordism category $\mathsf{Cob}_e = \mathsf{Cob}_e^{1+1}((0,1)^2)$ to the abstract $(1+1)$-dimensional cobordism category Cob^{1+1}. So, any Frobenius algebra induces a functor $\mathsf{Cob}_e \to \mathsf{Ab}$ by composing the corresponding abstract $(1+1)$-dimensional TQFT with the forgetful functor. (Here, we view the monoidal category Ab of abelian groups as a monoidal bicategory with only identity 2-morphisms.) In particular, the Khovanov Frobenius algebra $V = H^*(S^2)$ induces such a functor.

Hu-Kriz-Kriz [HKK16] observed that the Khovanov functor $V\colon \mathsf{Cob} \to \mathsf{Ab}$ lifts to a lax 2-functor $V_{HKK}\colon \mathsf{Cob}_e \to \mathscr{B}$:

$$\begin{array}{ccc} \mathsf{Cob}_e & \xrightarrow{V_{HKK}} & \mathscr{B} \\ \downarrow & & \downarrow \\ \mathsf{Cob} & \xrightarrow[V]{} & \mathsf{Ab}. \end{array} \tag{2.4}$$

In this section, we will describe this functor $V_{HKK}\colon \mathsf{Cob}_e \to \mathscr{B}$, following the treatment in our earlier paper [LLS17, Section 8.1].

Remark 2.58. The functor $\mathsf{Cob}_e \to \mathscr{B}$ from [HKK16, LLS17] actually did not lift the Khovanov functor V, but rather its opposite. That ensured that the *cohomology* of the space constructed in [LS14a, HKK16, LLS17] was isomorphic to the Khovanov homology.



However, in this paper we wish to construct a space-level refinement of Khovanov's arc algebras (among other things). If we stick to cohomology, we would either have to construct a co-ring spectrum whose cohomology is the Khovanov arc algebra, or define a Khovanov arc co-algebra first, and then construct a ring spectrum whose cohomology is the newly defined Khovanov arc co-algebra. Not fancying either route, in this paper instead construct space-level refinements whose *homologies* are Khovanov homology; that is, their cohomology is the Khovanov homology of the mirror knot (cf. [Kho00, Proposition 32]). Therefore, below we define a functor $V_{HKK}\colon \mathsf{Cob}_e \to \mathscr{B}$ that actually lifts the Khovanov functor $V\colon \mathsf{Cob} \to \mathsf{Ab}$, and not its opposite; in particular, it is not the functor described in [HKK16, LLS17], but rather, its opposite.

Remark 2.59. In [HKK16, LLS17], the functor to $\mathscr{B}$ was actually constructed from a larger category, where the additional restriction that each component of the cobordism intersects $\{1\} \times (0,1)^2$ was not imposed. However, in this paper we wish to make the embedded cobordism category strictly monoidal and strictly associative, and therefore we have quotiented out the objects and morphisms by Diff^1 and Diff^2, respectively. Unfortunately, Diff^2 can interchange some closed components of a cobordism, and therefore, we work with the subcategory where each component of the cobordism must intersect $\{1\} \times (0,1)^2$, ruling out closed components.

On objects, for $C \in \mathrm{Ob}(\mathsf{Cob}_e)$ a disjoint union of circles, $V_{HKK}(C)$ is the set of labelings of the circles in C by 1 or X, i.e., functions $\pi_0(C) \to \{1, X\}$. Note that Diff^1 can not interchange the components of C, so C, despite being a Diff^1-equivalence class, still has a notion of components.

To define V_{HKK} on morphisms, fix an embedded cobordism Σ from C_0 to C_1. Fix also a checkerboard coloring (2-coloring) of the complement of Σ; for definiteness, choose the coloring in which the region at ∞ (the region whose closure in $[0,1] \times (0,1)^2$ is non-compact) is colored white.

The value of $V_{HKK}(\Sigma)$ is the product over the components Σ' of Σ of $V_{HKK}(\Sigma')$ (with respect to the checkerboard coloring of the complement of Σ' that is induced from the checkerboard coloring of the complement of Σ by declaring that the two colorings agree in a neighborhood of Σ'), and the source and target maps respect this decomposition. (Once again, since Σ has no closed components, Diff^2 cannot interchange components, and so the notion of components descends to equivalence classes.)

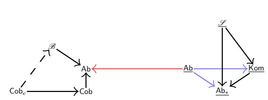
So, to define $V_{HKK}(\Sigma)$ we may assume Σ is connected, but the checkerboard coloring is now arbitrary (that is, the region at ∞ need not be colored white). Fix $x \in V_{HKK}(C_0)$ and $y \in V_{HKK}(C_1)$. If Σ has genus > 1 then $V_{HKK}(\Sigma) = \varnothing$. If Σ has genus 0 then we declare that $s^{-1}(x) \cap t^{-1}(y) \subset V_{HKK}(\Sigma)$ has 0 or 1 elements, and so $V_{HKK}(\Sigma)$ is determined by Formula (2.4). If Σ has genus 1 then $s^{-1}(x) \cap t^{-1}(y) \subset V_{HKK}(\Sigma)$ is empty unless x labels each circle in C_0 by 1 and y labels each circle in C_1 by X.

In the remaining genus 1 case, $V_{HKK}(\Sigma)$ has two elements, which we describe as follows. Let S^2 denote the one-point compactification of $(0,1)^2$. Let $B(([0,1] \times S^2) \setminus \Sigma)$ denote the black region in the checkerboard coloring (possibly extended to the new points at infinity). Let $B((\{0,1\} \times S^2) \setminus \Sigma) = (\{0,1\} \times S^2) \cap B(([0,1] \times S^2) \setminus \Sigma)$. Then $V_{HKK}(\Sigma)$ is the set of generators of

$$H_1(B(([0,1] \times S^2) \setminus \Sigma))/H_1(B((\{0,1\} \times S^2) \setminus \Sigma)) \cong \mathbb{Z}.$$

To define V_{HKK} on 2-morphisms, note that the definitions above are natural with respect to isotopies of the surface Σ.

The composition 2-isomorphism is obvious except when composing two genus 0 components Σ_0, Σ_1 to obtain a genus 1 component Σ. In this non-obvious case, it again suffices to assume Σ is connected. For any curve C on Σ, let C_b and C_w be its push-offs into $B((\{1/2\} \times S^2) \setminus \Sigma)$ (the black region) and $(\{1/2\} \times S^2 \setminus \Sigma) \setminus B((\{1/2\} \times S^2) \setminus \Sigma)$ (the white region), respectively. Now consider the a unique component C of $(\partial\Sigma_0) \cap (\partial\Sigma_1)$ that is non-separating in Σ and is labeled 1, and orient it as the boundary of $B((\{1/2\} \times S^2) \setminus \Sigma)$. One of the two push-offs C_b and C_w is a generator of $H_1(([0,1] \times S^2) \setminus \Sigma)/H_1((\{0,1\} \times S^2) \setminus \partial\Sigma) \cong \mathbb{Z}^2$ and the other



one is zero. If C_b is the generator, label Σ by $[C]$. If C_w is the generator, let D be a curve on Σ, oriented so that the algebraic intersection number $D \cdot C = 1$ (with Σ being oriented as the boundary of the black region); and label Σ by $[D_b]$.

3. Combinatorial tangle invariants

3.1. **A decoration with divides.** Let Cob_d be the following 2-category.

(1) An object of Cob_d is an equivalence class of the following data:
- A smooth, closed 1-manifold Z embedded in $(0,1)^2$.
- A compact 1-dimensional submanifold-with-boundary $A \subset Z$ satisfying the following: If I denotes the closure of $Z \setminus A$, then each of A and I is a disjoint union of closed intervals. We call components of A *active arcs* and components of I *inactive arcs*.

We call (Z, A) a *divided* 1*-manifold*. Two divided 1-manifolds (Z, A) and (Z', A) are equivalent if there is an orientation-preserving diffeomorphism $\phi \in \mathrm{Diff}^1$ so that $(\phi \times \mathrm{Id}_{(0,1)})(Z, A) = (Z', A')$.

We may sometimes suppress A from the notation.

See Figure 3.1 for some examples of divided 1-manifolds.

(2) A morphism from (Z, A) to (Z', A') is an equivalence class of pairs (Σ, Γ) where
- Σ is a smoothly embedded cobordism in $[0,1]\times(0,1)^2$ from Z to Z' (satisfying Convention 2.45).
- $\Gamma \subset \Sigma$ is a collection of properly embedded arcs in Σ (also satisfying Convention 2.45), with $(\partial A \cup \partial A') = \partial\Gamma$, and so that every component of $\Sigma \setminus \Gamma$ has one of the following forms:
  - (I) A rectangle, with two sides components of Γ and two sides components of $A \cup A'$.
  - (II) A $(2n+2)$-gon, with $(n+1)$ sides components of Γ, one side a component of I', and the other n sides components of I. (The integer n is allowed to be zero.)

  We call the components of Γ *divides*.

The pairs (Σ, Γ) and (Σ', Γ') are equivalent if there is a diffeomorphism $\phi \in \mathrm{Diff}^2$ so that $(\phi \times \mathrm{Id}_{(0,1)})(\Sigma) = \Sigma'$ and $(\phi \times \mathrm{Id}_{(0,1)})(\Gamma) = \Gamma'$.

We will call a morphism in Cob_d a *divided cobordism*. Again, we will sometimes suppress Γ from the notation.

See Figure 3.2 for some examples of divided cobordisms.

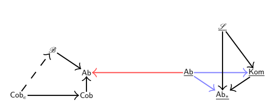
(3) There is a unique 2-morphism from (Σ, Γ) to (Σ', Γ') whenever (some representative of the equivalence class of) (Σ, Γ) is isotopic to (some representative of the equivalence class of) (Σ', Γ') rel boundary.

(4) Composition of divided cobordisms is defined as follows. Given $(\Sigma, \Gamma)\colon (Z, A) \to (Z', A')$ and $(\Sigma', \Gamma')\colon (Z', A') \to (Z'', A'')$, choose a representative of the equivalence class of (Z', A') and representatives of the equivalence classes (Σ, Γ) and (Σ', Γ') which end / start at this representative of (Z', A'). Define $(\Sigma', \Gamma') \circ (\Sigma, \Gamma)$ to be $(\Sigma' \circ \Sigma, \widetilde{\Gamma' \circ \Gamma})$.

It is not too hard to check that composition of divided cobordisms is indeed is a divided cobordism. To wit, Type (II) regions compose to produce Type (II) regions; in particular, since each divide has a Type (II) region on one side, we do not get any closed components in the set of divides after composing. While composing Type (I) rectangles, we glue them along their active boundaries to get new Type (I) rectangles. We do not get any annuli by gluing together such rectangles since that would produce closed divides.

It is also clear that the composition map extends uniquely to 2-morphisms.

Forgetting the divides does not immediately give a functor from the 2-category Cob_d to the 2-category Cob_e. While we do get maps on the objects and the 1-morphisms, there are no immediate maps on the 2-morphisms. However:



Lemma 3.1. *If (Σ_t, Γ_t) is a loop of divided cobordisms (rel boundary) then the induced map $\Sigma_0 \to \Sigma_1 = \Sigma_0$ is isotopic to the identity map.*

Proof. Since the loop is constant on the boundary, the induced map $\Sigma_0 \to \Sigma_0$ must take each connected component C of $\Sigma_0 \setminus \Gamma$ to itself. The map fixes $\partial\Sigma_0$ pointwise and the divides Γ setwise; but since there are no closed divides, it is isotopic to a map that fixes Γ pointwise. However, since C is a planar region (for both Types (I) and (II)), the mapping class group of C fixing the boundary is trivial. □

Proposition 3.2. *The lax 2-functor $V_{HKK}\colon \mathsf{Cob}_e \to \mathscr{B}$ induces a lax 2-functor $\mathsf{Cob}_d \to \mathscr{B}$.*

More precisely, there is an analogue $\widehat{\mathsf{Cob}_d}$ of Cob_d in which the set of 2-morphisms from Σ_0 to Σ_1 is the set of isotopy classes of isotopies of divided cobordisms from Σ_0 to Σ_1. There are forgetful maps $\Pi_{\mathsf{Cob}_d}\colon \widehat{\mathsf{Cob}_d} \to \mathsf{Cob}_d$ (collapsing the 2-morphism sets) and $\Pi_{\mathsf{Cob}_e}\colon \widehat{\mathsf{Cob}_d} \to \mathsf{Cob}_e$ (forgetting the divides). Proposition 3.2 asserts that the map $V_{HKK} \circ \Pi_{\mathsf{Cob}_e}$ descends to a functor $\mathsf{Cob}_d \to \mathscr{B}$, so that the following diagram commutes:

$$\begin{array}{ccc} \widehat{\mathsf{Cob}_d} & \xrightarrow{\Pi_{\mathsf{Cob}_e}} & \mathsf{Cob}_e \\ {\scriptstyle \Pi_{\mathsf{Cob}_e}}\downarrow & & \downarrow{\scriptstyle V_{HKK}} \\ \mathsf{Cob}_d & \dashrightarrow & \mathscr{B}. \end{array}$$

Proof of Proposition 3.2. We must check that if ϕ is an isotopy from (Σ, Γ) to itself then the induced map $V_{HKK}(\Sigma) \to V_{HKK}(\Sigma)$ is the identity map. The only interesting case, of course, is a genus 1 component of Σ. By Lemma 3.1, a loop induces the identity map on $H_1(\Sigma)$. Mayer-Vietoris theorem implies that the map $H_1(\Sigma) \to H_1(\overline{B(([0,1]\times S^2)\setminus\Sigma)}) \cong H_1(B(([0,1]\times S^2)\setminus\Sigma))$ is surjective, so the map on $H_1(B(([0,1]\times S^2)\setminus \Sigma))$ induced by ϕ is also the identity map. □

By a slight abuse of notation, we will let V_{HKK} denote the induced functor $\mathsf{Cob}_d \to \mathscr{B}$ as well.

Remark 3.3. It is interesting to compare Cob_d with Zarev's *divided surfaces* [Zar, Definition 3.1].

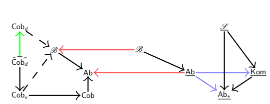
3.2. **A meeting of multicategories.**

3.2.1. *The Burnside multicategory.* We may treat the Burnside category $\mathscr{B}$ as a monoidal category with Cartesian product as the monoidal operation on objects. However, this operation is not strictly associative. We can make the monoidal structure strict by embedding the objects of $\mathscr{B}$ in standard Euclidean spaces, similarly to what we did for morphisms in Definition 2.56, and then define a multicategory $\underline{\mathscr{B}}$ induced from the monoidal structure.

More directly, define $\underline{\mathscr{B}}$ as the multicategory enriched in groupoids with:

- Objects pairs (k, X) where $k \in \mathbb{N}$ and X is a finite subset of $\mathbb{R}^k$. We will always suppress k from the notation.
- $\mathrm{Hom}_{\underline{\mathscr{B}}}(X_1, \dots, X_n; Y) = \mathrm{Hom}_{\mathscr{B}}(X_1 \times \cdots \times X_n, Y)$, the groupoid of maps in the Burnside category from $X_1 \times \cdots \times X_n$ to Y. (Note that since each X_i is a subset of $\mathbb{R}^{k_i}$, $(X_i \times X_{i+1}) \times X_{i+2} = X_i \times (X_{i+1} \times X_{i+2})$ identically.)

Multi-composition is defined in the obvious way. The special case $n = 0$ of the multimorphism sets seems worth spelling out. Let $1 = (0, \{0\})$ be the object in $\underline{\mathscr{B}}$ consisting of a single point embedded in $\mathbb{R}^0$. Note that for any object X in $\underline{\mathscr{B}}$, $1 \times X = X$. We declare that the empty product in the Burnside category is the object 1. So, for any object $X \in \mathrm{Ob}(\underline{\mathscr{B}})$, $\mathrm{Hom}_{\underline{\mathscr{B}}}(\varnothing; X) = \mathrm{Hom}_{\mathscr{B}}(1, X)$. In particular, an element of the set X gives a multimorphism $\varnothing \to X$.



Recall that we have a multicategory of abelian groups $\underline{\mathsf{Ab}}$ by defining $\mathrm{Hom}(V_1, V_2, \dots, V_n; V)$ to be the set of multilinear maps $V_1, \dots, V_n \to V$ (or equivalently, the set of maps $V_1 \otimes \cdots \otimes V_n \to V$). We can view $\underline{\mathsf{Ab}}$ as trivially enriched in groupoids. The forgetful functor $\mathscr{B} \to \mathsf{Ab}$ from Section 2.11 respects the monoidal structure on both $\mathscr{B}$ and Ab, and therefore, induces a forgetful functor $\mathcal{F}_{orget} \colon \underline{\mathscr{B}} \to \underline{\mathsf{Ab}}$.

3.2.2. *Shape multicategories.* Recall from Section 2.10 that B_n denotes the set of crossingless matchings on $2n$ points. Define $\mathcal{H}_n^0$ to be the shape multicategory associated to B_n (Definition 2.2). Specifically, the multicategory $\mathcal{H}_n^0$ has one object for each pair (a, b) of crossingless matchings of $2n$ points, and a unique multimorphism

$$(a_1, a_2), (a_2, a_3), \dots, (a_{k-1}, a_k) \to (a_1, a_k).$$

We will sometimes denote the unique morphism in $\mathrm{Hom}((a_1, a_2), (a_2, a_3), \dots, (a_{k-1}, a_k); (a_1, a_k))$ by $f_{a_1,\dots,a_k}$. In particular, the special case $k = 1$ of the zero-input multimorphism $\varnothing \to (a_1, a_1)$ is denoted f_{a_1}.

Similarly, define ${}_m\mathcal{T}_n^0$ to be the shape multicategory associated to the sequence of sets $(\mathsf{B}_m, \mathsf{B}_n)$ (Definition 2.3). Specifically, the multicategory ${}_m\mathcal{T}_n^0$ has three kinds of objects:

(1) Objects (a, b) where a, b are crossingless matchings on $2m$ points,
(2) Objects (a, b) where a, b are crossingless matchings on $2n$ points, and
(3) Objects (a, b) where a is a crossingless matching of $2m$ points and b is a crossingless matching of $2n$ points. For clarity we will write such objects instead as (a, T, b) where T is just a notational placeholder.

(In the special case $m = n$, the objects of types (1) and (2) are the same.) There is a unique multimorphism

$$(a_1, a_2), (a_2, a_3), \dots, (a_{k-1}, a_k) \to (a_1, a_k)$$

if $a_1, \dots, a_k$ are crossingless matchings on $2m$ points. There is a unique multimorphism

$$(b_1, b_2), (b_2, b_3), \dots, (b_{\ell-1}, b_\ell) \to (b_1, b_\ell)$$

if $b_1, \dots, b_\ell$ are crossingless matchings on $2n$ points. There is a unique multimorphism

$$(a_1, a_2), \dots, (a_{k-1}, a_k), (a_k, T, b_1), (b_1, b_2), \dots, (b_{\ell-1}, b_\ell) \to (a_1, T, b_\ell)$$

if $a_1, \dots, a_k$ are crossingless matchings on $2m$ points and $b_1, \dots, b_\ell$ are crossingless matchings on $2n$ points. (The special cases $k = 1$ and $\ell = 1$ are allowed.)

Note that $\mathcal{H}_m^0$ and $\mathcal{H}_n^0$ are full sub-multicategories of ${}_m\mathcal{T}_n^0$. Extending the notation $f_{a_1,\dots,a_k}$ from $\mathcal{H}_m^0$, we will sometimes denote the unique morphism in

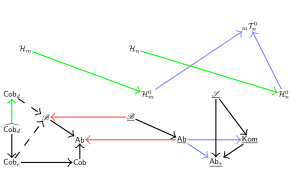
$$\mathrm{Hom}_{{}_m\mathcal{T}_n^0}\big((a_1, a_2), \dots, (a_{k-1}, a_k), (a_k, T, b_1), (b_1, b_2), \dots, (b_{\ell-1}, b_\ell); (a_1, T, b_\ell)\big)$$

by $f_{a_1,\dots,a_k,T,b_1,\dots,b_\ell}$.

Let $\mathcal{H}_n$ (respectively ${}_m\mathcal{T}_n$) be the canonical groupoid enrichment of $\mathcal{H}_n^0$ (respectively ${}_m\mathcal{T}_n^0$) from Section 2.4.1. See in particular Example 2.7 for some of the multimorphisms that appear in the groupoid enriched categories.

Now recall, from Section 2.10, Khovanov's arc algebra $\mathcal{H}^n$, and Khovanov's tangle invariant $\mathcal{C}_{Kh}(T)$, which is a *dg* $(\mathcal{H}^m, \mathcal{H}^n)$-bimodule. The algebra $\mathcal{H}^n$ is equipped with an orthogonal set of idempotents (Definition 2.4), one for each crossingless matching $a \in \mathsf{B}_n$, with the idempotent corresponding to a being the element of $V(a\overline{a})$ that labels each of the n circles by $1 \in V$. Therefore, via the equivalences from Section 2.3, we have the following:

Principle 3.4. The Khovanov arc algebra $\mathcal{H}^n$ may be viewed as a multifunctor $\mathcal{H}_n^0 \to \underline{\mathsf{Ab}}$. Composing with the inclusion $\underline{\mathsf{Ab}} \to \underline{\mathsf{Kom}}$ (which views an abelian group as a chain complex concentrated in grading 0), we can also view the Khovanov arc algebra as a multifunctor from $\mathcal{H}_n^0$ to chain complexes. Similarly, Khovanov's



tangle invariant $\mathcal{C}_{Kh}(T)$ may be viewed as a multifunctor ${}_m\mathcal{T}_n^0 \to \underline{\mathsf{Kom}}$ which restricts to $\mathcal{H}_m^0$ and $\mathcal{H}_n^0$ as the arc algebra multifunctors.

3.2.3. *The divided cobordism multicategory.* Next we turn to the multicategory $\widetilde{\underline{\mathsf{Cob}_d}}$ of divided cobordisms. The divided cobordism category Cob_d from Section 3.1 can be endowed with a disjoint union bifunctor $\amalg$ induced by concatenation in the first $(0,1)$-factor. Disjoint union is a strictly associative (non-symmetric) monoidal structure on Cob_d, since we have quotiented out objects by Diff^1 and morphisms by Diff^2. Therefore, we get an associated multicategory $\underline{\mathsf{Cob}_d}$. The groupoid enriched multicategory $\widetilde{\underline{\mathsf{Cob}_d}}$ is the canonical groupoid enrichment of $\underline{\mathsf{Cob}_d}$.

Fleshing out the definition, the objects of $\widetilde{\underline{\mathsf{Cob}_d}}$ are the same as the objects of Cob_d, i.e., Diff^1-equivalence classes of smooth, closed, embedded 1-manifolds in $(0,1)^2$ which are decomposed as unions of *active arcs* and *inactive arcs*.

A *basic multimorphism from* $(Z_1, \dots, Z_n)$ *to* Z is an element of $\mathrm{Hom}_{\mathsf{Cob}_d}(Z_1 \amalg \dots \amalg Z_n, Z)$. Now, an object of $\mathrm{Hom}_{\widetilde{\underline{\mathsf{Cob}_d}}}(Z_1, \dots, Z_n; Z)$ consists of:

- a tree $\pitchfork$;
- a labeling of each edge of $\pitchfork$ by an object of Cob_d, so that the input edges are labeled $Z_1, \dots, Z_n$ and the output edge is labeled Z; and
- a labeling of each internal vertex v of $\pitchfork$ with input edges labeled $Z'_1, \dots, Z'_k$ and output edge labeled Z' by a basic multimorphism from $(Z'_1, \dots, Z'_k)$ to Z' (i.e., an object in $\mathrm{Hom}_{\mathsf{Cob}_d}(Z'_1 \amalg \dots \amalg Z'_k, Z')$).

Composition of multimorphisms is induced by composition of trees; being a canonical thickening, this is automatically strictly associative and has strict units (the 0 internal vertex trees).

Given a multimorphism f in $\mathrm{Hom}_{\widetilde{\underline{\mathsf{Cob}_d}}}(Z_1, \dots, Z_n; Z)$, the *collapsing* f° of f is the result of composing the cobordisms associated to the vertices of the tree according to the edges of the tree, in some order compatible with the tree. Associativity of composition in $\widetilde{\underline{\mathsf{Cob}_d}}$ implies that the collapsing f° of f is well-defined, i.e., independent of the order that one composes vertices in the tree. Given multimorphisms $f, g \in \mathrm{Hom}_{\widetilde{\underline{\mathsf{Cob}_d}}}(Z_1, \dots, Z_n; Z)$ there is a unique morphism from f to g if and only if f° is isotopic to g°. It is clear that if $f \circ (g_1, \dots, g_n)$ is defined and there is a morphism from f to f' and from g_i to g'_i for $i = 1, \dots, n$ then there is a morphism from $f \circ (g_1, \dots, g_n)$ to $f' \circ (g'_1, \dots, g'_n)$.

Putting these observations together, we have proved:

Lemma 3.5. *These definitions make* $\widetilde{\underline{\mathsf{Cob}_d}}$ *into a multicategory.*

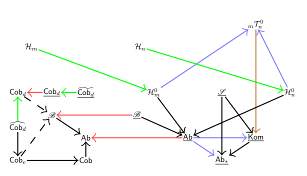
3.2.4. *Cubes.* To a non-flat tangle we will associate a cube of flat tangles, and hence, roughly, a cube of multifunctors between groupoid-enriched multicategories. In this section we make sense of this notion in enough generality for our applications.

Definition 3.6. Let $\underline{2}_0^N$, the *cube category*, be the category with objects $\{0,1\}^N$ and a unique morphism $v = (v_1, \dots, v_N) \to w = (w_1, \dots, w_N)$ whenever $v_i \leq w_i$ for all $1 \leq i \leq N$.

Remark 3.7. In our previous papers, we defined cube categories to be the opposite category of the above. However, since in this paper we are taking homology instead of cohomology (cf. Remark 2.58) we need the morphisms in the cube to go from 0 to 1.

We will define a groupoid-enriched multicategory $\underline{2}^N \tilde{\times} {}_m\mathcal{T}_n$, a kind of product of the cube $\underline{2}^N$ and ${}_m\mathcal{T}_n$. We first define its strictification $(\underline{2}^N \tilde{\times} {}_m\mathcal{T}_n)^0$ (Definition 2.9).

- Objects of $(\underline{2}^N \tilde{\times} {}_m\mathcal{T}_n)^0$ are pairs $(a,b) \in \mathrm{Ob}(\mathcal{H}_m) \cup \mathrm{Ob}(\mathcal{H}_n)$ or quadruples (v, a, T, b) where $v \in \{0,1\}^N$ and $(a, T, b) \in \mathrm{Ob}({}_m\mathcal{T}_n)$.



- For any objects $a_i \in \mathrm{Ob}(\mathcal{H}_m)$, $b_j \in \mathrm{Ob}(\mathcal{H}_n)$, and morphism $v \to w$ in $\underline{2}^n$, there are unique multimorphisms
$$\begin{aligned} (a_1, a_2), \dots, (a_{k-1}, a_k) &\to (a_1, a_k) \\ (a_1, a_2), \dots, (a_{k-1}, a_k), (v, a_k, T, b_1), (b_1, b_2), \dots, (b_{\ell-1}, b_\ell) &\to (w, a_1, T, b_\ell) \\ (b_1, b_2), \dots, (b_{\ell-1}, b_\ell) &\to (b_1, b_\ell) \end{aligned}$$
in $(\underline{2}^N \tilde{\times}_m \mathcal{T}_n)^0$ and no other multimorphisms.

Next define the *thick N-cube category of* ${}_m\mathcal{T}_n^0$, $\underline{2}^N \tilde{\times}_m \mathcal{T}_n$, as a multicategory enriched in groupoids with:

- Objects same as $\mathrm{Ob}((\underline{2}^N \tilde{\times}_m \mathcal{T}_n)^0)$.
- A *basic multimorphism* is one of:
  - A multimorphism in $\mathcal{H}_m$ or $\mathcal{H}_n$, or
  - A multimorphism of the form
  $$(a_1, a_2), \dots, (a_{k-1}, a_k), (v, a_k, T, b_1), (b_1, b_2), \dots, (b_{\ell-1}, b_\ell) \to (v, a_1, T, b_\ell)$$
  in $(\underline{2}^N \tilde{\times}_m \mathcal{T}_n)^0$, or
  - A morphism of the form $(v, a, T, b) \to (w, a, T, b)$ in $(\underline{2}^N \tilde{\times}_m \mathcal{T}_n)^0$.
- An objects of a multimorphism groupoid in $\underline{2}^N \tilde{\times}_m \mathcal{T}_n$ is a tree with n inputs, together with a labeling of:
  - each edge by an object of $\underline{2}^N \tilde{\times}_m \mathcal{T}_n$ and
  - each vertex by a basic multimorphism from the inputs of the vertex to the output of the vertex.
- Given a multimorphism in $\underline{2}^N \tilde{\times}_m \mathcal{T}_n$, there is a corresponding multimorphism in $(\underline{2}^N \tilde{\times}_m \mathcal{T}_n)^0$ by composing the basic multimorphisms according to the tree. Define the multimorphism groupoid to have a unique morphism $\pitchfork \to \pitchfork'$ if the corresponding multimorphisms in $(\underline{2}^N \tilde{\times}_m \mathcal{T}_n)^0$ are the same. Equivalently, there is a unique morphism $\pitchfork \to \pitchfork'$ if and only if $\pitchfork$ and $\pitchfork'$ have the same source and target.

The above definition ensures that $(\underline{2}^N \tilde{\times}_m \mathcal{T}_n)^0$ is indeed the strictification of $\underline{2}^N \tilde{\times}_m \mathcal{T}_n$.

Lemma 3.8. *The projection $\underline{2}^N \tilde{\times}_m \mathcal{T}_n \to (\underline{2}^N \tilde{\times}_m \mathcal{T}_n)^0$, which is the identity on objects and sends a tree with inputs $x_1, \dots, x_n$ and output y to the unique multimorphism $x_1, \dots, x_n \to y$, is a weak equivalence.*

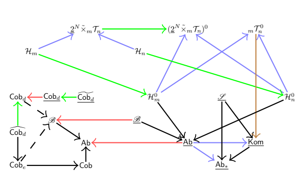
Proof. The proof is essentially the same as the proof of Lemma 2.8. ☐

The category $\underline{2}^{N+1} \tilde{\times}_m \mathcal{T}_n$ has the category $\underline{2}^N \tilde{\times}_m \mathcal{T}_n$ as a full subcategory in two distinguished ways: the full subcategory spanned by objects (a, b) and $(\{0\} \times v, a, T, b)$, which we denote $\{0\} \times \underline{2}^N \tilde{\times}_m \mathcal{T}_n$; and the full subcategory spanned by objects (a, b) and $(\{1\} \times v, a, T, b)$, which we denote $\{1\} \times \underline{2}^N \tilde{\times}_m \mathcal{T}_n$. The strictified product $(\underline{2}^{N+1} \tilde{\times}_m \mathcal{T}_n)^0$ has corresponding subcategories $(\{0\} \times \underline{2}^N \tilde{\times}_m \mathcal{T}_n)^0$ and $(\{1\} \times \underline{2}^N \tilde{\times}_m \mathcal{T}_n)^0$, both isomorphic to $(\underline{2}^N \tilde{\times}_m \mathcal{T}_n)^0$.

Remark 3.9. The groupoid-enriched multicategory $\underline{2}^N \tilde{\times}_m \mathcal{T}_n$ is related to a groupoid-enriched version of the Boardman-Vogt tensor product [BV73, Section II.3, Paragraph (2.15)], the main difference being that we have not multiplied the objects of the form (a, b) in ${}_m\mathcal{T}_n$ by $\underline{2}^N$.

3.3. **A cabinet of cobordisms.** In this section we enhance some of the topological objects used to define the Khovanov arc algebras and modules so that they lie in the category of divided cobordisms.

First, given crossingless matchings $a, b \in \mathsf{B}_n$, we make $a\overline{b}$ into a divided 1-manifold as follows. The inactive arcs are the connected components of a small neighborhood of $\partial a \subset a\overline{b}$ (so there are $2n$ inactive arcs), while the active arcs are the connected components of the complement of the inactive arcs (so there are also $2n$ active arcs). See Figure 3.1.



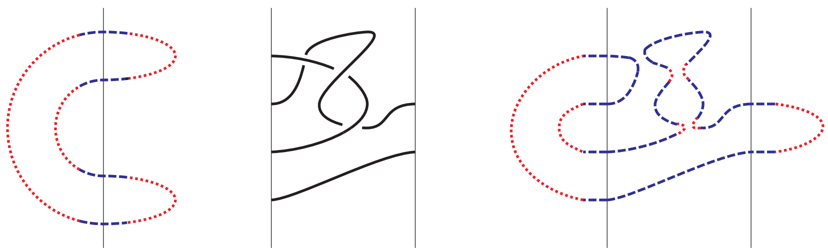


FIGURE 3.1. **Active arcs and flat tangles.** Left: $a\overline{b}$ for two crossingless matchings $a, b \in \mathsf{B}_2$ and the decomposition of $a\overline{b}$ into active (dotted) and inactive (dashed) arcs. Center: a $(4,2)$-tangle T. Right: $aT_v\overline{b}$ where $v = (1,0,0)$ and a is the same crossingless matching as on the left.

Given a oriented link diagram $K \in {}_0\mathsf{D}_0$ with N ordered crossings and a vector $v \in \{0,1\}^N$, we make the resolution K_v into a divided 1-manifold as follows. Let $\pi(K)$ denote the projection of K to $(0,1)^2$. For each $1 \leq i \leq N$, choose a small disk D_i around the i^{th} crossing of $\pi(K)$, so that ∂D_i intersects $\pi(K)$ transversely in 4 points and the D_i are disjoint. Choose the resolution K_v so that $\pi(K) \cap \big((0,1)^2 \setminus (\bigcup_i D_i)\big) = K_v \cap \big((0,1)^2 \setminus (\bigcup_i D_i)\big)$, i.e., so that $\pi(K)$ and K_v agree outside the disks D_i. The boundaries of the disks D_i divide K_v into $4N$ arcs, $2N$ inside the disks D_i and $2N$ outside. Declare the arcs outside the disks D_i to be inactive. Define the arcs inside D_i to be active if $v_i = 0$ and inactive if $v_i = 1$.

Combining the previous two cases, given a $(2m, 2n)$-tangle diagram $T \in {}_m\mathsf{D}_n$ with N ordered crossings, $a \in \mathsf{B}_m$, $b \in \mathsf{B}_n$, and $v \in \{0,1\}^N$ we make $aT_v\overline{b}$ into a divided 1-manifold as follows. Again, choose small, disjoint neighborhoods D_i of the crossings of $\pi(T)$, so that outside the disks D_i, T_v agrees with $\pi(T)$. Choose small neighborhoods of the endpoints of a and b. Then the active arcs of $aT_v\overline{b}$ are:

- The arcs inside the D_i with $v_i = 0$, and
- The arcs in a and $\overline{b}$ in the complement of the neighborhoods of the endpoints.

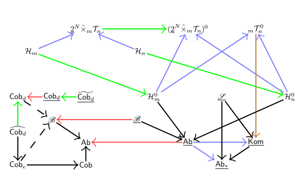
The remaining arcs of $aT_v\overline{b}$ are inactive. See Figure 3.1.

Next we turn to the divided cobordisms we will use as building blocks.

A *trivial cobordism* is a cobordism of the form $[0,1] \times Z$ where Z is a divided 1-manifold. If P is the set of endpoints of the active arcs in Z then the divides are given by $\Gamma = [0,1] \times P$.

Next, fix a divided 1-manifold Z and a disk D so that $D \cap Z$ consists of exactly two active arcs in Z. Call these four endpoints a, b, c, d, so that the arcs join $a \leftrightarrow b$ and $c \leftrightarrow d$, and a and d are consecutive around ∂D. Let Z' be a divided 1-manifold which agrees with Z outside D and consists of two arcs in $Z' \cap D$ connecting $a \leftrightarrow d$ and $b \leftrightarrow c$. Make Z' into a divided 1-manifold by declaring that the arcs inside D are inactive, and the other arcs of Z' are the same as the arcs of Z. A *saddle cobordism* is a cobordism Σ from Z to Z' so that:

- $\Sigma \cap [0,1] \times ((0,1)^2 \setminus D) = [0,1] \times (Z \setminus D)$,
- Inside $[0,1] \times D$, Σ consists of a single embedded saddle, and
- The dividing arcs Γ for Σ connect $a \leftrightarrow d$ and $b \leftrightarrow c$ inside the saddle, and agree with $[0,1] \times P$ outside the saddle, where P is the collection of endpoints of active arcs of Z'.



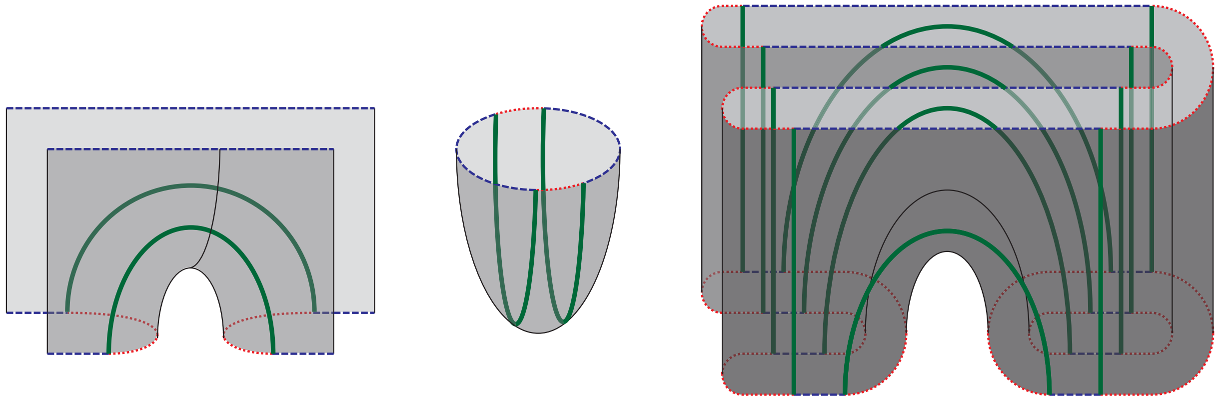


FIGURE 3.2. **Basic divided cobordisms.** Left: a saddle. Center: a cup. Curves Γ are **thick**, arcs in $I \cup I'$ are dashed, and arcs in $A \cup A'$ are dotted. Together with product cobordisms, these are the local pieces that the divided cobordisms of interest are built from. Right: the divided cobordism associated to a product on the Khovanov arc algebra.

(See Figure 3.2 for the local form of Σ in a neighborhood of D.) The cobordism Σ is well-defined up to unique isomorphism in Cob_d. We call D the *support* of the saddle cobordism. Note that a saddle cobordism $\Sigma\colon Z_1 \to Z_2$ is determined by Z_1 and the support of Σ (up to isotopy rel $(\{0\} \times Z_1) \cup ([0,1] \times (Z_1 \setminus D))$).

More generally, given a divided 1-manifold Z and a collection of disjoint disks D_i so that each $D_i \cap Z$ consists of two active arcs, a *multi-saddle cobordism* is a divided cobordism Σ from Z so that $\Sigma \cap ([0,1] \times D_i)$ is a saddle for each i and $\Sigma \setminus \left(\bigcup_i [0,1] \times D_i\right) = [0,1] \times Z$; and where the dividing arcs on Σ

- connect the points in $P \cap \partial D_i$ in pairs inside the saddles, as in Figure 3.2 (i.e., so that points not connected in $Z \cap D_i$ are connected by arcs in Γ) and
- are of the form $[0,1] \times \{p\}$ for $p \in (P \setminus \partial D_i)$ the ends of active arcs not involved in the saddles.

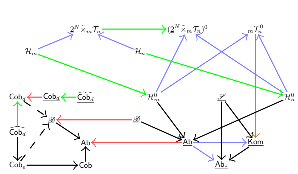
We call $\bigcup_i D_i$ the *support* of the multi-saddle cobordism.

Next, given crossingless matchings $a, b, c \in \mathsf{B}_n$, a *merge cobordism* $a\overline{b} \amalg b\overline{c} \to a\overline{c}$ is a composition of saddle cobordisms, one for each arc in b. Again, this merge cobordism is well-defined up to unique isomorphism in Cob_d. Similarly, given $a, b \in \mathsf{B}_m$, a flat $(2m, 2n)$-tangle $T \in {}_m\widehat{\mathsf{B}}_n$, and $c, d \in \mathsf{B}_n$ there are *merge cobordisms* $a\overline{b} \amalg bT\overline{c} \to aT\overline{c}$ and $bT\overline{c} \amalg c\overline{d} \to bT\overline{d}$. As usual, these merge cobordisms are well-defined up to unique isomorphisms. The *support* of a merge cobordism is the union of the supports of the sequence of saddle cobordisms. We will also call the union of a merge cobordism with a trivial cobordism a merge cobordism. More generally, a *multi-merge cobordism* is a composition of merge cobordisms.

A *birth cobordism* is a genus 0 decorated cobordism from the empty set to $a\overline{a}$, for some $a \in \mathsf{B}_n$. Birth cobordisms are unions of *cups*; see Figure 3.2. We call the union of the disks bounded by $a\overline{a}$ the *support* of the birth cobordism. A *multi-birth cobordism* is the union of finitely many birth cobordisms with disjoint supports and a trivial cobordism.

We note some commutation relations for cobordisms:

Proposition 3.10. *Let $\Sigma_1\colon Z_1 \to Z_2$ and $\Sigma_2\colon Z_2 \to Z_3$ be saddle cobordisms supported on disjoint disks D_1 and D_2. Let $\Sigma_2'\colon Z_1 \to Z_2'$ and $\Sigma_1'\colon Z_2' \to Z_3$ be saddle cobordisms supported on D_2 and D_1, respectively. Then $\Sigma_2 \circ \Sigma_1$ is isotopic to $\Sigma_1' \circ \Sigma_2'$ rel boundary.*



Proof. This is straightforward, and is left to the reader. □

We state a corollary somewhat informally; it can be formalized along the lines of the statement of Proposition 3.10, but the precise version seems more confusing than enlightening:

Corollary 3.11. *Suppose $\Sigma_1\colon Z_1 \to Z_2$ and $\Sigma_2\colon Z_2 \to Z_3$ is each a multi-saddle or a multi-merge cobordism, and the supports of Σ_1 and Σ_2 are disjoint. Then Σ_1 and Σ_2 commute up to isotopy, in the obvious sense.*

Finally, we note some relations involving births:

Proposition 3.12. *Birth and merge cobordisms satisfy the following relations:*

(1) *Let Z_2 be a divided 1-manifold and $Z_1 \subset Z_2$ a subset which is itself a divided 1-manifold. Then all multi-birth cobordisms from Z_1 to Z_2, in which the circles $Z_2 \setminus Z_1$ are born, are isotopic.*
(2) *If Σ_1 is a multi-birth, multi-merge cobordism or multi-saddle cobordism and Σ_2 is a multi-birth cobordism, and the supports of Σ_1 and Σ_2 are disjoint then Σ_1 and Σ_2 commute up to isotopy.*
(3) *If $\Sigma_1\colon aT\overline{b} \to a\overline{a}\amalg aT\overline{b}$ (respectively $\Sigma_1\colon aT\overline{b} \to aT\overline{b}\amalg b\overline{b}$) is a birth cobordism and $\Sigma_2\colon a\overline{a}\amalg aT\overline{b} \to aT\overline{b}$ (respectively $\Sigma_2\colon aT\overline{b}\amalg b\overline{b} \to aT\overline{b}$) is a merge cobordism then $\Sigma_2\circ\Sigma_1$ is isotopic to a trivial cobordism $aT\overline{b} \to aT\overline{b}$.*
(4) *If $\Sigma_1\colon aT\overline{b} \amalg bT'\overline{c} \to aT\overline{b} \amalg b\overline{b} \amalg bT'\overline{c}$ is a birth cobordism and $\Sigma_2\colon aT\overline{b} \amalg b\overline{b} \amalg bT'\overline{c} \to aTT'\overline{c}$ is a multi-merge cobordism then $\Sigma_2 \circ \Sigma_1$ is isotopic to a merge cobordism $aT\overline{b} \amalg bT'\overline{c} \to aTT'\overline{c}$.*

Proof. Parts (1) and (3) are straightforward from the definitions. Parts (2) and (4) follow from Parts (1) and (3). □

3.4. **A frenzy of functors.** Section 2.11 recalls the Khovanov-Burnside functor, which we can view as a multifunctor $\underline{V}_{HKK}\colon \underline{\widetilde{\mathsf{Cob}}_d} \to \underline{\mathscr{B}}$:

Lemma 3.13. *There is a strict multifunctor $\underline{V}_{HKK}\colon \underline{\widetilde{\mathsf{Cob}}_d} \to \underline{\mathscr{B}}$ defined as follows:*

- *On objects, $\underline{V}_{HKK}(Z) = V_{HKK}(Z)$, the set of labelings of Z by $\{1, X\}$.*
- *On basic multimorphisms, $\underline{V}_{HKK}\big(\Sigma\colon (Z_1,\dots,Z_n) \to Z\big)$ is the correspondence*

$$V_{HKK}(\Sigma)\colon \underline{V}_{HKK}(Z_1)\times\cdots\times\underline{V}_{HKK}(Z_n) \to \underline{V}_{HKK}(Z).$$

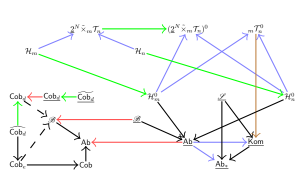
On general multimorphisms of $\underline{\widetilde{\mathsf{Cob}}_d}$ (which are trees with vertices labeled by basic multimorphisms), V_{HKK} is gotten by composing, in some order compatible with the tree, the correspondences $V_{HKK}(\Sigma_v)$ associated to the vertices v.

Given $f \in \mathrm{Hom}_{\underline{\widetilde{\mathsf{Cob}}_d}}(Z_1,\dots,Z_n;Z)$, we have two correspondences from $V_{HKK}(Z_1)\times\cdots\times V_{HKK}(Z_n)$ to $V_{HKK}(Z)$: the correspondence $\underline{V}_{HKK}(f)$, which is a composition of a sequence of correspondences associated to cobordisms, and the correspondence $V_{HKK}(f^\circ)$, which is the correspondence associated to the composition of those cobordisms. The coherence isomorphisms for the lax functor V_{HKK} give an isomorphism $C(f)\colon \underline{V}_{HKK}(f) \to V_{HKK}(f^\circ)$. Now, given $f,g \in \mathrm{Hom}_{\underline{\widetilde{\mathsf{Cob}}_d}}(Z_1,\dots,Z_n;Z)$ and $\phi \in \mathrm{Hom}(f,g)$, let ϕ° be the corresponding morphism in Cob_d from f° to g° and define

$$\underline{V}_{HKK}(\phi) = C(g)^{-1}\circ V_{HKK}(\phi^\circ)\circ C(f).$$

Proof. We must check that:

(1) Given $\phi \in \mathrm{Hom}(f,g)$ and $\psi \in \mathrm{Hom}(g,h)$, $\underline{V}_{HKK}(\psi\circ\phi) = \underline{V}_{HKK}(\psi)\circ\underline{V}_{HKK}(\phi)$, so that $\underline{V}_{HKK}$ defines a map of groupoids.
(2) The functor $\underline{V}_{HKK}$ respects the identity maps. This is trivial.
(3) The functor $\underline{V}_{HKK}$ respects composition of trees.



For Point (1), we have

$$\underline{V}_{HKK}(\psi)\circ\underline{V}_{HKK}(\phi) = C(h)^{-1}\circ V_{HKK}(\psi^\circ)\circ V_{HKK}(\phi^\circ)\circ C(f) = C(h)^{-1}\circ V_{HKK}(\psi^\circ\circ\phi^\circ)\circ C(f) = \underline{V}_{HKK}(\psi\circ\phi),$$

where the second equality uses functoriality of V_{HKK} (Proposition 3.2). For Point (3), at the level of objects of the multimorphism groupoids this is immediate from associativity of composition in $\mathscr{B}$. For morphisms in the multimorphism groupoids this uses naturality of the coherence maps $C(f)$. ☐

Lemma 3.14. *There is a multifunctor $\underline{\mathsf{MC}}_n\colon \mathcal{H}_n \to \underline{\widetilde{\mathsf{Cob}}_d}$ from the Multicategory $\mathcal{H}_n$ to the Cobordism multicategory $\underline{\widetilde{\mathsf{Cob}}_d}$ defined as follows:*

- *On objects, $\underline{\mathsf{MC}}_n((a,b)) = a\overline{b}$, which is a divided 1-manifold as described in Section 3.3.*
- *On basic multimorphisms, $\underline{\mathsf{MC}}_n$ sends $f_{a_1,\dots,a_k}\colon (a_1,a_2),\dots,(a_{k-1},a_k) \to (a_1,a_k)$ to some particular, chosen multi-merge cobordism*

  $$\underline{\mathsf{MC}}_n(f_{a_1,\dots,a_k})\colon a_1\overline{a_2} \amalg \cdots \amalg a_{k-1}\overline{a_k} \to a_1\overline{a_k}$$

  if $k>1$ and to the birth cobordism

  $$\underline{\mathsf{MC}}_n(f_{a_1})\colon \varnothing \to a_1\overline{a_1}$$

  if $k=1$. The functor $\underline{\mathsf{MC}}_n$ assigns to an object in $\mathrm{Hom}_{\mathcal{H}_n}((a_1,a_2),\dots,(a_{k-1},a_k);(a_1,a_k))$ with underlying tree $\pitchfork$ the composition (in $\underline{\widetilde{\mathsf{Cob}}_d}$), according to $\pitchfork$, of the multi-merge or birth cobordisms chosen for each vertex.

Proof. We must check that $\underline{\mathsf{MC}}_n$ extends to the morphisms in the multimorphism groupoids (i.e., 2-morphisms), and that it respects multi-compositions. The fact that $\underline{\mathsf{MC}}_n$ extends to 2-morphisms follows from Corollary 3.11 and Proposition 3.12 (the second of which is only relevant when stumps are involved). The fact that $\underline{\mathsf{MC}}_n$ respects composition is purely formal on the level of 1-multimorphisms (from the definition of the canonical thickening). At the level of 2-morphisms, it follows from the fact that given multimorphisms Σ,Σ' in $\underline{\widetilde{\mathsf{Cob}}_d}$, there is at most one 2-morphism from Σ to Σ'. ☐

Given a flat $(2m,2n)$-tangle $T \in {}_m\widehat{\mathsf{B}}_n$ there is a multifunctor $\underline{\mathsf{MC}}^\flat_T\colon {}_m\mathcal{T}_n \to \underline{\widetilde{\mathsf{Cob}}_d}$ defined similarly to $\underline{\mathsf{MC}}_n$. Indeed, on the subcategories $\mathcal{H}_m,\mathcal{H}_n \subset {}_m\mathcal{T}_n$ the functor $\underline{\mathsf{MC}}^\flat_T$ is exactly $\underline{\mathsf{MC}}_m$, $\underline{\mathsf{MC}}_n$. On objects (a,T,b), let $\underline{\mathsf{MC}}^\flat_T((a,T,b)) = aT\overline{b}$. On the basic multimorphisms

$$f_{a_1,\dots,a_i,T,b_1,\dots,b_j}\colon (a_1,a_2),\dots,(a_{i-1},a_i),(a_i,T,b_1),(b_1,b_2),\dots,(b_{j-1},b_j) \to (a_1,T,b_j)$$

the functor $\underline{\mathsf{MC}}^\flat_T(f_{a_1,\dots,a_i,T,b_1,\dots,b_j})$ is some chosen multi-merge cobordism corresponding to the obvious merging. As usual, this extends formally to general objects in the multimorphism groupoids.

Lemma 3.15. *This construction extends uniquely to a multifunctor $\underline{\mathsf{MC}}^\flat_T\colon {}_m\mathcal{T}_n \to \underline{\widetilde{\mathsf{Cob}}_d}$.*

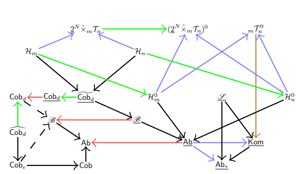
Proof. The proof is essentially the same as the proof of Lemma 3.14, and is left to the reader. ☐

Next, fix a $(2m,2n)$-tangle diagram $T \in {}_m\mathsf{D}_n$ with N ordered crossings. We associate to T a multifunctor

$$\underline{\mathsf{MC}}_T\colon \underline{2}^N \tilde{\times}\, {}_m\mathcal{T}_n \to \underline{\widetilde{\mathsf{Cob}}_d}$$

as follows. First, choose a collection of disjoint disks D_i around the crossings of T, and for each $v \in \{0,1\}^N$ choose a particular flat tangle T_v representing the v-resolution of T, so that T_v agrees with (the projection of) T outside the disks D_i.

Now, objects of $\underline{2}^N \tilde{\times}\, {}_m\mathcal{T}_n$ are of three kinds:



- Pairs (a, b) where $a, b \in \mathsf{B}_m$. In this case we define $\underline{\mathsf{MC}}_T(a, b) = a\bar{b}$, which we give the structure of a divided 1-manifold as described in Section 3.3.
- Pairs (a, b) where $a, b \in \mathsf{B}_n$. In this case we (again) define $\underline{\mathsf{MC}}_T(a, b) = a\bar{b}$.
- Quadruples (v, a, T, b) where $v \in \{0, 1\}^N$, $a \in \mathsf{B}_m$ and $b \in \mathsf{B}_n$. In this case, we define $\underline{\mathsf{MC}}_T(v, a, T, b) = aT_v\bar{b}$. We give $aT_v\bar{b}$ the structure of a divided 1-manifold as described in Section 3.3.

As always, defining $\underline{\mathsf{MC}}_T$ on multimorphism groupoids takes more work. To define $\underline{\mathsf{MC}}_T$ on objects of the multimorphism groupoids it suffices to define $\underline{\mathsf{MC}}_T$ for the following two elementary morphisms:

- A basic multimorphism coming from a morphism in $\underline{2}^N$, i.e., a map $f_{\underline{2}^N} : (v, a, T, b) \to (w, a, T, b)$. Define $\underline{\mathsf{MC}}_T(f_{\underline{2}^N})$ to be a multi-saddle cobordism from $aT_v\bar{b}$ to $aT_w\bar{b}$ (see Section 3.3).
- A basic multimorphism coming from a morphism $f_{{}_m\mathcal{T}_n}$ in ${}_m\mathcal{T}_n^0$. In this case define $\underline{\mathsf{MC}}_T(f_{{}_m\mathcal{T}_n})$ to be the cobordism $\underline{\mathsf{MC}}^\flat_{T_v}(f_{{}_m\mathcal{T}_n})$ (associated to the flat tangle T_v).

On a general object, $\underline{\mathsf{MC}}_T$ is defined by composing these multimorphisms according to the tree. (Since this composition happens in $\widetilde{\underline{\mathsf{Cob}}_d}$, given a multimorphism f in $\underline{2}^N \tilde{\times}_m \mathcal{T}_n$ with underlying tree $\pitchfork$, $\underline{\mathsf{MC}}_T(f)$ is the same tree $\pitchfork$ with vertices labeled by the divided cobordisms corresponding to the labels in f.)

Since there is a unique isomorphism between isotopic divided cobordisms, to extend $\underline{\mathsf{MC}}_T$ to morphisms in the multimorphism groupoids it suffices to show that if two morphisms $\pitchfork$, $\pitchfork'$ in $\underline{2}^N \tilde{\times}_m \mathcal{T}_n$ have a morphism between them the divided cobordisms $\underline{\mathsf{MC}}_T(\pitchfork)^\circ$ and $\underline{\mathsf{MC}}_T(\pitchfork')^\circ$ are isotopic.

Lemma 3.16. *If $\pitchfork$ and $\pitchfork'$ are multimorphisms in $\underline{2}^N \tilde{\times}_m \mathcal{T}_n$ with the same source and target then the divided cobordisms $\underline{\mathsf{MC}}_T(\pitchfork)^\circ$ and $\underline{\mathsf{MC}}_T(\pitchfork')^\circ$ are isotopic.*

Proof. Both $\underline{\mathsf{MC}}_T(\pitchfork)^\circ$ and $\underline{\mathsf{MC}}_T(\pitchfork')^\circ$ are compositions of:

- multi-merge cobordisms of crossingless matchings,
- saddle cobordisms supported on small disks around certain crossings of T, which are disjoint from the crossingless matchings being merged, and
- multi-birth cobordisms, corresponding to stump leaves, each of which is followed by a multi-merge cobordism.

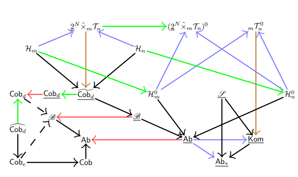
By Proposition 3.12, if we let $\pitchfork_0$ (respectively $\pitchfork'_0$) be the result of removing all stump leaves from $\pitchfork$ then $\underline{\mathsf{MC}}_T(\pitchfork)^\circ$ and $\underline{\mathsf{MC}}_T(\pitchfork_0)^\circ$ are isotopic, as are $\underline{\mathsf{MC}}_T(\pitchfork')^\circ$ and $\underline{\mathsf{MC}}_T(\pitchfork'_0)^\circ$. Now, since the source and target of $\pitchfork_0$ and $\pitchfork'_0$ are the same, the cobordisms $\underline{\mathsf{MC}}_T(\pitchfork_0)^\circ$ and $\underline{\mathsf{MC}}_T(\pitchfork'_0)^\circ$ have saddles at the same crossings and merge the same crossingless matchings. Thus, the result follows from Corollary 3.11 and the fact that all multi-merge cobordisms with the same source and target are isotopic. ☐

Proposition 3.17. *The map $\underline{\mathsf{MC}}_T$ does, indeed, define a multifunctor $\underline{2}^N \tilde{\times}_m \mathcal{T}_n \to \widetilde{\underline{\mathsf{Cob}}_d}$.*

Proof. By Lemma 3.16, the map $\underline{\mathsf{MC}}_T$ is well-defined. We must check that it respects multi-composition. At the level of objects of the multimorphism groupoids, since we defined $\underline{\mathsf{MC}}_T(f)$ by composing the values of $\underline{\mathsf{MC}}_T$ on basic multimorphisms, this is immediate from the definition. Since each 2-morphism set in $\widetilde{\underline{\mathsf{Cob}}_d}$ is empty or has 1 element, at the level of morphisms of the multimorphism groupoids there is nothing to check. ☐

3.5. **The initial invariant.** In this section, we will construct combinatorial tangle invariants as equivalence classes of multifunctors to the Burnside multicategory. Explicity, to the $2n$ points $[2n]_{\mathrm{std}} \subset (0, 1)$, we associate the functor from the Multicategory $\mathcal{H}_n$ to the Burnside multicategory $\underline{\mathscr{B}}$

$$\underline{\mathsf{MB}}_n \coloneqq \underline{V}_{HKK} \circ \underline{\mathsf{MC}}_n : \mathcal{H}_n \to \underline{\mathscr{B}},$$



and to a tangle diagram $T \in {}_m\mathsf{D}_n$ connecting $\{0\} \times \{0\} \times [2m]_{\mathrm{std}}$ to $\{0\} \times \{1\} \times [2n]_{\mathrm{std}}$, we associate the pair $(\underline{\mathsf{MB}}_T, N_+)$, where $\underline{\mathsf{MB}}_T$ is the functor

$$\underline{\mathsf{MB}}_T \coloneqq \underline{V}_{HKK} \circ \underline{\mathsf{MC}}_T \colon \underline{2}^N \tilde{\times}_m \mathcal{T}_n \to \underline{\mathscr{B}}$$

and N_+ is the number of positive crossings in the oriented tangle diagram T. We will refer to this sort of pairs often, so we give it a name:

Definition 3.18. A *stable functor* from $\underline{2}^N \tilde{\times}_m \mathcal{T}_n$ to $\underline{\mathscr{B}}$ is a pair

$$(\text{functor } F\colon \underline{2}^N \tilde{\times}_m \mathcal{T}_n \to \underline{\mathscr{B}}, \text{integer } S)$$

so that the restriction of F to the subcategory $\mathcal{H}_m$ (respectively $\mathcal{H}_n$) of $\underline{2}^N \tilde{\times}_m \mathcal{T}_n$ is $\underline{\mathsf{MB}}_m$ (respectively $\underline{\mathsf{MB}}_n$).

3.5.1. *Recovering the Khovanov invariants.* Given a functor $F_n \colon \mathcal{H}_n \to \underline{\mathscr{B}}$, we can compose with the forgetful functor $\mathcal{F}_{orget} \colon \underline{\mathscr{B}} \to \underline{\mathsf{Ab}}$ to obtain a functor $\mathcal{F}_{orget} \circ F_n \colon \mathcal{H}_n \to \underline{\mathsf{Ab}}$. Since $\underline{\mathsf{Ab}}$ is trivially enriched, the functor $\mathcal{F}_{orget} \circ F_n$ descends to an un-enriched multifunctor, still denoted $\mathcal{F}_{orget} \circ F_n$, from the strictification $\mathcal{H}_n^0$ to $\underline{\mathsf{Ab}}$.

Similary, given a stable functor $(F\colon \underline{2}^N \tilde{\times}_m \mathcal{T}_n \to \underline{\mathscr{B}}, S)$ we get a functor $\mathcal{F}_{orget} \circ F \colon (\underline{2}^N \tilde{\times}_m \mathcal{T}_n)^0 \to \underline{\mathsf{Ab}}$. We can associate to the pair $(\mathcal{F}_{orget} \circ F, S)$ a functor

$$\mathsf{Tot}(\mathcal{F}_{orget} \circ F, S) \colon {}_m\mathcal{T}_n^0 \to \underline{\mathsf{Kom}},$$

which restricts to $\mathcal{F}_{orget} \circ F_m$ and $\mathcal{F}_{orget} \circ F_n$ on the subcategories $\mathcal{H}_m$ and $\mathcal{H}_n$, as follows. Given an object $(a, b) \in \mathrm{Ob}({}_m\mathcal{T}_n)^0$ we let

$$\mathsf{Tot}(\mathcal{F}_{orget} \circ F, M)(a, b) = (\mathcal{F}_{orget} \circ F)(a, b),$$

viewed as a chain complex concentrated in grading 0. Given an object $(a, T, b) \in \mathrm{Ob}({}_m\mathcal{T}_n^0)$ there is an associated subcategory $\underline{2}^N \times (a, T, b)$ of $(\underline{2}^N \tilde{\times}_m \mathcal{T}_n)^0$ isomorphic to the cube $\underline{2}^N$: it is the full subcategory spanned by objects of the form (v, a, T, b). Let $\mathsf{Tot}(\mathcal{F}_{orget} \circ F, M)(a, T, b)$ be the totalization of the cube of abelian groups $\mathcal{F}_{orget} \circ F|_{\underline{2}^N \times (a,T,b)}$, cf. Equation (2.2), followed by a downward grading shift by the integer S (so that the chain complex is supported in gradings $[-S, N - S]$).

Lemma 3.19. *The Khovanov arc algebra $\mathcal{H}^m$ (respectively $\mathcal{H}^n$) is the functor $\mathcal{F}_{orget} \circ \underline{\mathsf{MB}}_m \colon \mathcal{H}_m^0 \to \underline{\mathsf{Ab}}$ (respectively $\mathcal{F}_{orget} \circ \underline{\mathsf{MB}}_n \colon \mathcal{H}_n^0 \to \underline{\mathsf{Ab}}$) which is the restriction of $\mathsf{Tot}(\mathcal{F}_{orget} \circ \underline{\mathsf{MB}}_T, N_+)$ to $\mathcal{H}_m^0$ (respectively $\mathcal{H}_n^0$), and the Khovanov tangle invariant $\mathcal{C}_{Kh}(T)$ is the functor $\mathsf{Tot}(\mathcal{F}_{orget} \circ \underline{\mathsf{MB}}_T, N_+) \colon {}_m\mathcal{T}_n \to \underline{\mathsf{Kom}}$, reinterpreted per Principle 3.4.*

Proof. This is an exercise in unwinding the definitions. □

3.5.2. *Invariance.* Next we describe in what sense is the functor $\underline{\mathsf{MB}}_n \colon \mathcal{H}_n \to \underline{\mathscr{B}}$ an invariant of $2n$ points, and in what sense is the stable functor $(\underline{\mathsf{MB}}_T \colon \underline{2}^N \tilde{\times}_m \mathcal{T}_n \to \underline{\mathscr{B}}, N_+)$ an invariant for the underlying tangle. First we do $\underline{\mathsf{MB}}_n$.

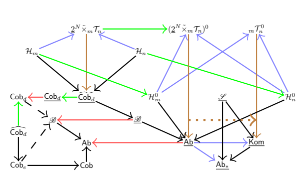
Superficially, the functor $\underline{\mathsf{MB}}_n \colon \mathcal{H}_n^0 \to \underline{\mathscr{B}}$ depended on a number of choices:

(C-1) The choice of curves representing each isotopy class of crossingless matching in B_n.
(C-2) The choice of divided multi-merge cobordisms.
(C-3) The choice of embeddings in the definitions of the Burnside multicategory (Section 3.2.1).

To deal with this, we could make specific once-and-for-all choices; or we can invoke the following:

Definition 3.20. A *natural isomorphism* η between multifunctors F, G from a groupoid enriched multicategory $\mathscr{C}$ to $\underline{\mathscr{B}}$ is a collection of bijections $\eta_x \colon F(x) \to G(x)$ for all objects $x \in \mathrm{Ob}(\mathscr{C})$, and $\eta_\phi \colon F(\phi) \to G(\phi)$ for all multimorphisms $\phi \in \mathrm{Hom}(x_1, \dots, x_n; y)$ which are compatible with the 2-morphisms and the source and the target maps in the following sense: for any objects $x_1, \dots, x_n, y \in \mathrm{Ob}(\mathscr{C})$, any multimorphisms $\phi, \psi \in \mathrm{Hom}(x_1, \dots, x_n; y)$, any 2-morphism $\kappa \colon \phi \to \psi$, and any element $w \in F(\phi)$,

$$\eta_\psi \circ F(\kappa)(w) = G(\kappa) \circ \eta_\phi(w), \qquad (\eta_{x_1}, \dots, \eta_{x_n}) \circ s(w) = s \circ \eta_\phi(w), \qquad \eta_y \circ t(w) = t \circ \eta_\phi(w).$$



Lemma 3.21. *Let $\underline{\mathsf{MB}}_n^1, \underline{\mathsf{MB}}_n^2 \colon \mathcal{H}_n \to \underline{\mathscr{B}}$ be the functors associated to two different choices of curves, multi-merge cobordisms, and embeddings of associated sets. Then there is a natural isomorphism $\eta^{12} \colon \underline{\mathsf{MB}}_n^1 \to \underline{\mathsf{MB}}_n^2$. Further, these maps η form a transitive system, in the sense that η^{11} is the identity and if $\underline{\mathsf{MB}}_n^3 \colon \mathcal{H}_n \to \underline{\mathscr{B}}$ is the functor associated to a third collection of choices then $\eta^{13} = \eta^{23} \circ \eta^{12}$.*

Proof. Since the 2-morphisms in the Burnside multicategory pay no attention to the embeddings of the correspondences, the identity 2-morphisms give a transitive system of natural isomorphisms associated to changing the embeddings of correspondences. Similarly, any two choices of divided multi-merge cobordisms are uniquely isomorphic (because isotopic divided cobordisms are uniquely isomorphic), so different choices of decorated cobordisms give naturally isomorphic functors, and these natural isomorphisms are transitive. Finally, any two choices of representatives of the crossingless matchings are related by an essentially cylindrical divided cobordism, and this divided cobordism is unique up to unique isomorphism; independence from the choice of curves representing the crossingless matchings follows. □

Now, consider the groupoid $\mathscr{C}$ with:

- Objects sets of choices (C-1)–(C-3).
- A unique morphism between each pair of objects.

Lemma 3.21 asserts that we have a functor $\mathscr{C} \to \mathrm{Fun}(\mathcal{H}_n, \underline{\mathscr{B}})$, where $\mathrm{Fun}(\mathcal{H}_n, \underline{\mathscr{B}})$ is the category of functors from $\mathcal{H}_n \to \underline{\mathscr{B}}$ with morphisms being natural isomorphisms. Existence of this functor on the contractible groupoid $\mathscr{C}$ expresses the fact that different choices are canonically isomorphic.

Following the standard colimit procedure, we can harness the above fact to construct $\underline{\mathsf{MB}}_n$ as a functor independent of choices. For any object x and any multimorphism ϕ of $\mathcal{H}_n$, define

$$\underline{\mathsf{MB}}_n(x) = \coprod_{i \in \mathrm{Ob}(C)} \underline{\mathsf{MB}}_n^i(x)/\sim \ \text{ and } \underline{\mathsf{MB}}_n(\phi) = \coprod_{i \in \mathrm{Ob}(C)} \underline{\mathsf{MB}}_n^i(\phi)/\sim,$$

where the equivalence relation $\sim$ identifies $u \in \underline{\mathsf{MB}}_n^i(x)$ (respectively, $w \in \underline{\mathsf{MB}}_n^i(\phi)$) with $\eta_x^{i,j}(u) \in \underline{\mathsf{MB}}_n^j(x)$ (respectively, $\eta_\phi^{i,j}(w) \in \underline{\mathsf{MB}}_n^i(\phi)$) for any $i, j \in \mathrm{Ob}(\mathscr{C})$, with the source, target, and 2-morphism maps defined componentwise.

For the rest of the paper, we will elide the fact that $\underline{\mathsf{MB}}_n \colon \mathcal{H}_n \to \underline{\mathscr{B}}$ depended on choices, and expect the reader to either assume we made once-and-for-all choices in defining $\underline{\mathsf{MB}}_n$, or insert the discussion above where appropriate.

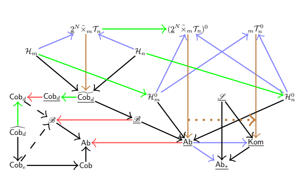
Next we turn to $\underline{\mathsf{MB}}_T$.

Definition 3.22. Given multifunctors $F, G \colon \underline{2}^N \tilde{\times}_m \mathcal{T}_n \to \underline{\mathscr{B}}$, and any integer S, a *natural transformation* connecting the stable functors (F, S) to (G, S) is a multifunctor $H \colon \underline{2}^{N+1} \tilde{\times}_m \mathcal{T}_n \to \underline{\mathscr{B}}$ so that $H|_{\{0\} \times \underline{2}^N \tilde{\times}_m \mathcal{T}_n} = F$ and $H|_{\{1\} \times \underline{2}^N \tilde{\times}_m \mathcal{T}_n} = G$. A natural transformation from (F, S) to (G, S) induces a homomorphism of *dg* modules $\mathsf{Tot}(\mathcal{F}_{orget} \circ F, S) \to \mathsf{Tot}(\mathcal{F}_{orget} \circ G, S)$ in an obvious way, where $\mathsf{Tot}(\mathcal{F}_{orget} \circ F, S)$ and $\mathsf{Tot}(\mathcal{F}_{orget} \circ G, S)$ are being viewed as *dg* bimodules as per Section 2.3. We call H a *quasi-isomorphism* if the induced chain map is a quasi-isomorphism.

Proposition 3.23. *Up to quasi-isomorphism, the stable functor $(\underline{\mathsf{MB}}_T, N_+)$ is independent of the choices of resolutions and cobordisms in the definition of $\underline{\mathsf{MC}}_T$.*

Proof. Fix choices $\underline{\mathsf{MC}}_T^0$ and $\underline{\mathsf{MC}}_T^1$. We will define a natural transformation $H \colon \underline{2}^{N+1} \tilde{\times}_m \mathcal{T}_n \to \widetilde{\underline{\mathsf{Cob}_d}}$ from $\underline{\mathsf{MC}}_T^0$ to $\underline{\mathsf{MC}}_T^1$ and then compose with $\underline{V}_{HKK}$ to get a natural transformation from $\underline{\mathsf{MB}}_T^0$ to $\underline{\mathsf{MB}}_T^1$.

On the subcategories $\mathcal{H}_m$ and $\mathcal{H}_n$ of $\underline{2}^N \tilde{\times}_m \mathcal{T}_n$, $\underline{\mathsf{MC}}_T^0$ and $\underline{\mathsf{MC}}_T^1$ already agree. From the definition of $\tilde{\times}$, to define H on the objects of multimorphism groupoids, it suffices to define H on maps $f_{\underline{2}^{N+1}} \times \mathrm{Id}_{(a,T,b)}$, where $f_{\underline{2}^{N+1}} \colon (0, v) \to (1, w)$ is a morphism from $\{0\} \times \underline{2}^N$ to $\{1\} \times \underline{2}^N$, since H has already been defined on



the other type of elementary morphisms. Define $H(f_{\underline{2}^{N+1}} \times \mathrm{Id}_{(a,T,b)})$ to be any multi-saddle cobordism from the resolution T_v with respect to the first set of choices to the resolution T_w with respect to the second set of choices. The extension of H to morphisms in the multimorphism groupoids proceeds without incident as in the construction of $\underline{\mathsf{MC}}_T$ using Lemma 3.16.

The induced diagram of chain complexes $\mathsf{Tot}(\mathcal{F}_{orget} \circ \underline{V}_{HKK} \circ H, N_+)$ sends the arrows $(0,v) \to (1,v)$ to identity maps. Thus, the map $\mathsf{Tot}(\mathcal{F}_{orget} \circ \underline{\mathsf{MB}}^0_T, N_+) \to \mathsf{Tot}(\mathcal{F}_{orget} \circ \underline{\mathsf{MB}}^1_T, N_+)$ is the identity map, and hence is a quasi-isomorphism (indeed, an isomorphism). □

Definition 3.24. A *face inclusion* is a functor $i\colon \underline{2}^M \to \underline{2}^N$ that is injective on objects and preserves relative gradings (see [LLS, Definition 5.5]). Let $|i|$ be the absolute grading shift of i, given by $|i(v)| - |v|$ for any $v \in \mathrm{Ob}(\underline{2}^M)$, where $|\cdot|$ denotes the height (number of 1's) in the cube. Given a stable functor $(F\colon \underline{2}^M \tilde{\times}_m \mathcal{T}_n \to \underline{\mathscr{B}}, S)$ and a face inclusion $i\colon \underline{2}^M \hookrightarrow \underline{2}^N$ there is an induced stable functor $(i_! F\colon \underline{2}^N \tilde{\times}_m \mathcal{T}_n \to \underline{\mathscr{B}}, S+N-M-|i|)$, where $i_! F$ is defined as follows:

- On objects of the form (a,b), $(i_!F)(a,b) = F(a,b)$. On objects of the form (v,a,T,b)
$$(i_!F)(v,a,T,b) = \begin{cases} F(u,a,T,b) & \text{if } v = i(u) \text{ is in the image of } i, \\ \varnothing & \text{otherwise.} \end{cases}$$
- On multimorphisms, if all of the input and output leaves of a tree $\pitchfork$ are labeled by elements (v,a,T,b) with v in the image of i then the same must be true for all intermediate edges and vertices, so there is a tree $\pitchfork'$ with $i(\pitchfork') = \pitchfork$ (in the obvious sense), and we define $(i_!F)(\pitchfork) = F(\pitchfork')$. Otherwise, $(i_!F)(\pitchfork)$ is the empty correspondence. (Note that, in the second case, at least one of the source or target of $(i_!F)(\pitchfork)$ is the empty set.)

We call $(i_!F, S+N-M-|i|)$ a *stabilization* of (F,S) and (F,S) a *destabilization* of $(i_!F, S+N-M-|i|)$. The *dg* bimodules $\mathsf{Tot}(\mathcal{F}_{orget} \circ F, S)$ and $\mathsf{Tot}(\mathcal{F}_{orget} \circ i_!F, S+N-M-|i|)$ are isomorphic, and the isomorphism is canonical up to an overall sign.

Call stable functors $(F\colon \underline{2}^M \tilde{\times}_m \mathcal{T}_n \to \underline{\mathscr{B}}, R)$ and $(G\colon \underline{2}^N \tilde{\times}_m \mathcal{T}_n \to \underline{\mathscr{B}}, S)$ *stably equivalent* if (F,R) and (G,S) are related by a sequence of quasi-isomorphisms, stabilizations, and destabilizations.

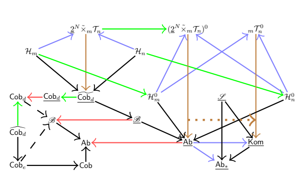
There are some particularly convenient ways to produce equivalences:

Definition 3.25. Given a functor $F\colon \underline{2}^N \tilde{\times}_m \mathcal{T}_n \to \underline{\mathscr{B}}$, an *insular subfunctor* of F is a collection of subsets $G(v,a,T,b) \subset F(v,a,T,b)$, such that for any $x_i \in F(a_i,a_{i+1})$, $y \in G(u,a_k,T,b_1)$, $z_i \in F(b_i,b_{i+1})$, $w \in F(v,a_1,T,b_\ell) \setminus G(v,a_1,T,b_\ell)$, and
$$f \in \mathrm{Hom}((a_1,a_2),\dots,(a_{k-1},a_k),(u,a_k,T,b_1),(b_1,b_2),\dots,(b_{\ell-1},b_\ell);(v,a_1,T,b_\ell)),$$
$$s^{-1}(x_1,\dots,x_{k-1},y,z_1,\dots,z_{\ell-1}) \cap t^{-1}(w) = \varnothing \subset F(f). \tag{3.1}$$

Extend G to a functor $G\colon \underline{2}^N \tilde{\times}_m \mathcal{T}_n \to \underline{\mathscr{B}}$ by defining $G(a,b) = F(a,b)$ for $(a,b) \in \mathrm{Ob}(\mathcal{H}_m) \cup \mathrm{Ob}(\mathcal{H}_n)$ and, for $f \in \mathrm{Hom}(p_1,\dots,p_n;q)$,
$$G(f) = s^{-1}(G(p_1) \times \cdots \times G(p_n)) \cap t^{-1}(G(q)) \subset F(f)$$
with the obvious source and target maps, and 2-morphisms induced by F in the obvious way. The fact that G respects composition follows from Equation (3.1).

Given an insular subfunctor G of F there is a *quotient functor* $F/G\colon \underline{2}^N \tilde{\times}_m \mathcal{T}_n \to \underline{\mathscr{B}}$ defined by:

- $(F/G)(a,b) = F(a,b)$,
- $(F/G)(v,a,T,b) = F(v,a,T,b) \setminus G(v,a,T,b)$,



- $(F/G)(f) = s^{-1}((F/G)(p_1) \times \cdots \times (F/G)(p_n)) \cap t^{-1}((F/G)(q)) \subset F(f)$ for $f \in \mathrm{Hom}(p_1, \dots, p_n; q)$, and
- the value of F/G on 2-morphisms is induced by F.

Again, the fact that this defines a functor follows from Equation (3.1).

Given an insular subfunctor G of F, and any integer S, there is an induced short exact sequence of *dg* bimodules

$$0 \to \mathsf{Tot}(\mathcal{F}_{orget} \circ G, S) \hookrightarrow \mathsf{Tot}(\mathcal{F}_{orget} \circ F, S) \twoheadrightarrow \mathsf{Tot}(\mathcal{F}_{orget} \circ (F/G), S) \to 0.$$

Lemma 3.26. *Fix any integer S. If G is an insular subfunctor of F then there is a natural transformation η from (G, S) to (F, S) so that the induced map of differential bimodules is the inclusion map defined above. There is also a natural transformation θ from (F, S) to $(F/G, S)$ so that the induced map of differential bimodules is the quotient map defined above. In particular, if the inclusion (respectively quotient) map of chain complexes is a quasi-isomorphism then the map η (respectively θ) is an equivalence.*

Proof. To define η (respectively θ), for

$$f \in \mathrm{Hom}((a_1, a_2), \dots, (a_{k-1}, a_k), ((0, u), a_k, T, b_1), (b_1, b_2), \dots, (b_{\ell-1}, b_\ell); ((1, v), a_1, T, b_\ell))$$

a basic multimorphism there is a corresponding basic multimorphism

$$\widetilde{f} \in \mathrm{Hom}((a_1, a_2), \dots, (a_{k-1}, a_k), (u, a_k, T, b_1), (b_1, b_2), \dots, (b_{\ell-1}, b_\ell); (v, a_1, T, b_\ell)).$$

Define $\eta(f) = G(\widetilde{f})$ (respectively $\theta(f) = (F/G)(\widetilde{f})$). Similarly, on 2-morphisms η (respectively θ) is induced by G (respectively F/G). It is straightforward to verify that these definitions make η and θ into natural transformations with the desired properties. □

Theorem 3. *The stable equivalence class of $\underline{\mathsf{MB}}_T$ is invariant under Reidemeister moves, and so gives a tangle invariant. Further, the chain map*

$$\mathsf{Tot}(\mathcal{F}_{orget} \circ \underline{\mathsf{MB}}_{T_1}, N_+(T_1)) \to \mathsf{Tot}(\mathcal{F}_{orget} \circ \underline{\mathsf{MB}}_{T_2}, N_+(T_2))$$

induced by a sequence of Reidemeister moves relating T_1 and T_2 agrees, up to a sign and homotopy, with Khovanov's invariance maps [Kho02, Section 4].

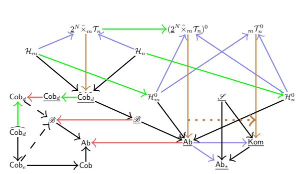
Proof. This is essentially a translation of the invariance proof for the Khovanov homotopy type [LS14a, Section 6] (itself a modest extension of invariance proofs for Khovanov homology) to the language of this paper.

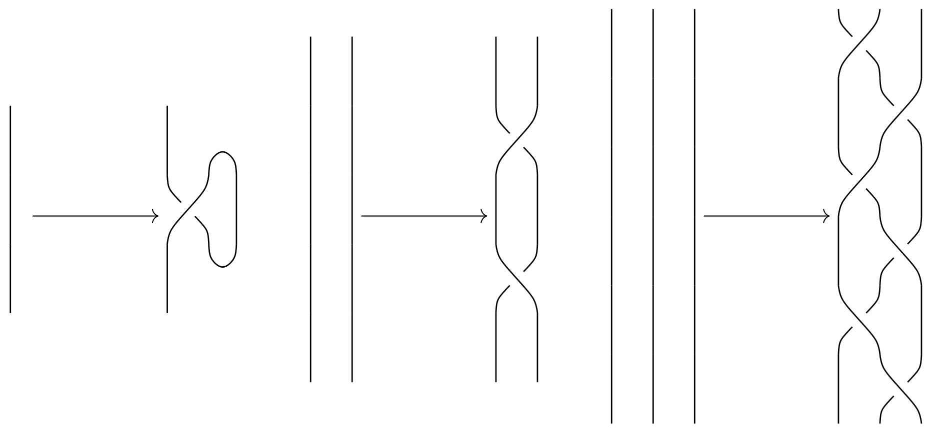
It suffices to verify invariance under reordering of the crossings and the three Reidemeister moves shown in Figure 3.3, because this Reidemeister I and the Reidemeister II move generate the other Reidemeister I move, and the usual Reidemeister III move is generated by this braid-like Reidemeister III move and Reidemeiester II moves (see [Bal11, Section 7.3]).

If $T \in {}_m\mathsf{D}_n$ is a $(2m, 2n)$-tangle diagram with N ordered crossings, and if $T' \in {}_m\mathsf{D}_n$ is the same tangle diagram, but with its crossings reordered by some permutation $\sigma \in \mathfrak{S}_N$, then the stable functor $(\underline{\mathsf{MC}}_{T'}, N_+)$ is the stabilization $(i_!\underline{\mathsf{MC}}_T, N_+)$, where $i \colon \underline{2}^N \to \underline{2}^N$ is the face inclusion $(v_1, \dots, v_N) \mapsto (v_{\sigma(1)}, \dots, v_{\sigma(N)})$.

Next we turn to the Reidemeister I move. Let $T \in {}_m\mathsf{D}_n$ be a $(2m, 2n)$-tangle diagram with N ordered crossings, of which N_+ are positive, and T' the result of performing a Reidemeister I move to T as in Figure 3.3, so T' has one more positive crossing c than T; assume c is the $(N+1)^{\text{st}}$ crossing of T'. Note that the 1-resolution of c gives a tangle isotopic to T and the 0-resolution of c gives the disjoint union of T and a small circle C. For each object $(v, a, T, b) \in \mathrm{Ob}(\underline{2}^{N+1} \tilde{\times}_m \mathcal{T}_n)$ define $G(v, a, T, b) \subset \underline{\mathsf{MC}}_{T'}(v, a, T, b)$ as

$$G(v, a, T, b) = \begin{cases} \underline{\mathsf{MC}}_{T'}(v, a, T, b) & \text{if } v_{N+1} = 1 \\ \{w \in \underline{\mathsf{MC}}_{T'}(v, a, T, b) \mid w \text{ assigns 1 to } C\} & \text{if } v_{N+1} = 0. \end{cases}$$





a: RI. **b:** RII. **c:** RIII.

FIGURE 3.3. **Reidemeiester moves.** The orientations of the strands are arbitrary. This figure originally appeared in [LS14a].

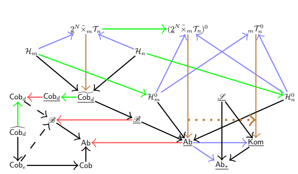
(Compare [LS14a, Figure 6.2].) We claim that G is an insular subfunctor of $\underline{\mathsf{MC}}_{T'}$ and that the chain complex associated to G is acyclic. The second statement is clear. For the first, note that every element $w \in \underline{\mathsf{MC}}_{T'}(v,a,T,b) \setminus G(v,a,T,b)$ is supported over the 0-resolution at c, and assigns X to the small circle C. The maps associated to the algebra action respect the labeling of C, and the edges in the cube go from the 0-resolution to the 1-resolution, and hence either do not change the crossing c or map to a resolution in which $G(v,a,T,b) = \underline{\mathsf{MC}}_{T'}(v,a,T,b)$.

Thus, by Lemma 3.26, $(\underline{\mathsf{MC}}_{T'}, N_+ + 1)$ is stably equivalent to $(\underline{\mathsf{MC}}_{T'}/G, N_+ + 1)$. If $i\colon \underline{2}^N \to \underline{2}^{N+1}$ is the face inclusion $(v_1,\dots,v_N) \mapsto (v_1,\dots,v_N,0)$, forgetting the circle C gives an isomorphism from $(\underline{\mathsf{MC}}_{T'}/G, N_+ + 1)$ to $(i_!\underline{\mathsf{MC}}_T, N_+ + 1)$, which is stabilization of $(\underline{\mathsf{MC}}_T, N_+)$.

The proofs of Reidemeister II and III invariance are similar adaptations of the proofs from our previous paper [LS14a, Propositions 6.3 and 6.4]. For Reidemeister II invariance, that proof defines a contractible insular subfunctor G_1 of $\underline{\mathsf{MC}}_{T'}$ and an insular subfunctor G_3 of the quotient $G_2 = \underline{\mathsf{MC}}_{T'}/G_1$ so that the quotient $G_4 = G_2/G_3$ is contractible, and G_3 is isomorphic to $\underline{\mathsf{MC}}_T$ modulo the correct grading shifts. (See particularly [LS14a, Figure 6.3], where circles labeled 1 are denoted $+$ and circles labeled X are denoted $-$.) The new point is that all of these subsets are preserved by the algebra action; but this is obvious from their definitions, which only involve restricting to certain vertices of the cube or restricting the labels of certain closed circles. Similarly, for Reidemeister III invariance the old proof gives a sequence of insular subfunctors inducing equivalences. Further details are left to the reader.

The second part of the statement follows from the fact that, locally, up to sign there is a unique homotopy class of homotopy equivalences of $(\mathcal{H}^n, \mathcal{H}^{n-2})$-bimodules (respectively $(\mathcal{H}^n, \mathcal{H}^n)$-bimodules) corresponding to a Reidemeister I move (respectively II or III move). (See Figure 3.4.) Both the map on the chain complexes induced by the construction above and Khovanov's map respect composition of tangles and so are induced from local maps. See our previous paper [LS14b, Proposition 3.4] for further details. □

4. FROM COMBINATORICS TO TOPOLOGY



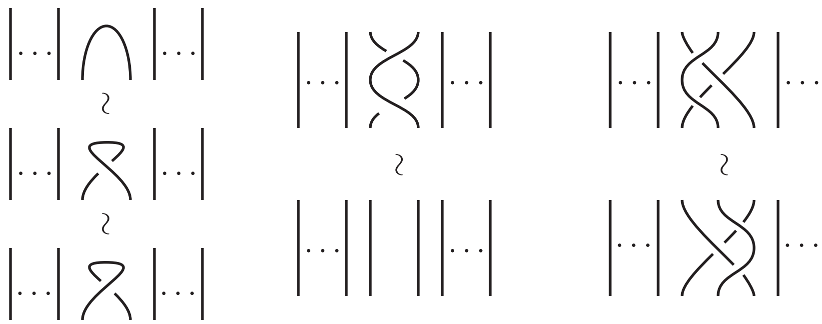


FIGURE 3.4. **Local Reidemeister moves.** Khovanov's invariance proof shows that the bimodules before and after each Reidemeister move are quasi-isomorphic; and in fact there is an essentially unique, up to sign, quasi-isomorphism between them.

4.1. **Construction of the spectral categories and bimodules.** We warm up by giving a functor $G\colon \mathcal{H}_n^0 \to \underline{\mathscr{S}}$ refining the arc algebras. In Section 3.5 we defined a functor $\underline{\mathsf{MB}}_n\colon \mathcal{H}_n \to \underline{\mathscr{B}}$. The Burnside multicategory maps to the multicategory of permutative categories Permu, by taking a set X to the category Sets/X of finite sets over X, and a correspondence $A\colon X \to Y$ to the functor $\mathsf{Sets}/X \to \mathsf{Sets}/Y$ given by fiber product with A (cf. Section 2.8). Elmendorf-Mandell define a multifunctor $\mathsf{Permu} \to \underline{\mathscr{S}}$, K-theory, where $\underline{\mathscr{S}}$ is the multicategory of symmetric spectra (with multicategory structure induced by the smash product) [EM06, Theorem 1.1]. (Again, see Section 2.8.) So, composing with this functor gives us a functor

$$\mathcal{H}_n \to \underline{\mathscr{S}}.$$

Rectification as in Definition 2.42 combined with Lemma 2.8, turns this into a functor

$$G\colon \mathcal{H}_n^0 \to \underline{\mathscr{S}}. \tag{4.1}$$

The story for tangles is similar. Given a tangle diagram $T \in {}_m\mathsf{D}_n$ (with N ordered crossings, of which N_+ are positive), in Section 3.5 we defined a stable functor $(\underline{\mathsf{MB}}_T\colon \underline{2}^N \tilde{\times}_m \mathcal{T}_n \to \underline{\mathscr{B}}, N_+)$. Compose $\underline{\mathsf{MB}}_T$ with the map $\underline{\mathscr{B}} \to \mathsf{Permu}$ to get a functor $\underline{2}^N \tilde{\times}_m \mathcal{T}_n \to \mathsf{Permu}$. Applying Elmendorf-Mandell's K-theory functor [EM06, Theorem 1.1] as before gives us a functor

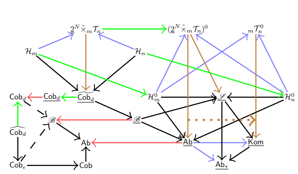
$$\underline{2}^N \tilde{\times}_m \mathcal{T}_n \to \underline{\mathscr{S}}.$$

Rectification as in Definition 2.42 turns this into a functor

$$F\colon (\underline{2}^N \tilde{\times}_m \mathcal{T}_n)^0 \to \underline{\mathscr{S}}$$

from the strictified product. Note that $\mathcal{H}_m \cup \mathcal{H}_n$ is a blockaded subcategory of ${}_m\mathcal{T}_n$, so by Lemma 2.44, on $\mathcal{H}_m^0 \cup \mathcal{H}_n^0$ the functor F agrees with the map G from Equation 4.1.

Recall from Section 3.5 that for each pair of crossingless matchings $a \in \mathsf{B}_m$ and $b \in \mathsf{B}_n$ we have a cube $\underline{2}^N \times (a, T, b)$ in $(\underline{2}^N \tilde{\times}_m \mathcal{T}_n)^0$. The restriction of F to $\underline{2}^N \times (a, T, b)$ is a functor $F|_{(a,T,b)}\colon \underline{2}^N \to \underline{\mathscr{S}}$. Next we take the iterated mapping cone of $F|_{(a,T,b)}$. That is, adjoin an additional object $*$ to $\underline{2}^1$ with a single morphism $0 \to *$, to obtain a larger category $\underline{2}^1_+$. (This category is denoted P in Corollary 2.15.) Let $\underline{2}^N_+ = (\underline{2}^1_+)^N$. Extend $F|_{(a,T,b)}$ to $F|^+_{(a,T,b)}\colon \underline{2}^N_+ \to \underline{\mathscr{S}}$ by declaring that $F|^+_{(a,T,b)}(x) = \{\mathrm{pt}\}$, a single point, if $x \notin \mathrm{Ob}(\underline{2}^N)$. Then the iterated mapping cone of $F|_{(a,T,b)}$ is the homotopy colimit $\operatorname{hocolim} F|^+_{(a,T,b)}$.



Now, define

$$G\colon {}_m\mathcal{T}_n^0 \to \underline{\mathscr{S}}$$

by defining

$$G(a,b) = F(a,b)$$
$$G(a,T,b) = \mathrm{sh}^{-N_+} \operatorname{hocolim}_{\underline{2}_+^N} F|^+_{(a,T,b)}.$$

In fact, on the entire subcategory $\mathcal{H}_m^0 \cup \mathcal{H}_n^0$, define G to agree with F (and hence also the map G from Equation (4.1)). The map

$$G(f_{a_1,\dots,a_k,T,b_1,\dots,b_\ell})\colon G(a_1,a_2)\wedge\cdots\wedge G(a_k,T,b_1)\wedge\cdots\wedge G(b_{\ell-1},b_\ell) \to G(a_1,T,b_\ell)$$

is the composition

$$\begin{aligned} &G(a_1,a_2)\wedge\cdots\wedge G(a_k,T,b_1)\wedge\cdots\wedge G(b_{\ell-1},b_\ell)\\ &\qquad = F(a_1,a_2)\wedge\cdots\wedge\left[\mathrm{sh}^{-N_+} \operatorname{hocolim}_{\underline{2}_+^N} F|^+_{(a_k,T,b_1)}\right]\wedge\cdots\wedge F(b_{\ell-1},b_\ell)\\ &\qquad \cong \mathrm{sh}^{-N_+} \operatorname{hocolim}_{\underline{2}_+^N}\left[F(a_1,a_2)\wedge\cdots\wedge F|^+_{(a_k,T,b_1)}\wedge\cdots\wedge F(b_{\ell-1},b_\ell)\right]\\ &\qquad \to \mathrm{sh}^{-N_+} \operatorname{hocolim}_{\underline{2}_+^N} F|^+_{(a_1,T,b_\ell)} = G(a_1,T,b_\ell), \end{aligned}$$

where the last map comes from naturality of the shift functor and homotopy colimits (see Propositions 2.34 and 2.10) and the fact that F is a multifunctor.

Lemma 4.1. *This definition makes G into a multifunctor.*

Proof. Again, this follows from naturality of shift functors and homotopy colimits, and the fact that F is a multifunctor. □

Proposition 4.2. *Composing G and the chain functor $\underline{\mathscr{S}} \to \underline{\mathsf{Kom}}$ gives a map ${}_m\mathcal{T}_n^0 \to \underline{\mathsf{Kom}}$ which is quasi-isomorphic to the Khovanov tangle invariant (reinterpreted as in Section 2.3).*

The following result will be useful in proving Proposition 4.2.

Lemma 4.3. *Suppose K is any multifunctor $\mathscr{C} \to \underline{\mathsf{Kom}}$. Then there are natural transformations*

$$K \leftarrow \tau_{\geq 0}\circ K \to H_0 \circ K$$

of multifunctors $\mathscr{C} \to \underline{\mathsf{Kom}}$. If, for any $x \in \mathrm{Ob}(\mathscr{C})$, the complex $K(x)$ has no homology in negative (respectively positive, nonzero) degrees, the left-hand map (respecitvely the right-hand map, each of the maps) is a natural quasi-isomorphism.

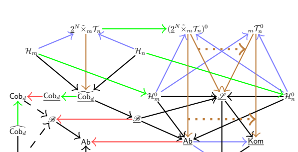
Proof. Let $\tau_{\geq 0}$ be the *connective cover* functor on $\underline{\mathsf{Kom}}$, sending a complex C to the following subcomplex:

$$(\tau_{\geq 0}C)_k = \begin{cases} C_k & \text{if } k>0\\ \ker(d_0) & \text{if } k=0\\ 0 & \text{if } k<0.\end{cases}$$

Then $\tau_{\geq 0}$ is a multifunctor $\underline{\mathsf{Kom}} \to \underline{\mathsf{Kom}}$ with a natural transformation $\tau_{\geq 0} \to \mathrm{Id}$, inducing an isomorphism on homology in non-negative degrees. Similarly, there is a natural transformation $\tau_{\geq 0} \to H_0$ of multifunctors, inducing an isomorphism on H_0.

Putting these together, for a functor K as described the composite maps

$$K \leftarrow \tau_{\geq 0}\circ K \to H_0 \circ K$$



are natural transformations of multifunctors $\mathscr{C} \to \underline{\mathsf{Kom}}$; and the left-hand (resp. right-hand) arrow is a quasi-isomorphism if K has homology groups supported in non-negative (resp. non-positive) degrees. ☐

Proof of Proposition 4.2. We begin by observing that the functor

$$C_* \circ F \colon (\underline{2}^N \tilde{\times}_m \mathcal{T}_n)^0 \to \underline{\mathsf{Kom}}$$

has homology concentrated in degree zero: the spectra $G(a,b)$ and $F|_{(a,T,b)}(v)$ are wedge sums of copies of the sphere spectrum $\mathbb{S}$. Therefore, the previous lemma provides us with a quasi-isomorphism between the multifunctor $C_* \circ F$ and the multifunctor $H_0 \circ F$.

The identification $H_0(G(a,b)) \cong 1_a\mathcal{H}^n 1_b$ is obvious: $F(a,b)$ is a wedge sum of spheres, one for each Khovanov generator. (See Section 2.8.) Similarly, for each vertex $v \in \underline{2}^N$, $F|_{(a,T,b)}(v)$ is a wedge sum of copies of the sphere spectrum $\mathbb{S}$, one for each element of $\underline{\mathsf{MB}}_T(a,T,b)$, so $H_0(F|_{(a,T,b)}(v)) \cong \mathcal{F}_{orget}(\underline{\mathsf{MB}}_T(v,a,T,b))$. Further, the map on homology associated to each edge $v \to w$ of the cube is the map $\mathcal{F}_{orget}(\underline{\mathsf{MB}}_T((v,a,T,b) \to (w,a,T,b))$.

We must check that the composition maps agree with the Khovanov composition maps. For definiteness, consider the map $F(a,b) \wedge F(b,T,c) \to F(a,T,c)$. There is a corresponding map

$$(H_0 \circ F)|_{(a,b)} \otimes (H_0 \circ F)|_{(a,T,c)}(v) \to (H_0 \circ F)|_{(b,T,c)}(v)$$

that is natural in $v \in \underline{2}^N$. Tracing through the isomorphisms above, this is exactly the Khovanov multiplication

$$1_a\mathcal{H}^n 1_b \otimes 1_b\mathcal{C}_{Kh}(T_v)1_c \to 1_a\mathcal{C}_{Kh}(T_v)1_c.$$

Thus, the multifunctor $H_0 \circ F$ represents (up to shift) precisely the cubical diagram of bimodules over the arc algebras whose totalization is $1_a\mathcal{C}_{Kh}(T)1_b$. As quasi-isomorphisms preserve shifts and homotopy colimits (see Proposition 2.10), our quasi-isomorphism from F to $H_0 \circ F$ becomes a quasi-isomorphism

$$C_*G(a,T,b) \simeq \operatorname{hocolim}_{\underline{2}^N_+}(H_0 \circ F)|_{(a,T,b)}[-N_+]. \tag{4.2}$$

By Corollary 2.15, this homotopy colimit is precisely the total complex $\mathsf{Tot}(\mathcal{F}_{orget} \circ \underline{\mathsf{MB}}_T|_{(a,T,b)}), N_+)$, which is the bimodule $1_a\mathcal{C}_{Kh}(T)1_b$. Since the quasi-isomorphisms respected composition and Equation (4.2) is natural, the identification $C_*G(a,T,b) \simeq 1_a\mathcal{C}_{Kh}(T)1_b$ respects multiplication. This proves the result. ☐

We could stop here, and define G to be our space-level refinement of the Khovanov tangle invariants, but we can make the invariant look a little closer to Khovanov's invariant by reinterpreting it as a spectral category. That is, we will refine $\mathcal{H}^n$ to a category $\mathscr{H}^n$ with:

- Objects crossingless matchings.
- $\operatorname{Hom}(a,b)$ a symmetric spectrum.
- Composition a map $\operatorname{Hom}(b,c) \wedge \operatorname{Hom}(a,b) \to \operatorname{Hom}(a,c)$.
- Identity elements which are maps $\mathbb{S} \to \operatorname{Hom}(a,a)$.

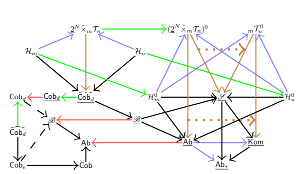
(This is a spectrum-level analogue of a linear category, cf. Section 2.3. See [BM12] for a more in-depth review of spectral categories.) Associated to a $(2m,2n)$-tangle T we will construct a left-$\mathscr{H}^m$, right-$\mathscr{H}^n$ bimodule $\mathscr{X}(T)$, i.e., a functor $\mathscr{X}(T)\colon (\mathscr{H}^m)^{\mathrm{op}} \times \mathscr{H}^n \to \mathscr{S}$.

We construct $\mathscr{H}^n$ as follows. Let

$$\operatorname{Hom}_{\mathscr{H}^n}(a,b) = G(a,b).$$

Composition is defined by

$$\operatorname{Hom}_{\mathscr{H}^n}(b,c) \wedge \operatorname{Hom}_{\mathscr{H}^n}(a,b) = G(b,c) \wedge G(a,b) \cong G(a,b) \wedge G(b,c) \xrightarrow{G(f_{a,b,c})} G(a,c) = \operatorname{Hom}_{\mathscr{H}^n}(a,c).$$



Identity elements are given by

$$\mathbb{S} \xrightarrow{G(f_a)} G(a,a) = \mathrm{Hom}_{\mathscr{H}^n}(a,a).$$

Turning to $\mathscr{X}(T)$, let

$$\mathscr{X}(T)(a,b) = G(a,T,b).$$

On morphisms, the map is given by

$$\begin{aligned}\mathrm{Hom}_{(\mathscr{H}^m)^{\mathrm{op}}\times\mathscr{H}^n}((a,b),(a',b')) \wedge \mathscr{X}(T)(a,b) &= G(a',a)\wedge G(b,b')\wedge G(a,T,b)\\ &\cong G(a',a)\wedge G(a,T,b)\wedge G(b,b') \xrightarrow{G(f_{a',a,T,b,b'})} G(a',T,b') = \mathscr{X}(T)(a',b').\end{aligned}$$

Lemma 4.4. *These definitions make $\mathscr{H}^n$ into a spectral category and $\mathscr{X}(T)$ into a $(\mathscr{H}^m,\mathscr{H}^n)$-bimodule.*

Proof. We only need to check the associativity and identity axioms, which are immediate from the definitions and the fact that G was a strict multifunctor. □

Note that, in a similar spirit to Section 2.3, we can reinterpret $\mathscr{H}^n$ as a ring spectrum

$$\mathscr{H}^n_{ring} = \bigvee_{a,b\in\mathrm{Ob}(\mathscr{H}^n)} \mathrm{Hom}_{\mathscr{H}^n}(a,b)$$

with multiplication given by composition when defined and trivial when composition is not defined. (Our ordering convention is that the product $a\cdot b$ stands for $b\circ a$.) Similarly, $\mathscr{X}(T)$ induces an $(\mathscr{H}^m_{ring},\mathscr{H}^n_{ring})$-bimodule spectrum

$$\mathscr{X}_{module}(T) = \bigvee_{\substack{a\in\mathrm{Ob}(\mathscr{H}^m)\\ b\in\mathrm{Ob}(\mathscr{H}^n)}} \mathscr{X}(T)(a,b).$$

Finally, we will use the following technical lemma, to simplify the definition of the derived tensor product and topological Hochschild homology:

Lemma 4.5. *The spectral categories $\mathscr{H}^n$ and spectral bimodules $\mathscr{X}(T)$ are pointwise cofibrant. That is, $\mathrm{Hom}_{\mathscr{H}^n}(x,y)$ and $\mathscr{X}(T)(x,y)$ are cofibrant symmetric spectra for all pairs of objects x,y.*

Proof. This is clear since the spectra are produced by rectification from Definition 2.42, which gives a cofibrant diagram which is hence pointwise cofibrant (Lemma 2.41), and then taking homotopy colimits and shifting, which preserves cofibrancy (Lemma 2.40). □

4.2. **Invariance of the bimodule associated to a tangle.** Before turning to the bimodule, consider invariance of the spectral category $\mathscr{H}^n$. Superficially, the functor $G\colon \mathcal{H}^0_n \to \underline{\mathscr{S}}$, and hence the spectral category $\mathscr{H}^n$, depended on a number of choices:

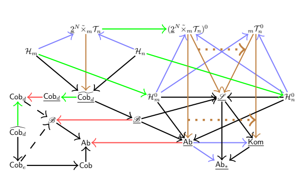
(1) The choices (C-1)–(C-3) from Section 3.5.2.
(2) Any choices in the Elmendorf-Mandell machine and the rectification procedure.

As noted in Sections 2.8 and 2.9, Choice (2) is, in fact, canonical. As discussed in Section 3.5.2, Choices (C-1)–(C-3) can be made canonical by a colimit-type construction. So, $\mathscr{H}^n$ is, in fact, completely well-defined.

Turning next to $\mathscr{X}(T)$, we will show that this spectral bimodule is well-defined up to the following equivalence:

Definition 4.6. Given spectral categories $\mathscr{C}$ and $\mathscr{D}$ and spectral $(\mathscr{C},\mathscr{D})$-bimodules $\mathscr{M}$ and $\mathscr{N}$, a *homomorphism* $\mathscr{F}\colon \mathscr{M}\to\mathscr{N}$ is a natural transformation from $\mathscr{M}$ to $\mathscr{N}$. A homomorphism is an *equivalence* if for each $a\in\mathrm{Ob}(\mathscr{C})$ and $b\in\mathrm{Ob}(\mathscr{D})$, the map

$$\mathscr{F}(a,b)\colon \mathscr{M}(a,b)\to\mathscr{N}(a,b)$$



is an equivalence of spectra. The symmetric, transitive closure of this notion of equivalence is an equivalence relation; two bimodules are *equivalent* if they are related by this equivalence relation (i.e., if there is a zig-zag of equivalences between them).

Proposition 4.7. *If* $(F_1 : \underline{2}^{N_1}\tilde{\times}_m\mathcal{T}_n \to \underline{\mathscr{B}}, S_1)$ *and* $(F_2 : \underline{2}^{N_2}\tilde{\times}_m\mathcal{T}_n \to \underline{\mathscr{B}}, S_2)$ *are stably equivalent functors then the induced spectral bimodules* $\mathscr{G}_1$ *and* $\mathscr{G}_2$ *over* $(\mathscr{H}^m, \mathscr{H}^n)$ *are equivalent.*

Proof. It is clear from the iterated mapping cone construction that stabilizations and destabilizations give equivalent spectral bimodules. Next, given a quasi-isomorphism F_{12} from F_1 to F_2, the construction from Section 4.1 gives a functor G_{12} to $\underline{\mathscr{S}}$ and an induced spectral bimodule $\mathscr{G}_{12}$ with the following properties:

(1) Each of the spaces $G_{12}(a, T, b)$ is acyclic. Thus, $\mathscr{G}_{12}(a, b)$ is contractible for each (a, b).
(2) There is a cofibration sequence
$$\cdots \to G_1(a, T, b) \to G_2(a, T, b) \to G_{12}(a, T, b) \to \Sigma G_1(a, T, b) \to \dots$$
and these maps are natural in the obvious sense.

These maps induce a long exact sequence
$$\cdots \to \pi_n\mathscr{G}_1 \to \pi_n\mathscr{G}_2 \to \pi_n\mathscr{G}_{12} \to \pi_{n-1}\mathscr{G}_1 \to \cdots .$$
Since $\mathscr{G}_{12}(a, b)$ is contractible for all (a, b), this implies that $\mathscr{G}_2$ and $\mathscr{G}_1$ are equivalent. □

Theorem 4. *Up to equivalence of* $(\mathscr{H}^m, \mathscr{H}^n)$*-bimodules,* $\mathscr{X}(T)$ *is an invariant of the isotopy class of the* $(2m, 2n)$*-tangle* T*. Further, the maps on homology induced by a sequence of Reidemeister moves agree, up to a sign, with Khovanov's invariance maps* [Kho02, Section 4].

Proof. This is immediate from Theorem 3 and Proposition 4.7. □

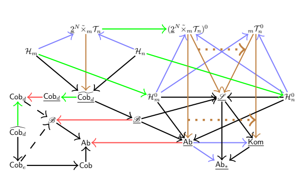
5. Gluing

In this section we prove that gluing tangles corresponds to the derived tensor product of spectral bimodules (Theorem 5). We start by introducing one more shape multicategory, adapted to study triples of tangles (T_1, T_2, T_1T_2). We then recall the tensor product of spectral bimodules and, with these tools in hand, prove the gluing theorem.

Fix non-negative integers m, n, p. The *gluing multicategory* $\mathcal{G}^0_{m,n,p}$, which is the shape multicategory associated to $(\mathsf{B}_m, \mathsf{B}_n, \mathsf{B}_p)$ (cf. Definition 2.3). Explicitly, $\mathcal{G}^0_{m,n,p}$ has objects:

- Pairs (a, b) of crossingless matchings on $2m$ points.
- Pairs (a, b) of crossingless matchings on $2n$ points.
- Pairs (a, b) of crossingless matchings on $2p$ points.
- Triples (a, T_1, b) where a is a crossingless matching of $2m$ points, b is a crossingless matching of $2n$ points, and T_1 is a placeholder (a mnemonic for a $(2m, 2n)$ tangle).
- Triples (a, T_2, b) where a is a crossingless matching of $2n$ points, b is a crossingless matching of $2p$ points, and T_2 is a placeholder (a mnemonic for a $(2n, 2p)$ flat tangle).
- Triples (a, T_1T_2, b) where a is a crossingless matching of $2m$ points, b is a crossingless matching of $2p$ points, and T_1T_2 is a placeholder (a mnemonic for the composition of T_1 and T_2).

(In the special case that $m = n$ or $m = p$ or $n = p$, some of the objects of different types in the list above are the same.) So, the objects of ${}_m\mathcal{T}^0_n$, ${}_n\mathcal{T}^0_p$, and ${}_m\mathcal{T}^0_p$ are contained in the gluing multicategory, and in fact we let these three multicategories be full subcategories of the gluing multicategory. There is one more kind of multimorphism in the gluing multicategory: a unique multimorphism
$$(a_1, a_2), \dots, (a_{i-1}, a_i), (a_i, T_1, b_1), (b_1, b_2), \dots, (b_{j-1}, b_j), (b_j, T_2, c_1), (c_1, c_2), \dots, (c_{k-1}, c_k) \to (a_1, T_1T_2, c_k)$$



where the a_ℓ (respectively b_ℓ, c_ℓ) are crossingless matchings of $2m$ (respectively $2n$, $2p$) points. Let $\mathcal{G}_{m,n,p}$ be the canonical groupoid enrichment of $\mathcal{G}^0_{m,n,p}$.

Next we define a category $\underline{2}^{N_1|N_2}\tilde{\times}\mathcal{G}_{m,n,p}$ similar to (and extending) $\underline{2}^N\tilde{\times}_m\mathcal{T}_n$. The objects of $\underline{2}^N\tilde{\times}\mathcal{G}_{n,n,p}$ are of the following forms:

- Pairs (a,b) in $\mathrm{Ob}(\mathcal{H}_m)$ or $\mathrm{Ob}(\mathcal{H}_n)$ or $\mathrm{Ob}(\mathcal{H}_p)$.
- Quadruples (v,a,T_1,b) where $v\in\mathrm{Ob}(\underline{2}^{N_1})$, $a\in\mathsf{B}_m$, and $b\in\mathsf{B}_n$.
- Quadruples (v,a,T_2,b) where $v\in\mathrm{Ob}(\underline{2}^{N_2})$, $a\in\mathsf{B}_n$, and $b\in\mathsf{B}_p$.
- Quadruples (v,a,T_1T_2,b) where $v\in\mathrm{Ob}(\underline{2}^{N_1+N_2})$, $a\in\mathsf{B}_m$, and $b\in\mathsf{B}_p$.

So,

$$\mathrm{Ob}(\underline{2}^{N_1|N_2}\tilde{\times}\mathcal{G}_{n,n,p})=\mathrm{Ob}(\underline{2}^{N_1}\tilde{\times}_m\mathcal{T}_n)\cup\mathrm{Ob}(\underline{2}^{N_2}\tilde{\times}_n\mathcal{T}_p)\cup\mathrm{Ob}(\underline{2}^{N_1+N_2}\tilde{\times}_m\mathcal{T}_p).$$

A *basic multimorphism* for $\underline{2}^{N_1|N_2}\tilde{\times}\mathcal{G}_{m,n,p}$ is one of:

- A basic multimorphism in $\underline{2}^{N_1}\tilde{\times}_m\mathcal{T}_n$, $\underline{2}^{N_2}\tilde{\times}_n\mathcal{T}_p$, or $\underline{2}^{N_1+N_2}\tilde{\times}_m\mathcal{T}_p$, or
- A (unique) multimorphism

$$(a_1,a_2),\dots,(a_{j-1},a_j),(v,a_j,T_1,b_1),(b_1,b_2),\dots,(b_{k-1},b_k),(w,b_k,T_2,c_1),(c_1,c_2)\dots,(c_{\ell-1},c_\ell)\\ \to((v,w),a_1,T_1T_2,c_\ell).$$

The multimorphisms in $\underline{2}^{N_1|N_2}\tilde{\times}\mathcal{G}_{m,n,p}$ are planar, rooted trees whose edges are decorated by objects in $\underline{2}^{N_1|N_2}\tilde{\times}\mathcal{G}_{m,n,p}$ and whose vertices are decorated by basic multimorphisms compatible with the decorations on the edges. If two multimorphisms have the same source and target then we declare that there is a unique morphism in the corresponding multimorphism groupoid between them.

Let $(\underline{2}^{N_1|N_2}\tilde{\times}\mathcal{G}_{m,n,p})^0$ be the strictification of $\underline{2}^{N_1|N_2}\tilde{\times}\mathcal{G}_{m,n,p}$. We have the following analogue of Lemma 3.8:

Lemma 5.1. *The projection $\underline{2}^{N_1|N_2}\tilde{\times}\mathcal{G}_{m,n,p}\to(\underline{2}^{N_1|N_2}\tilde{\times}\mathcal{G}_{m,n,p})^0$ is a weak equivalence.*

Proof. The proof is essentially the same as the proofs of Lemmas 2.8 and 3.8. ☐

Given an $(2m,2n)$-tangle T_1 with N_1 crossings and an $(2n,2p)$-tangle T_2 with N_2 crossings, let T_1T_2 denote the composition of T_1 and T_2. We have multifunctors $\underline{\mathsf{MC}}_{T_1}\colon\underline{2}^{N_1}\tilde{\times}_m\mathcal{T}_n\to\underline{\widetilde{\mathsf{Cob}}_d}$, $\underline{\mathsf{MC}}_{T_2}\colon\underline{2}^{N_2}\tilde{\times}_n\mathcal{T}_p\to\underline{\widetilde{\mathsf{Cob}}_d}$, and $\underline{\mathsf{MC}}_{T_1T_2}\colon\underline{2}^{N_1+N_2}\tilde{\times}_m\mathcal{T}_p\to\underline{\widetilde{\mathsf{Cob}}_d}$.

Lemma 5.2. *There is a multifunctor $\underline{G}\colon\underline{2}^{N_1|N_2}\tilde{\times}\mathcal{G}_{m,n,p}\to\underline{\widetilde{\mathsf{Cob}}_d}$ extending $\underline{\mathsf{MC}}_{T_1}$, $\underline{\mathsf{MC}}_{T_2}$, and $\underline{\mathsf{MC}}_{T_1T_2}$.*

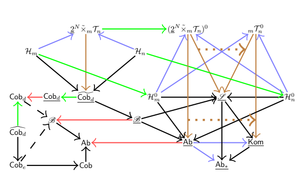
Proof. This is a straightforward adaptation of the construction of $\underline{\mathsf{MC}}_T$, and is left to the reader. ☐

Composing $\underline{G}$ with the Khovanov-Burnside functor gives a functor $\underline{V}_{HKK}\circ\underline{G}\colon\mathcal{G}_{m,n,p}\to\underline{\mathscr{B}}$. Proceeding as in the construction of the tangle invariants in Section 4.1 we obtain a functor

$$Gl\colon(\mathcal{G}_{m,n,p})^0\to\underline{\mathscr{S}}.$$

The functor Gl restricts to G_{T_1} on ${}_m\mathcal{T}^0_n$ and G_{T_2} on ${}_n\mathcal{T}^0_p$. (This uses the fact that ${}_m\mathcal{T}_n$ and ${}_n\mathcal{T}_p$ are blockaded subcategories of $\mathcal{G}_{m,n,p}$ and Lemma 2.44.) By Lemma 2.43, on ${}_m\mathcal{T}^0_p$, the functor Gl is naturally equivalent to $G_{T_1T_2}$, but because of the rectification step, may not agree with $G_{T_1T_2}$ exactly. Since there are no morphisms out of the subcategory ${}_m\mathcal{T}^0_p$, we can compose Gl with the equivalence from $Gl|_{{}_m\mathcal{T}^0_p}$ to $G_{T_1T_2}$ to obtain a new functor whose restriction to $Gl|_{{}_m\mathcal{T}^0_p}$ agrees with $G_{T_1T_2}$. Abusing notation, from now on we use Gl to denote this new functor.

We recall two notions of tensor product of modules over a spectral category:



Definition 5.3. Let $\mathscr{C}$, $\mathscr{D}$, and $\mathscr{E}$ be spectral categories, $\mathscr{M}$ a $(\mathscr{C},\mathscr{D})$-bimodule and $\mathscr{N}$ a $(\mathscr{D},\mathscr{E})$-bimodule. Assume that $\mathscr{D}$, $\mathscr{M}$ and $\mathscr{N}$ are pointwise cofibrant (cf. Lemma 4.5). The *tensor product* of $\mathscr{M}$ and $\mathscr{N}$ over $\mathscr{D}$, $\mathscr{M}\otimes_{\mathscr{D}}\mathscr{N}$, is the $(\mathscr{C},\mathscr{E})$-bimodule P where $P(a,c)$ is the coequalizer of the diagram

$$\coprod_{b,b'\in\mathrm{Ob}(\mathscr{D})}\mathscr{M}(a,b)\wedge\mathrm{Hom}_{\mathscr{D}}(b,b')\wedge\mathscr{N}(b',c)\rightrightarrows\coprod_{b\in\mathrm{Ob}(\mathscr{D})}\mathscr{M}(a,b)\wedge\mathscr{N}(b,c).$$

(Here, the two maps correspond to the action of $\mathrm{Hom}(b,b')$ on $\mathscr{M}(a,b)$ and on $\mathscr{N}(b',c)$, respectively.)

The *derived tensor product* of $\mathscr{M}$ and $\mathscr{N}$ over $\mathscr{D}$, $\mathscr{M}\otimes^{\mathbb{L}}_{\mathscr{D}}\mathscr{N}$, is

$$\begin{aligned}P(a,c)=\mathrm{hocolim}\Big(\cdots &\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}\coprod_{b,b',b''\in\mathrm{Ob}(\mathscr{D})}\mathscr{M}(a,b)\wedge\mathrm{Hom}_{\mathscr{D}}(b,b')\wedge\mathrm{Hom}_{\mathscr{D}}(b',b'')\wedge\mathscr{N}(b'',c)\\ &\substack{\rightarrow\\\rightarrow\\\rightarrow}\coprod_{b,b'\in\mathrm{Ob}(\mathscr{D})}\mathscr{M}(a,b)\wedge\mathrm{Hom}_{\mathscr{D}}(b,b')\wedge\mathscr{N}(b',c)\\ &\rightrightarrows\coprod_{b\in\mathrm{Ob}(\mathscr{D})}\mathscr{M}(a,b)\wedge\mathscr{N}(b,c)\Big).\end{aligned}$$

There is an evident quotient map $\mathscr{M}\otimes^{\mathbb{L}}_{\mathscr{D}}\mathscr{N}\to\mathscr{M}\otimes_{\mathscr{D}}\mathscr{N}$.

The derived tensor product is functorial and preserves equivalences in the following sense. Given a map $\mathscr{D}\to\mathscr{D}'$, modules $\mathscr{M}$ and $\mathscr{N}$ over $\mathscr{D}$, modules $\mathscr{M}'$ and $\mathscr{N}'$ over $\mathscr{D}'$, and maps $\mathscr{M}\to\mathscr{M}'$ and $\mathscr{N}\to\mathscr{N}'$ intertwining the actions of $\mathscr{D}$ and $\mathscr{D}'$, there is a map

$$\mathscr{M}\otimes^{\mathbb{L}}_{\mathscr{D}}\mathscr{N}\to\mathscr{M}'\otimes^{\mathbb{L}}_{\mathscr{D}'}\mathscr{N}'.$$

If the maps $\mathscr{D}\to\mathscr{D}'$, $\mathscr{M}\to\mathscr{M}'$, and $\mathscr{N}\to\mathscr{N}'$ are equivalences this map of derived tensor products is an equivalence.

Replacing smash products with tensor products gives the derived tensor product of chain complexes (assuming that the constituent complexes are all flat over $\mathbb{Z}$). Again, the derived tensor product is functorial and preserves quasi-isomorphisms of complexes.

Reinterpreting Gl, for each triple of crossingless matchings a,b,c we have a map

$$Gl((a,T_1,b),(b,T_2,c)\to(a,T_1T_2,c))\colon G(a,T_1,b)\wedge G(b,T_2,c)\to G(a,T_1T_2,c).$$

Lemma 5.4. *The map Gl induces a map of bimodules $\mathscr{X}(T_1)\otimes_{\mathscr{H}^n}\mathscr{X}(T_2)\to\mathscr{X}(T_1T_2)$.*

Proof. By definition,

$$\big(\mathscr{X}(T_1)\otimes_{\mathscr{H}^n}\mathscr{X}(T_2)\big)(a,c)=\coprod_{b\in\mathsf{B}_n}G_{T_1}(a,T,b)\wedge G_{T_2}(b,T,c)/\sim.$$

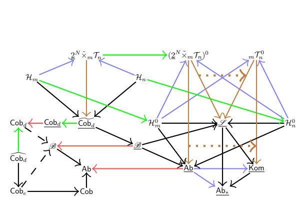
The map Gl gives maps

$$\textstyle\coprod_{b\in\mathsf{B}_n}G_{T_1}(a,T,b)\wedge G_{T_2}(b,T,c)\xrightarrow{\coprod_b Gl((a,T_1,b),(b,T_2,c)\to(a,T_1T_2,c))}G_{T_1T_2}(a,T,c).$$

We must check that these maps respect the equivalence relation $\sim$ and the actions of $\mathscr{H}^m$ and $\mathscr{H}^p$; but both statements are immediate from the fact that the map Gl is a multifunctor (and the definition of $\mathcal{G}^0_{m,n,p}$). ☐

Composing with the quotient map $\mathscr{X}(T_1)\otimes^{\mathbb{L}}_{\mathscr{H}^n}\mathscr{X}(T_2)\to\mathscr{X}(T_1)\otimes_{\mathscr{H}^n}\mathscr{X}(T_2)$ gives a map $\mathscr{X}(T_1)\otimes^{\mathbb{L}}_{\mathscr{H}^n}\mathscr{X}(T_2)\to\mathscr{X}(T_1T_2)$.

We recall a fact about the classical Khovanov bimodules:



Lemma 5.5. *If T is an $(2m, 2n)$ flat tangle then the bimodule $\mathcal{C}_{Kh}(T)$ is left-projective and right-projective. So, given a $(2m, 2n)$-tangle T_1 and a $(2n, 2p)$-tangle T_2 there are quasi-isomorphisms*

$$\mathcal{C}_{Kh}(T_1) \otimes^{\mathbb{L}}_{\mathcal{H}^n} \mathcal{C}_{Kh}(T_2) \simeq \mathcal{C}_{Kh}(T_1) \otimes_{\mathcal{H}^n} \mathcal{C}_{Kh}(T_2) \simeq \mathcal{C}_{Kh}(T_1T_2).$$

Proof. The first half of the statement was proved by Khovanov (who uses the word *sweet* for finitely-generated bimodules which are left-projective and right-projective) [Kho02, Proposition 3]. For the second half of the statement, the first quasi-isomorphism follows from the definition of the derived tensor product and sweetness, and the second quasi-isomorphism is Khovanov's gluing theorem (repeated above as Proposition 2.53). ☐

Lemma 5.6. *Given an $(2m, 2n)$-tangle T_1 and an $(2n, 2p)$-tangle T_2, there is a commutative diagram of isomorphisms in the derived category of complexes*

$$\begin{array}{ccccc}
C_*\big(\mathscr{X}(T_1) \otimes^{\mathbb{L}}_{\mathscr{H}^n} \mathscr{X}(T_2)\big) & \longleftarrow & C_*(\mathscr{X}(T_1)) \otimes^{\mathbb{L}}_{C_*(\mathscr{H}^n)} C_*(\mathscr{X}(T_2)) & \longrightarrow & \mathcal{C}_{Kh}(T_1) \otimes^{\mathbb{L}}_{\mathcal{H}^n} \mathcal{C}_{Kh}(T_2) \\
& \searrow^{Gl} & \downarrow & & \downarrow \\
& & C_*\big(\mathscr{X}(T_1T_2)\big) & \longrightarrow & \mathcal{C}_{Kh}(T_1T_2),
\end{array}$$

where the right-hand horizontal arrows are the induced by the quasi-isomorphisms of Proposition 4.2 and the right-most vertical arrow is the quasi-isomorphism from Lemma 5.5.

Proof. We begin by applying C_* to the diagram defining the derived tensor product $\mathscr{X}(T_1) \otimes^{\mathbb{L}}_{\mathscr{H}^n} \mathscr{X}(T_2)$. Using both the natural quasi-isomorphism $\operatorname{hocolim} C_* \to C_* \operatorname{hocolim}$ and monoidality of C_*, we get the quasi-isomorphism

$$C_*(\mathscr{X}(T_1)) \otimes^{\mathbb{L}}_{C_*(\mathscr{H}^n)} C_*(\mathscr{X}(T_2)) \to C_*\big(\mathscr{X}(T_1) \otimes^{\mathbb{L}}_{\mathscr{H}^n} \mathscr{X}(T_2)\big)$$

giving the map $C_*(\mathscr{X}(T_1)) \otimes^{\mathbb{L}}_{C_*(\mathscr{H}^n)} C_*(\mathscr{X}(T_2)) \to C_*(\mathscr{X}(T_1T_2))$.

We now address the right-hand square. Recall that Lemma 4.3 constructs natural transformations of multifunctors $\underline{\mathscr{S}} \to \underline{\mathsf{Kom}}$

$$C_* \leftarrow \tau_{\geq 0} \circ C_* \to H_0,$$

where the left-hand arrow is always an isomorphism in nonnegative homology degrees and the right-hand one is always an isomorphism in homology degree zero. In particular, this gives us natural quasi-isomorphisms of dg-categories

$$C_*\mathscr{H}^n \leftarrow \tau_{\geq 0} C_*\mathscr{H}^n \to H_0\mathscr{H}^n,$$

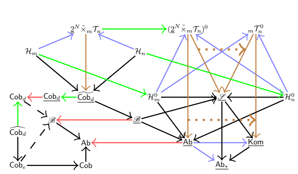
where the right-hand term is Khovanov's arc algebra $\mathcal{H}^n$. Similarly, we can apply these trunction transformations to the spectral bimodule $\mathscr{X}(T)$, obtaining quasi-isomorphisms

$$\begin{aligned}
C_*\mathscr{X}(T) &= C_*\big(\operatorname{sh}^{-N_+} \operatorname{hocolim}_{\underline{2}^N_+} F|^+_{(a,T,b)}\big) \\
&\leftarrow \operatorname{hocolim}_{\underline{2}^N_+} C_*(F|^+_{(a,T,b)})[-N_+] \\
&\leftarrow \operatorname{hocolim}_{\underline{2}^N_+} \big(\tau_{\geq 0} C_*(F|^+_{(a,T,b)})\big)[-N_+] \\
&\to \operatorname{hocolim}_{\underline{2}^N_+} \big(H_0 \circ F|^+_{(a,T,b)}\big)\big)[-N_+] \\
&= \mathcal{C}_{Kh}(T).
\end{aligned}$$

These maps are compatible with bimodule structures: all terms are bimodules over $(\tau_{\geq 0} C_*\mathscr{H}^m, \tau_{\geq 0} C_*\mathscr{H}^n)$, and these bimodule structures are compatible with the structure of a bimodule over the untruncated chain complex $(C_*\mathscr{H}^m, C_*\mathscr{H}^n)$ on $C_*\mathscr{X}(T)$ and of a bimodule over the arc algebras $(\mathcal{H}^m, \mathcal{H}^n)$ on $\mathcal{C}_{Kh}(T)$.

Let

$$D_*(T) = \operatorname{hocolim}_{\underline{2}^N_+} \big(\tau_{\geq 0} C_*(F|^+_{(a,T,b)})\big)[-N_+].$$



We now apply derived tensor products and the gluing pairing Gl, obtaining a diagram

$$\begin{array}{ccccc} C_*\mathscr{X}(T_1)\otimes^{\mathbb{L}}_{C_*\mathscr{H}^n} C_*\mathscr{X}(T_2) & \longleftarrow & D_*(T_1)\otimes^{\mathbb{L}}_{\tau_{\geq 0}C_*\mathscr{H}^n} D_*(T_2) & \longrightarrow & \mathcal{C}_{Kh}(T_1)\otimes^{\mathbb{L}}_{\mathcal{H}^n}\mathcal{C}_{Kh}(T_2) \\ \downarrow & & \downarrow & & \downarrow \\ C_*\mathscr{X}(T_1T_2) & \longleftarrow & D_*(T_1T_2) & \longrightarrow & \mathcal{C}_{Kh}(T_1T_2). \end{array}$$

As just shown, the bottom horizontal maps are quasi-isomorphisms. As the derived tensor product preserves homotopy colimits, the top horizontal maps are also quasi-isomorphisms. Finally, the right-hand vertical map is a quasi-isomorphism from the case of ordinary Khovanov homology (Lemma 5.5), and commutativity of the right-hand square is clear from the definition of Khovanov's gluing map (see Proposition 2.53). ☐

Theorem 5. *The gluing functor $\mathscr{X}(T_1)\otimes^{\mathbb{L}}_{\mathscr{H}^n}\mathscr{X}(T_2)\to\mathscr{X}(T_1T_2)$ is an equivalence of bimodules.*

Proof. Lemma 5.6 shows that the induced map of chain complexes agrees with the map $\mathcal{C}_{Kh}(T_1)\otimes^{\mathbb{L}}_{\mathcal{H}^n}\mathcal{C}_{Kh}(T_2)\to\mathcal{C}_{Kh}(T_1T_2)$, which is a quasi-isomorphism. As the spectra in question are connective, the result follows from the homology Whitehead theorem (Theorem 2.18). ☐

6. Quantum gradings

So far, we have suppressed the quantum gradings; in this section we reintroduce them.

Definition 6.1. The *grading multicategory* $\mathscr{G}$ has:

- One object for each integer n, and
- A unique multimorphism $(m_1,\dots,m_k)\to m_1+\cdots+m_k$ for each $m_1,\dots,m_k\in\mathbb{Z}$.

As usual, we can view the grading multicategory as trivially enriched in groupoids.

Definition 6.2. The *naive product* of multicategories $\mathscr{C}$ and $\mathscr{D}$, $\mathscr{C}\times\mathscr{D}$, has objects pairs $(c,d)\in\mathrm{Ob}(\mathscr{C})\times\mathrm{Ob}(\mathscr{D})$, multimorphism sets

$$\mathrm{Hom}_{\mathscr{C}\times\mathscr{D}}((c_1,d_1),\dots,(c_n,d_n);(c,d))=\mathrm{Hom}_{\mathscr{C}}(c_1,\dots,c_n;c)\times\mathrm{Hom}_{\mathscr{D}}(d_1,\dots,d_n;d),$$

and the obvious composition and identity maps.

Given a multicategory $\mathscr{C}$ and a multifunctor $F\colon\mathscr{G}\times\mathscr{C}\to\underline{\mathscr{B}}$ satisfying

(F) for all objects $x\in\mathrm{Ob}(\mathscr{C})$, $F(n,x)$ is empty for all but finitely many n,

there is an associated multifunctor $\coprod F\colon\mathscr{C}\to\mathscr{D}$ defined by

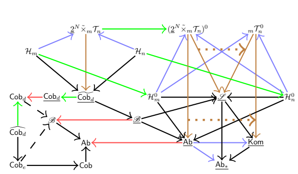
$$(\textstyle\coprod F)(x)=\coprod_{n\in\mathbb{Z}}F(n,x)$$

and, given $f\in\mathrm{Hom}_{\mathscr{C}}(x_1,\dots,x_k;y)$, the correspondence

$$\begin{aligned}(\textstyle\coprod F)(f)\colon \Big(\coprod_{m_1\in\mathbb{Z}}F(m_1,x_1)\Big)\times\cdots\times\Big(\coprod_{m_k\in\mathbb{Z}}F(m_k,x_k)\Big)\\ =\coprod_{(m_1,\dots,m_k)\in\mathbb{Z}^k}F(m_1,x_1)\times\cdots\times F(m_k,x_k)\longrightarrow\coprod_{n\in\mathbb{Z}}F(n,y)\end{aligned}$$

satisfies

$$s^{-1}\left(F(m_1,x_1)\times\cdots\times F(m_k,x_k)\right)\cap t^{-1}\left(F(n,y)\right)=\begin{cases}F(((m_1,\dots,m_k)\to n)\times f) & \text{if } n=m_1+\cdots+m_k\\ \varnothing & \text{otherwise.}\end{cases}$$



We will lift the functors $\underline{\mathsf{MB}}_m\colon \mathcal{H}_m \to \underline{\mathscr{B}}$ and $\underline{\mathsf{MB}}_T\colon \underline{2}^N\tilde{\times}_m\mathcal{T}_n \to \underline{\mathscr{B}}$ to functors

$$\underline{\mathsf{MB}}^\bullet_m\colon \mathscr{G}\times\mathcal{H}_m \to \underline{\mathscr{B}}$$
$$\underline{\mathsf{MB}}^\bullet_T\colon \mathscr{G}\times(\underline{2}^N\tilde{\times}_m\mathcal{T}_n) \to \underline{\mathscr{B}}.$$

By "lift" we mean that there are natural isomorphisms

$$\coprod \underline{\mathsf{MB}}^\bullet_m \cong \underline{\mathsf{MB}}_m \qquad\qquad \coprod \underline{\mathsf{MB}}^\bullet_T \cong \underline{\mathsf{MB}}_T. \tag{6.1}$$

We start by defining the lifts at the level of objects, by copying Khovanov's definitions of the quantum gradings on the arc algebras and modules. Specifically, given an object $(a,b)\in \mathrm{Ob}(\mathcal{H}_m)$ and an element $x\in\underline{\mathsf{MB}}_m(a,b)$ which labels $p(x)$ circles by 1 and $n(x)$ circles by X, we define the *quantum grading*

$$\mathrm{gr}_q(x) = n(x) - p(x) + m \tag{6.2}$$

and let

$$\underline{\mathsf{MB}}^\bullet_m(k,(a,b)) = \{x\in\underline{\mathsf{MB}}_m(a,b) \mid \mathrm{gr}_q(x)=k\}.$$

Similarly, for $(v,a,T,b)\in\mathrm{Ob}(\underline{2}^N\tilde{\times}_m\mathcal{T}_n)$ and $x\in\underline{\mathsf{MB}}_T(v,a,T,b)$ which labels $p(x)$ circles by 1 and $n(x)$ circles by X we define

$$\mathrm{gr}_q(x) = n(x) - p(x) + n - |v|, \tag{6.3}$$

where $|v|$ is the number of 1s in v, and let

$$\underline{\mathsf{MB}}^\bullet_T(k,(v,a,T,b)) = \{x\in\underline{\mathsf{MB}}_T(v,a,T,b) \mid \mathrm{gr}_q(x)=k\}.$$

Example 6.3. For $(a,a)\in\mathrm{Ob}(\mathcal{H}_m)$, the quantum grading of an element $x\in\underline{\mathsf{MB}}_m(a,b)$ is 2 times the number of circles labeled X, and in particular ranges between 0 and $2m$. The unit element, in which all circles are labeled 1, is in quantum grading 0.

Lemma 6.4. *These definitions of $\underline{\mathsf{MB}}^\bullet_m$ and $\underline{\mathsf{MB}}^\bullet_T$ extend uniquely to the morphism groupoids of $\underline{\mathsf{MB}}^\bullet_m$ and $\underline{\mathsf{MB}}^\bullet_T$ satisfying Equations* (6.1).

Proof. Uniqueness is clear. Existence follows from the fact that the multiplication on the Khovanov arc algebras and bimodules respects the quantum gradings. □

Using $\underline{\mathsf{MB}}^\bullet_m$ and $\underline{\mathsf{MB}}^\bullet_T$ in place of $\underline{\mathsf{MB}}_m$ and $\underline{\mathsf{MB}}_T$ in Section 4.1 gives functors

$$G^\bullet\colon \mathscr{G}\times\mathcal{H}^0_n \to \underline{\mathscr{S}} \qquad\text{and}\qquad G^\bullet\colon \mathscr{G}\times{}_m\mathcal{T}^0_n \to \underline{\mathscr{S}}.$$

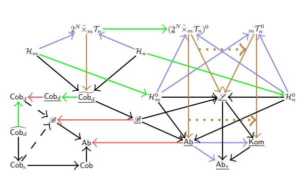
These give a graded spectral category $\mathscr{H}^n$ and graded $(\mathscr{H}^m,\mathscr{H}^n)$-bimodule $\mathscr{X}(T)$, with same objects, by setting

$$\mathrm{Hom}_{\mathscr{H}^n}(a,b)_k = G^\bullet(k,(a,b))$$
$$\mathscr{X}(T)(a,b)_k = G^\bullet(k,(a,T,b))$$

(where the subscript k denotes the k^{th} graded part). These refine the spectral category and bimodule introduced in Section 4.1 in the sense that

$$\mathrm{Hom}_{\mathscr{H}^n}(a,b) \simeq \bigvee_k \mathrm{Hom}_{\mathscr{H}^n}(a,b)_k$$
$$\mathscr{X}(T)(a,b) \simeq \bigvee_k \mathscr{X}(T)(a,b)_k,$$

canonically, where the left side is the definition in Section 4.1 and the right side is the definition in this section. So, the fact that we are using the same notation for the definitions in this section and in Section 4.1 will not cause confusion.



The proof of invariance (Sections 3.5.2 and 4.2) goes through without essential changes. The graded analogue of the gluing theorem is:

Theorem 6. *The gluing map induces an equivalence of graded spectral bimodules*

$$\mathscr{X}(T_1) \otimes^{\mathbb{L}}_{\mathscr{H}^n} \mathscr{X}(T_2) \simeq \mathscr{X}(T_1T_2).$$

The proof differs from the proof of Theorem 5 only in that the notation is more cumbersome.

Remark 6.5. There is an asymmetry in Formula (6.3): the number of points $2n$ on the right of the tangle appears, but the number of points $2m$ on the left of the tangle does not.

Remark 6.6. The quantum gradings we have defined agree with the gradings in Khovanov's paper on the arc algebras [Kho02], but not with those in his first paper on Khovanov homology [Kho00]. See also Remark 2.55.

7. Some computations and applications

7.1. **The connected sum theorem.** We start by noting that our previous connected sum theorem can be understood as a special case of tangle gluing. Recall:

Theorem 7 ([LLS17, Theorem 8])**.** *Given any knots K_1, K_2 there are $\mathscr{H}^1$-module structures on $\mathscr{X}(K_i)$ so that $\mathscr{X}(K_1\#K_2) \simeq \mathscr{X}(K_1) \otimes^{\mathbb{L}}_{\mathscr{H}^1} \mathscr{X}(K_2)$.*

Proof. Delete a small interval from K_i to obtain a $(0,2)$-tangle T_1 and a $(2,0)$-tangle T_2. Since there is a unique crossingless matching c of 2 points, $\mathscr{X}(T_i)$ consists of a single spectrum $\mathscr{X}(K_1) \simeq \mathscr{X}(T_1)(\varnothing, c)$ (respectively $\mathscr{X}(K_2) \simeq \mathscr{X}(T_2)(c,\varnothing)$), together with a map

$$\mathscr{X}(T_1)(\varnothing, c) \wedge \mathrm{Hom}_{\mathscr{H}^1}(c,c) \to \mathscr{X}(T_1)(\varnothing, c)$$
$$\mathrm{Hom}_{\mathscr{H}^1}(c,c) \wedge \mathscr{X}(T_1)(c,\varnothing) \to \mathscr{X}(T_2)(c,\varnothing)$$

making $\mathscr{X}(T_1)(\varnothing, c)$ (respectively $\mathscr{X}(T_2)(c,\varnothing)$) into a module spectrum over the ring spectrum $\mathrm{Hom}_{\mathscr{H}^1}(c,c)$. So, the statement is immediate from Theorem 5. ☐

Remark 7.1. In [LLS17, Theorem 8], the derived tensor product over $\mathscr{H}^1$ was denoted $\otimes_{\mathbb{H}^1}$, and the Khovanov spectra were denoted $\mathcal{X}_{Kh}(K_i)$. The construction of this paper is the 'opposite' of the construction of the previous paper (see Remark 2.58) and therefore $\mathscr{X}(K_i) = \mathcal{X}_{Kh}(m(K_i))$ where $m(K_i)$ is the mirror knot.

Next we note that the Künneth spectral sequence for structured spectra implies a Künneth spectral sequence for Khovanov generalized homology (e.g., Khovanov K-theory, Khovanov bordism, ...):

Theorem 8. *Suppose K is decomposed as a union of a $(0,2n)$-tangle T_1 and a $(2n,0)$-tangle T_2. Then for any generalized homology theory h_* there is a spectral sequence*

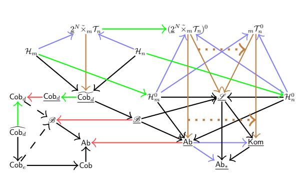
$$\mathrm{Tor}^{h_*(\mathscr{H}^n)}_{p,q}(h_*(\mathscr{X}(T_1)), h_*(\mathscr{X}(T_2))) \Rightarrow h_{p+q}(\mathscr{X}(K)).$$

Proof. This is a corollary [EKMM97, Theorem 6.4], after using the equivalence of symmetric spectra and EKMM spectra. ☐

7.2. **Hochschild homology and links in $S^1 \times S^2$.** Using Hochschild homology, Rozansky defined a knot homology for links in $S^1 \times S^2$ with even winding number around S^1 [Roz]. In this section we note that Rozansky's invariant admits a stable homotopy refinement, and conjecture that the refinement is a knot invariant.

Given an (n,n)-tangle T in $[0,1] \times \mathbb{D}^2$, there are three ways one can close T:

(1) Identify $(0,p) \sim (1,p)$ to obtain a knot $K_{S^1\times\mathbb{D}^2} \subset S^1 \times \mathbb{D}^2$.
(2) Include $S^1 \times \mathbb{D}^2$ as a neighborhood of the unknot in S^3, and let $K_{S^3} \subset S^3$ be the image of $K_{S^1\times\mathbb{D}^2}$.



(3) Include $S^1 \times \mathbb{D}^2$ in $S^1 \times S^2 = (S^1 \times \mathbb{D}^2) \cup_\partial (S^1 \times \mathbb{D}^2)$, and let $K_{S^1 \times S^2} \subset S^1 \times S^2$ be the image of $K_{S^1 \times \mathbb{D}^2}$.

It is clear that every link in $S^1 \times \mathbb{D}^2$, S^3, and $S^1 \times S^2$ arises this way. If we require that n be even (which we shall) then the links which arise in $S^1 \times \mathbb{D}^2$ and $S^1 \times S^2$ are exactly those with even winding number around S^1.

Rozansky's invariant of a knot K in $S^1 \times S^2$ is the Hochschild homology of $\mathcal{C}_{Kh}(T)$, where T is a tangle whose closure is K. Correspondingly, the space-level lift is the topological Hochschild homology of $\mathscr{X}(T)$, the definition of which we recall briefly:

Definition 7.2. Given a pointwise cofibrant spectral category $\mathscr{C}$ and a $(\mathscr{C},\mathscr{C})$-bimodules $\mathscr{M}$, the *topological Hochschild homology* $\mathrm{THH}_{\mathscr{C}}(\mathscr{M}) = \mathrm{THH}(\mathscr{M})$ of $\mathscr{M}$ is the homotopy colimit of the diagram

$$\cdots \substack{\to\\\to\\\to\\\to} \coprod_{a_1,a_2,a_3 \in \mathrm{Ob}(\mathscr{C})} \mathscr{M}(a_3,a_1) \wedge \mathscr{C}(a_1,a_2) \wedge \mathscr{C}(a_2,a_3) \substack{\to\\\to\\\to} \coprod_{a_1,a_2 \in \mathrm{Ob}(\mathscr{C})} \mathscr{M}(a_2,a_1) \wedge \mathscr{C}(a_1,a_2) \rightrightarrows \coprod_{a_1 \in \mathrm{Ob}(\mathscr{C})} \mathscr{M}(a_1,a_1)$$

where $\mathscr{C}(a,b)$ denotes $\mathrm{Hom}_{\mathscr{C}}(a,b)$ and the maps

$$d_i \colon \mathscr{M}(a_n,a_1) \wedge \mathscr{C}(a_1,a_2) \wedge \cdots \wedge \mathscr{C}(a_{n-1},a_n) \to \coprod \mathscr{M}(b_{n-1},b_1) \wedge \mathscr{C}(b_1,b_2) \wedge \cdots \wedge \mathscr{C}(b_{n-2},b_{n-1})$$

are given by composition $\mathscr{C}(a_i,a_{i+1}) \wedge \mathscr{C}(a_{i+1},a_{i+2}) \to \mathscr{C}(a_i,a_{i+2})$ if $1 \le i \le n-2$ and the actions $\mathscr{M}(a_n,a_1) \wedge \mathscr{C}(a_1,a_2) \to \mathscr{M}(a_n,a_2)$ and $\mathscr{C}(a_{n-1},a_n) \wedge \mathscr{M}(a_n,a_1) \to \mathscr{M}(a_{n-1},a_1)$ if $i = 0$ or $n-1$, respectively.

(Compare [BM12, Proposition 3.5]. Recall from Lemma 4.5 that $\mathscr{H}^n$ is pointwise cofibrant.)

Proposition 7.3. *If T and T' induce isotopic knots in $S^1 \times \mathbb{D}^2$ then for each $j \in \mathbb{Z}$,*

$$\mathrm{THH}(\mathscr{X}(T,j)) \simeq \mathrm{THH}(\mathscr{X}(T',j)).$$

Proof. Given a $(2n,2n)$-tangle T decomposed as a composition of two smaller tangles, $T = T_1 \circ T_2$, we will call the tangle $T_2 \circ T_1$ a *rotation of* T. If T and T' induce isotopic knots in $S^1 \times \mathbb{D}^2$ then T and T' are related by a sequence of Reidemeister moves and rotations. Topological Hochschild homology is invariant under quasi-isomorphisms of spectral bimodules [BM12, Proposition 3.7], so by Theorem 4 Reidemeister moves do not change $\mathrm{THH}(\mathscr{X}(T,j))$. Topological Hochschild homology is a trace, in the sense that given spectral categories $\mathscr{C}$, $\mathscr{D}$, a $(\mathscr{C},\mathscr{D})$-bimodule $\mathscr{M}$ and a $(\mathscr{D},\mathscr{C})$-bimodule $\mathscr{N}$,

$$\mathrm{THH}_{\mathscr{C}}(\mathscr{M} \otimes^{\mathbb{L}}_{\mathscr{D}} \mathscr{N}) \simeq \mathrm{THH}_{\mathscr{D}}(\mathscr{N} \otimes^{\mathbb{L}}_{\mathscr{C}} \mathscr{M})$$

[BM12, Proposition 6.2]. Thus, it follows from Theorem 5 that $\mathrm{THH}(\mathscr{X}(T,j))$ is invariant under rotation as well. ☐

Remark 7.4. Since we have only defined an invariant of a $(2m,2n)$-tangle, any link in $S^1 \times \mathbb{D}^2$ which arise from our construction has even winding number.

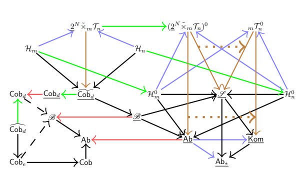
Proposition 7.5. *The singular homology of* $\mathrm{THH}(\mathscr{X}(T))$ *is Rozansky's invariant* $\mathrm{H}^{\mathrm{st}}(\mathbb{S}^2 \times \mathbb{S}^1, K_{S^2 \times S^1})$.

Proof. The proof is similar to the proof of Proposition 4.2, and is left to the reader. ☐

Conjecture 7.6. If T and T' induce isotopic knots in $S^1 \times S^2$ then for each $j \in \mathbb{Z}$,

$$\mathrm{THH}(\mathscr{X}(T,j)) \simeq \mathrm{THH}(\mathscr{X}(T',j)).$$

As Rozansky notes, given Proposition 7.3, to verify Conjecture 7.6 it suffices to verify that $\mathrm{THH}(\mathscr{X}(T,j))$ is invariant under dragging the first strand around the others [Roz, Theorem 2.2].



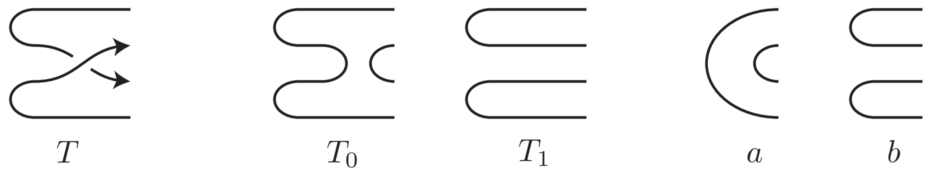


FIGURE 7.1. **Where have all the ladybugs gone?** Left: a tangle T. Center: the resolutions T_0 and T_1 of T. Right: the crossingless matchings a and b.

7.3. **Where the ladybug matching went: an example.** Our longtime readers will recall that a key step in the construction of $\mathscr{X}(K)$ is the ladybug matching, which provides an identification across each 2-dimensional face in the cube of resolutions. (This matching is equivalent to the rule for composing genus 0 cobordisms to get a genus 1 cobordism in Section 2.11.) In particular, the ladybug matching is relevant for certain pairs of crossings in a diagram K. Such readers may also wonder where the ladybug matching has gone, now that the Khovanov homotopy type can be constructed by composing a sequence of 1-crossing tangles. We answer this question, with an example.

Consider the $(0,4)$-tangle T shown in Figure 7.1. If we let a and b be the two crossingless matchings on 4 strands, labeled as in that figure, then

$$\begin{aligned}\mathscr{X}(T)(a) &= \mathrm{Cone}\left(\mathbb{S}_{a,1\otimes 1} \vee \mathbb{S}_{a,1\otimes X} \vee \mathbb{S}_{a,X\otimes 1} \vee \mathbb{S}_{a,X\otimes X} \longrightarrow \mathbb{S}_{a,1} \vee \mathbb{S}_{a,X}\right)\\ &= \mathrm{Cone}\left(\mathbb{S}_{a,1\otimes 1} \to \mathbb{S}_{a,1}\right) \vee \mathrm{Cone}\left(\mathbb{S}_{a,1\otimes X} \vee \mathbb{S}_{a,X\otimes 1} \to \mathbb{S}_{a,X}\right) \vee \mathrm{Cone}\left(\mathbb{S}_{a,X\otimes X} \to \mathrm{pt}\right),\\ \mathscr{X}(T)(b) &= \mathrm{Cone}\left(\mathbb{S}_{b,1} \vee \mathbb{S}_{b,X} \longrightarrow \mathbb{S}_{b,1\otimes 1} \vee \mathbb{S}_{b,1\otimes X} \vee \mathbb{S}_{b,X\otimes 1} \vee \mathbb{S}_{b,X\otimes X}\right)\\ &= \left(\mathbb{S}_{b,1\otimes 1}\right) \vee \mathrm{Cone}\left(\mathbb{S}_{b,1} \to \mathbb{S}_{b,1\otimes X} \vee \mathbb{S}_{b,X\otimes 1}\right) \vee \mathrm{Cone}\left(\mathbb{S}_{b,X} \to \mathbb{S}_{b,X\otimes X}\right),\end{aligned}$$

where we have used subscripts to indicate the Khovanov generator corresponding to each summand. These mapping cones are indicated in Figure 7.2 (where $\mathbb{S}$ has been depicted as S^1).

Consider now the space B_1. The operation $\mathscr{X}(T)(b) \otimes \mathrm{Hom}_{\mathscr{H}^2}(b,a) \to \mathscr{X}(T)(a)$ gives a map

$$B_1 \wedge \mathbb{S}_{b\overline{a},1} \to A_2,$$

where $\mathbb{S}_{b\overline{a},1}$ is the wedge summand of $\mathrm{Hom}_{\mathscr{H}^2}(b,a)$ which labels the single circle in $b\overline{a}$ by 1 (which lives in quantum grading 1). This map sends half of B_1 to the top half in A_2 and half of B_1 to the bottom half in A_2. Which half is sent to which half is determined by the ladybug matching. The two maps are, of course, homotopic, by rotating the sphere A_2 by π or $-\pi$, but the homotopy is not canonical.

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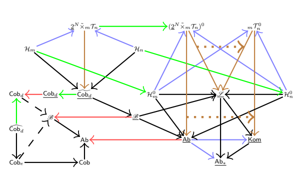

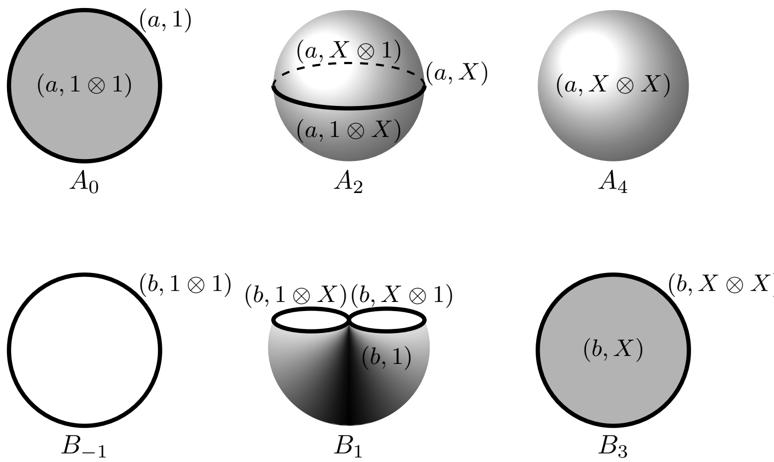

FIGURE 7.2. **Some mapping cones.** The space $\mathscr{X}(T)(a)$ is the wedge sum of the spaces A_0, A_2, and A_4, while $\mathscr{X}(T)(b)$ is the wedge sum of the spaces B_{-1}, B_1, and B_3 (the subscripts denote the quantum gradings). A cellular decomposition is shown with the cells labeled by the corresponding Khovanov generators. The space A_2 is built from one 1-cell labeled (a, X) two 2-cells labeled $(a, 1 \otimes X)$ and $(a, X \otimes 1)$. The space B_1 has two 1-cells labeled $(b, 1 \otimes X)$ and $(b, X \otimes 1)$ and one 2-cell labeled $(b, 1)$.

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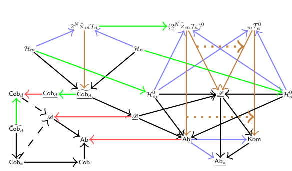

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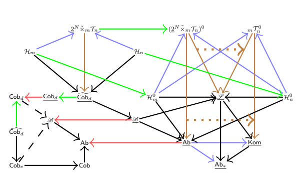

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