MATH 2243: LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS SAMPLE FINAL EXAM

INSTRUCTOR: SASHA VORONOV

Some of the problems on the actual final will be multiple-choice. However, you just have to approach them the same way as other problems. Just do not forget to mark a choice, even if you have not solved the problem.

You may not use a calculator, notes, books, etc. Only the exam paper and a pencil or pen may be kept on your desk during the test. You must show all work.

Good luck!

Problem 1. Suppose that some exotic nuclear waste contains five times as much dysprosium-159 (^{159}Dy, half-life = 144 days) as gold-195 (^{195}Au, half-life = 186 days). When will there be equal amounts of these two isotopes?

Problem 2. Suppose that a is a constant and consider the IVP

$$y' - y = e^{at}, \qquad y(0) = 0.$$

(1) Find the solution if $a \neq 1$.
(2) Find the solution if $a = 1$.

Problem 3. Jane finds that since she entered the U of M (at time $t = 0$), she learns facts $F(t)$ at a rate equal to $F(t)^2$ facts a day. Unfortunately, she also forgets things she has learned at a constant rate of 1 fact per day. Derive a differential equation for the total number of cats Jane knows at time t. Solve the differential equation, assuming the initial condition $F(0) = F_0$. (You may leave your answer in implicit form.)

Problem 4. Let $W = \{(x_1, x_2) : x_1 + |x_2| = 0\}$ be a set of vectors in $\mathbb{R}^2$. Is W a subspace of $\mathbb{R}^2$?

Problem 5. The motion of a mass attached to a spring on a table can be described by a solution of the IVP:

$$mx'' + cx' + kx = f(t), \qquad x(0) = x_0, \quad x'(0) = v_0.$$

The mass weighs 8 lbs and would stretch the spring by 1/2 ft, if hung freely on the spring. Now back to the spring-mass system on the table. The mass is attached to a viscous damper with a damping constant of 2 lb-sec/ft and acted upon by an external force of $\cos 3t$ lbs. The mass is displaced 2 in toward the spring and released.

(1) Set up an IVP describing the motion.
(2) Solve the IVP.

Date: December 14, 2008.

Problem 6. Solve the initial value problem

$$y'' + y' = e^{-t}, \qquad y(0) = 0, y'(0) = 0.$$

Problem 7. Use Gaussian elimination to solve the system

$$\begin{aligned} 2x + 8y + 4z &= 2, \\ 2x + 10y - z &= 5, \\ 4x + 10y - z &= 1. \end{aligned}$$

Problem 8.

$$\text{Let} \quad A = \begin{bmatrix} 1 & 2 & k \\ 2 & 1 & 3 \\ 1 & 3 & k+1 \end{bmatrix}.$$

For which values of k does the system $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution? Justify your answer.

Problem 9. Let $f(t) = t$, $g(t) = e^t$, and $h(t) = t^3$. Are these three functions linearly independent? Can $a(t) = t^2$ be expressed as a linear combination of f, g, and h? Justify your answers.

Problem 10.

$$\text{Let} \quad A = \begin{bmatrix} -2 & 1 \\ -5 & 4 \end{bmatrix}.$$

(1) Find a matrix P diagonalizing A.
(2) Solve the equation $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(0) = (1, 3)$.

Problem 11. Find the general solution for $\mathbf{x}(t)$ of the following 2×2 system:

$$\mathbf{x}' = \begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix} \mathbf{x}.$$

Problem 12. Given $\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$, use the Laplace transform to solve the following IVP:

$$y'' + 4y' + 4y = 0, \qquad y(0) = 0, \quad y'(0) = -2.$$