

**MATH 2243: LINEAR ALGEBRA AND DIFFERENTIAL
EQUATIONS
SAMPLE MIDTERM TEST I**

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You may not use a calculator, notes, books, etc. Only the exam paper and a pencil or pen may be kept on your desk during the test.

Good luck!

Problem 1. Solve the initial value problem

$$xy' + 5y = 7x^2, \quad y(2) = 5.$$

Answer: $y = x^2 + 32/x^5$.

Problem 2. A cake is removed from an oven at 210°F and left to cool at room temperature, which is 70°F. After 30 minutes the temperature of the cake is 140°F. When will it be 105°F? Assume Newton's law of cooling holds.

Answer: After 60 minutes.

Problem 3. A commercial fishery is estimated to have carrying capacity of 10 thousand pounds of certain kind of fish. Suppose the annual growth rate of the total fish population P , measured in thousand pounds, is governed by the logistic equation

$$\frac{dP}{dt} = \left(1 - \frac{P}{10}\right)P,$$

and initially there is a total of 2 thousand lbs of fish.

- (1) What is the fish population after 1 year? (Find a formula for $P(t)$ and plug in $t = 1$ to get $10/(1 + 4e^{-1}) \approx 4.048$ thousand pounds.)

Answer: $P(t) = \frac{10}{1+4e^{-t}}$, $P(1) = 10/(1 + 4e^{-1}) \approx 4.048$.

Suppose after waiting a certain period of time, the owner decides to harvest 2.4 thousand lbs of fish annually at a constant rate. Then

- (2) What is the differential equation governing the fish population now?

Answer:

$$\frac{dP}{dt} = -2.4 + \left(1 - \frac{P}{10}\right)P.$$

- (3) The owner wants to wait such period of time before harvesting, so that the fish population does not start decreasing right away. What approximately is the minimal waiting period (before harvesting) you would recommend to the owner? Why?

Answer: The right-hand side of the equation in (2) must be positive at the time the owner starts harvesting fish. Solving the quadratic equation

$$-2.4 + \left(1 - \frac{P}{10}\right)P = 0,$$

we get $P = 4$ or 6 . Since the parabola $-24 + 10P - P^2$ is positive between the roots 4 and 6, there must be at least 4 thousand lbs of fish to start harvesting with a positive rate of change of population. Thus, the owner has to wait about 1 year, when the fish population will grow up to slightly more than 4 thousand lbs, before harvesting.

Problem 4. (1) Use Euler's method with step size $h = 1$ to approximate the solution to the initial value problem

$$\frac{dy}{dx} = x^2 + y^3, \quad y(0) = 1,$$

on the interval $[0, 1]$.

Answer: $y_1 = 1 + 1 \cdot 1 = 2$.

(2) Use the improved Euler method with same step size for the same problem.

Answer: $u_1 = 1 + 1 \cdot 1 = 2$, $y_1 = 1 + \frac{1}{2} \cdot 1(1 + 1 + 8) = 6$.