Math 4242
Spring 2017
Final, section 1
Time Limit: 120 minutes

Name (Print): SOLUTIONS

Student ID:

This exam contains 13 pages (including this cover page) and 7 problems. Check to see if any pages are missing.

You may not use your books or calculators in this exam, and you may not bring any notes other than **two letter-sized double sided cheat sheets**.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided.
- **Mysterious or unsupported answers will *not* receive full credit.** A correct answer, unsupported by calculations or explanations will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- **To cite a result** from class or the textbook, you should paraphrase the result and note it as a prior result.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	16	
2	13	
3	5	
4	15	
5	12	
6	4	
7	15	
Total:	80	

1. (16 points) For each statement below, determine whether it is true or false and give a brief explanation.

   (a) The function

   $$\det : \mathcal{M}_{n \times n} \to \mathbb{R}$$

   that takes an $n \times n$ matrix to its determinant is linear.

   False

   $$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

   $\det = 0$ $\det = 0$ $\det = 1$

   But $0 + 0 \neq 1$

   (b) The formula

   $$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| = \min(|x|, |y|)$$

   defines a norm on $\mathbb{R}^2$.

   False: $\left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = 0$ but $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq \underline{0}$

   (c) The formula

   $$\left\langle \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\rangle = w_1 \overline{z_1} + \overline{w_2} z_2$$

   defines an inner product on $\mathbb{C}^2$.

   False: should have $\langle \underline{w}, \lambda \underline{z} \rangle = \bar{\lambda} \langle \underline{w}, \underline{z} \rangle$

   But $\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = \lambda$ should be $\bar{\lambda}$

   (d) A square matrix whose diagonal entries are all negative may have a positive eigenvalue.

   True: take $A = \begin{pmatrix} -1 & 1 \\ 4 & -1 \end{pmatrix}$

   Then $\det(A - \lambda I) = \lambda^2 + 2\lambda - 3 = (\lambda + 3)(\lambda - 1)$

   evals -3, $+1$

(e) If A is a nonsingular symmetric matrix, then A^{-1} is also symmetric.

True: $I = (A^{-1}A) = (A^{-1}A)^T = A^T(A^{-1})^T = A(A^{-1})^T$

So $(A^{-1})^T = A^{-1}$

(f) If A is any $n \times n$ matrix, then $|\det A|$ is the product of the singular values of A.

False: if A is singular, $\det A = 0$

But all singular values are positive.

(g) If A is a symmetric matrix, then its singular values are equal to its eigenvalues.

False: the singular values are the absolute values of the nonzero eigenvalues.

(h) If a vector space V has an inner product, it must be finite dimensional.

False For example, $L^2([0,1])$ is infinite dim.

2. (13 points) Let

$$A = \begin{pmatrix} 0 & 2 & -1 \\ 0 & 2 & -1 \\ -1 & 5 & -2 \end{pmatrix}.$$

Find an invertible matrix S such that $S^{-1}AS$ is in Jordan normal form, and write down that Jordan normal form.

$$\det(A-\lambda I) = \det\begin{pmatrix} -\lambda & 2 & -1 \\ 0 & 2-\lambda & -1 \\ -1 & 5 & -2-\lambda \end{pmatrix}$$

$$= -\lambda(2-\lambda)(-2-\lambda) - 5\lambda + 2 - (2-\lambda)$$

$$= -\lambda^3 \qquad \text{all evals } 0$$

But A clearly has rank 2 $\Rightarrow \dim \ker(A) = 1$

$\ker(A) = \text{Span} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \underline{v}_1$

Must be a Jordan block of size 3, eval 0

Solve $A\underline{v}_2 = \underline{v}_1 \qquad \underline{v}_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

$A\underline{v}_3 = \underline{v}_2 \qquad \underline{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

So $S = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 2 & -1 & 0 \end{pmatrix}$ works $\qquad S^{-1}AS = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

3. (7 points) Find a QR decomposition of

$$A = \begin{pmatrix} 3 & 7 \\ 4 & 1 \end{pmatrix}.$$

(Hint: if you can find Q, you can easily compute R via the formula

$$R = Q^T A.)$$

Q: columns obtained from using GS on columns of A to get an orthonormal basis

$$\underline{w}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \underline{w}_2 = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$$

$$\underline{v}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \underline{v}_2 = \underline{w}_2 - \frac{\langle \underline{w}_2, \underline{v}_1 \rangle}{\|\underline{v}_1\|^2} \underline{v}_1 = \begin{pmatrix} 7 \\ 1 \end{pmatrix} - \frac{25}{25} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

normalize

$$\underline{u}_1 = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} \quad \underline{u}_2 = \begin{pmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{pmatrix} \qquad Q = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$$

$$R = Q^T A = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix} \begin{pmatrix} 3 & 7 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 0 & 5 \end{pmatrix}$$

4. Let $(\mathbb{R}^n)^* = \mathcal{L}(\mathbb{R}^n, \mathbb{R})$ be the dual space of $\mathbb{R}^n$, defined as the space of linear maps from $\mathbb{R}^n$ to $\mathbb{R}$. Recall that $(\mathbb{R}^n)^*$ can be thought of as the space of length n row vectors.

If $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ is a basis of $\mathbb{R}^n$, the row vectors

$$\mathbf{v}_1^*, \mathbf{v}_2^*, \cdots, \mathbf{v}_n^* \in (\mathbb{R}^n)^*$$

are defined by

$$\mathbf{v}_i^*(\mathbf{v}_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

(a) (5 points) Let

$$A = (\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n)$$

be the matrix formed by taking the $\mathbf{v}_i$ as columns. Express the row vectors $\mathbf{v}_1^*, \mathbf{v}_2^*, \cdots, \mathbf{v}_n^*$ in terms of A. Justify your answer.

They are the rows of A^{-1}

If $A^{-1} = \begin{pmatrix} \underline{r}_1 \\ \underline{r}_2 \\ \vdots \\ \underline{r}_n \end{pmatrix}$ then $I = A^{-1}A = \begin{pmatrix} \underline{r}_1\underline{v}_1 & \underline{r}_1\underline{v}_2 & \cdots & \underline{r}_1\underline{v}_n \\ \vdots & & & \\ \underline{r}_n\underline{v}_1 & & \cdots & \underline{r}_n\underline{v}_n \end{pmatrix}$

So $\underline{r}_i\underline{v}_j = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$ so $\underline{r}_i = \underline{v}_i^*$

(b) (3 points) Show that $\mathbf{v}_1^*, \mathbf{v}_2^*, \cdots, \mathbf{v}_n^*$ is a basis of $(\mathbb{R}^n)^*$. (This is called the *dual basis* to $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$.)

A^{-1} is nonsingular

Rows of a nonsingular matrix form a basis

(c) (4 points) Let

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Find $\mathbf{v}_1^*$ and $\mathbf{v}_2^*$. (This may be done independently of parts a) and b), but part a) might help.)

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\underline{v}_1^* = \begin{pmatrix} 0 & 1 \end{pmatrix} \qquad \underline{v}_2^* = \begin{pmatrix} 1 & -1 \end{pmatrix}$$

5. Let

$$K = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 5 & 3 \\ -2 & 3 & x \end{pmatrix}.$$

(a) (5 points) For which values of x is K positive definite?

$\frac{1}{2}$ row 1 + row 2

row 1 + row 3

$$K \to \begin{pmatrix} 2 & -1 & -2 \\ 0 & \frac{9}{2} & 2 \\ 0 & 2 & x-2 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & -2 \\ 0 & \frac{9}{2} & 2 \\ 0 & 0 & x-2-\frac{8}{9} \end{pmatrix}$$

row 3 − $\frac{4}{9}$ row 2

So to have positive pivots, we must have $x > 2 + \frac{8}{9} = \frac{26}{9}$

(b) (8 points) Suppose $x = 3$. Find a basis of $\mathbb{R}^3$ which is orthogonal with respect to the inner product $\langle\, ,\, \rangle_K$, where

$$\langle \mathbf{v}, \mathbf{w}\rangle_K = \mathbf{v}^T K \mathbf{w}.$$

$$K = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 5 & 3 \\ -2 & 3 & 3 \end{pmatrix}$$

Take $\quad \underline{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \underline{w}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \underline{w}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Then $\quad \underline{v}_1 = \underline{w}_1 \qquad \underline{v}_2 = \underline{w}_2 - \dfrac{\langle \underline{w}_2, \underline{v}_1\rangle_K}{\|\underline{v}_1\|_K^2}\underline{v}_1$

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{-1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$$

$\|\underline{v}_2\|_K^2 = \frac{1}{4}\cdot 2 + 2\cdot\frac{1}{2}\cdot(-1) + 5 = \frac{9}{2} \qquad \langle \underline{w}_3, \underline{v}_2\rangle_K = -2\cdot\frac{1}{2} + 1\cdot 3 = 2$

$$\underline{v}_3 = \underline{w}_3 - \frac{\langle \underline{w}_3, \underline{v}_2\rangle_K}{\|\underline{v}_2\|_K^2}\underline{v}_2 - \frac{\langle \underline{w}_3, \underline{v}_1\rangle_K}{\|\underline{v}_1\|_K^2}\underline{v}_1$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{\left(\frac{9}{2}\right)}\begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} - \frac{-2}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{7}{9} \\ -\frac{4}{9} \\ 1 \end{pmatrix}$$

$\underline{v}_1, \underline{v}_2, \underline{v}_3$ are orthogonal.

6. (4 points) Suppose A is a 3×3 matrix such that $\mathrm{tr}(A) = -4$, $\det(A) = -6$, and there is some vector $\mathbf{v} \in \mathbb{R}^3$ such that

$$A\mathbf{v} = \mathbf{v}.$$

What are the eigenvalues of A and their multiplicities?

$A\underline{v} = \underline{v} \Rightarrow \underline{v}$ is an evec with eval 1

Say evals are $\lambda_1 = 1$, λ_2, λ_3

$\lambda_1 + \lambda_2 + \lambda_3 = \mathrm{tr}(A) = -4 \Rightarrow \lambda_2 + \lambda_3 = -5$

$\lambda_1 \lambda_2 \lambda_3 = \det(A) = -6 \Rightarrow \lambda_2 \lambda_3 = -6$

So $\lambda_2 = 1$ $\lambda_3 = -6$ (or other way)

$\Rightarrow$ evals $1, 1, -6$

multiplicity 2

7. (a) (3 points) Give a condition on a, b and c for $(a\ b\ c)^T$ to belong to the range of

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}$$

Show that $(4\ 2\ \ -4)^T$ is not in the range of A.

Let $A = (\underline{v}_1\ \underline{v}_2)$

Then if $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = x\underline{v}_1 + y\underline{v}_2$

must have $x = a$, $y = b$

$\Rightarrow c = a - b$. This is the condition

$-4 \neq 4-2$ so $\begin{pmatrix} 4 \\ 2 \\ -4 \end{pmatrix}$ is not in the range

(b) (9 points) Find the pseudoinverse of A.

$K = A^TA = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ $\det(K - \lambda I) = (2-\lambda)^2 - 1$
$= \lambda^2 - 4\lambda + 3$
$= (\lambda - 1)(\lambda - 3)$

evals of K $\lambda_1 = 3$ $\lambda_2 = 1$

sing vals $\sigma_1 = \sqrt{3}$ $\sigma_2 = 1$

sing vecs $\underline{q}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$ $\underline{q}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$\Rightarrow$

Blank space for calculations.

$$p_1 = \frac{A q_1}{\sigma_1} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} \end{pmatrix} \qquad p_2 = \frac{A q_2}{\sigma_2} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

So in the SVD, $A = P\Sigma Q^T$ where

$$P = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \sqrt{\frac{2}{3}} & 0 \end{pmatrix} \quad \Sigma = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{pmatrix} \quad Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Pseudoinverse $A^+ = Q\Sigma^{-1}P^T$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

(c) (3 points) Using part b), or otherwise, find the least squares approximate solution to the linear system

$$x = 4$$
$$y = 2$$
$$x - y = -4.$$

Least squares approx. to $A\underline{x} = \underline{b}$ is given by $\underline{x}^* = A^+\underline{b}$

$$= \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$x = 2 \quad y = 3$