MATH 4281: INTRODUCTION À L'ALGÈBRE MODERNE SELECTED SOLUTIONS TO HOMEWORK 12

INSTRUCTOR: ALEX VORONOV

6.2.12: Solution: For noncontinuous functions, the answer is functions which take values 0 and 1 on points of X. The functions on $X = [0,1]$ which take values 0 or 1 and are continuous will be just constant, because of the intermediate value theorem: otherwise, they will have to take all values between 0 and 1.

6.2.14: Solution: Use Proposition 6.2.28 (b,d). We need to see that $R = Re + R(1-e)$, which is true because for each $r \in R$, we have $r = re + r(1-e)$. Then if $re + r'(1-e) = 0$, then, multiplying by e in the right, we get $re + 0 = 0$, i.e., $re = 0$. Similarly, $r'(1-e) = 0$.

6.2.15: Solution: A nontrivial idempotent might be found as $e = [15]$ by going through elements of $\mathbb{Z}_{35}$ one by one and checking if $[2]^2 = [2]$, $[3]^2 = [3]$, and so on. Then $\mathbb{Z}_{35}e = [15]\mathbb{Z}_{35}$ and $\mathbb{Z}_{35}(1-e) = [-14]\mathbb{Z}_{35} = [21]\mathbb{Z}_{35}$. Let us define maps $\phi : \mathbb{Z}_5 \to [21]\mathbb{Z}_{35}$ and $\psi : \mathbb{Z}_7 \to [15]\mathbb{Z}_{35}$: $\phi(m \pmod 5) := 21m \pmod{35}$ and $\psi(n \pmod 7) := 15n \pmod{35}$. Observe that these maps are well-defined, that is $\phi(5k) = 0$ and $\psi(7l) = 0$, ring homomorphisms, and have inverses: $\phi^{-1}(21[m]) := [m]$, and $\psi^{-1}(15[n]) := [n]$, which are also well defined and homomorphisms. Thus, ϕ and ψ are isomorphisms, which add to an isomorphism $\mathbb{Z}_5 \oplus \mathbb{Z}_7 \to [21]\mathbb{Z}_{35} \oplus [15]\mathbb{Z}_{35}$.

6.3.6: Solution: Proposition 6.3.7 gives a bijection between ideals in R containing M and ideals in R/M. Thus, if there are no ideals in R strictly in between M and R, then there are no ideals in R/M strictly in between 0 and R/M, and vice versa.

6.3.7: Solution: (a) $n\mathbb{Z}$ is NOT a maximal ideal, if and only if there exists an ideal I such that $n\mathbb{Z} \subset I \subset \mathbb{Z}$. By Proposition 6.2.17(b), $I = m\mathbb{Z}$ for some $m \in \mathbb{Z}$. $n\mathbb{Z} \subset m\mathbb{Z}$, if and only if $n = mk$, i.e., $m|n$. This happens, if and only if n is not a prime.

(b) The same argument, but for $K[x]$ instead of $\mathbb{Z}$ and Proposition 6.2.17(c) instead of (b).

(c) Obvious in view of Corollary 6.3.13.

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