MATH 4281: INTRODUCTION À L'ALGÈBRE MODERNE
SOLUTION TO PROBLEM 6.1.16

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Solution: According to Definition 6.1.10, all we need to show is that the map $f(a + b\xi) = (a + b, a - b)$ satisfies the following conditions:

(1) f is bijective;
(2) $f(x + y) = f(x) + f(y)$ for all x and y;
(3) $f(xy) = f(x)f(y)$ for all x and y.

(1) The map f is bijective, because it has an inverse $f^{-1}(u, v) = \frac{1}{2}(u + v) + \frac{1}{2}(u - v)\xi$. (As an alternative, you could show that f is injective and surjective.)

(2) $f(a+b\xi+a'+b'\xi) = f((a+a')+(b+b')\xi) = (a+a'+b+b', a+a'-b-b') = (a + b, a - b) + (a' + b', a' - b') = f(a + b\xi) + f(a' + b'\xi)$.

(3) $f((a+b\xi)(a'+b'\xi)) = f((aa'+bb')+(ab'+ba')\xi) = (aa'+bb'+ab'+ba', aa'+bb' - ab' - ba') = (a + b, a - b)(a' + b', a' - b') = f(a + b\xi)f(a' + b'\xi)$.

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