MATH 5335: GÉOMÉTRIE UNE
COMPUTING POINCARÉ LENGTH: HINTS TO PROBLEMS 9.9.23-26

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Here are some examples of computing Poincaré length, which you can view as hints to homework Problems 9.9.23-26.

Example 1. Find the length of the Euclidean segment between $(1,2)$ and $(4,6)$.

Solution. That length is the same as the Euclidean distance $\|(4,6)-(1,2)\| = \|(3,4)\| = \sqrt{9+16} = 5$. □

Example 2. Find the length of the Poincaré segment between $(-1,2)$ and $(-1,1/3)$.

Solution. That length is the same as the Poincaré distance along a vertical line:

$$\left|\int_2^{1/3} dy/y\right| = |\ln(1/3) - \ln 2| = |-\ln 3 - \ln 2| = \ln 6.$$

□

Example 3. Find the length of the Poincaré segment between $(0,5)$ and $(-5/2, 5\sqrt{3}/2)$.

Solution. That length is the same as the Poincaré distance along a Poincaré line which is a Euclidean circle. First, let us find a an equation of the circle. It must be of the form $(x-\omega)^2 + y^2 = \rho^2$. The conditions are that our given points must satisfy this equation:

$$\begin{cases} \omega^2 + 5^2 & = \rho^2, \\ (5/2+\omega)^2 + 5^2 \cdot 3/2^2 & = \rho^2. \end{cases}$$

Solve this system by subtracting the first equation from the second:

$$5\omega + 25/4 - 25/4 = 0,$$

whence $\omega = 0$ and $\rho = 5$. Now, find a parametric equation of the corresponding arc:

$$(0,0) + \rho(\cos t, \sin t), \qquad \pi/2 \le t \le \arccos(-1/2) = 2\pi/3.$$

Now use the Poincaré length formula (Definition 9.4.7):

$$\begin{aligned} \left|\int_{\pi/2}^{2\pi/3} 5dt/y\right| &= \left|\int_{\pi/2}^{2\pi/3} 5dt/5\sin t\right| \\ &= |\ln((\csc 2\pi/3 - \cot 2\pi/3)/(\csc \pi/2 - \cot \pi/2))| \\ &= \left|\ln((2/\sqrt{3} + 1/\sqrt{3})/(1-0))\right| = \frac{1}{2}\ln 3. \end{aligned}$$

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The evaluation of the integral of $\csc t = 1/\sin t$ can be found in your favorite Calculus text. The answer $\ln((\csc t_2 - \cot t_2)/(\csc t_1 - \cot t_1))$ is given in Proposition 9.4.8. However, you do not need to know how to compute that integral or memorize the answer for this class: I will provide the answer on the coming exam, if needed.

$\square$