

MATH 5335: GEOMETRY I
SOLUTIONS TO SAMPLE EXAM III

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Problem (10.7). These lines should have one of their direction indicators equal to one of the direction indicators $(1, 0)$, $(3, 0)$ for the first Poincaré line and the other direction indicator equal to one of the direction indicators $(-3, 0)$, $(-1, 0)$ of the second Poincaré line. Thus these four lines are as follows:

Direction indicators $(1, 0)$ and $(-3, 0)$: $(x + 1)^2 + y^2 = 4$;

Direction indicators $(1, 0)$ and $(-1, 0)$: $x^2 + y^2 = 1$;

Direction indicators $(3, 0)$ and $(-3, 0)$: $x^2 + y^2 = 9$;

Direction indicators $(3, 0)$ and $(-1, 0)$: $(x - 1)^2 + y^2 = 4$.

Problem (10.12). $y \geq |x|$ is the region above the graph of the function $y = |x|$, whereas $0 < y < 1$ selects the part of that region within the strip between $y = 0$ and $y = 1$ not including the boundary. Thus, the region will be the triangular region bounded by $y = |x|$ and $y = 1$, not including $y = 1$ and the origin. I would shade the interior of the triangle and make the side along $y = 1$ dotted and a small circle around the origin to indicate they are not part of the region.

Now, the area:

$$\int_0^1 \int_{-y}^y \frac{dx}{y^2} dy = \int_0^1 \frac{2y}{y^2} dy = 2 \int_0^1 \frac{dy}{y} = 2(\ln 1 - \ln 0) = 2(0 - (-\infty)) = \infty.$$

Remark: If we chose outside integration with respect to x , we would have to use different limits of integration in y , depending on whether $x < 0$ or $x \geq 0$.

Problem (10.30). You should sketch the region first. It will be the Poincaré triangle bounded by the arcs of circles $x^2 + y^2 = u^2 + 1$, $(x + u/2)^2 + y^2 = (u/2)^2 + 1$, and $(x - u/2)^2 + y^2 = (u/2)^2 + 1$. The radii on the right-hand sides of these equations are figured out using the Pythagorean theorem. The plan is as follows. Let us find the angular measure α of the angle at vertex $(-u, 1)$ and the angular measure β between the right side of the triangle at vertex $(0, 1)$ and the ray $\{(0, y) : y \geq 1\}$ at same vertex. Because the picture is symmetric with respect to the y axis, the sum of the interior angular measures will be $2\alpha + 2\beta$, and thereby the area will be $\pi - 2(\alpha + \beta)$.

Computation of α : Differentiate the equations of the arcs at $(-u, 1)$ using implicit differentiation:

$$\frac{d}{dx}(x^2 + y^2 = u^2 + 1) : \quad 2x + 2yy' = 0,$$

or $y' = -x/y$, which is equal to u at $(-u, 1)$. Then $(1, u)$ is a vector in this direction, and $U = \frac{1}{\sqrt{1+u^2}}(1, u)$ is the unit vector in this direction.

$$\frac{d}{dx}\left(\left(x + \frac{u}{2}\right)^2 + y^2 = \frac{u^2}{4} + 1\right) : \quad 2\left(x + \frac{u}{2}\right) + 2yy' = 0,$$

or $y' = -(x + \frac{u}{2})/y$, which is equal to $u/2$ at $(-u, 1)$. Then $(1, u/2)$ is a vector in this direction, and $V = \frac{1}{\sqrt{4+u^2}}(2, u)$ is the unit vector in this direction.

$$\text{Now } \alpha = \arccos\langle U, V \rangle = \arccos \frac{2+u^2}{\sqrt{(u^2+1)(u^2+4)}}.$$

Computation of β : The unit tangent vector to the arc of $(x - u/2)^2 + y^2 = (u/2)^2 + 1$ at $(0, 1)$ is just the same as V above, because this arc is just a translation of the arc of $(x + u/2)^2 + y^2 = (u/2)^2 + 1$ for which V was computed. The unit tangent vector for the ray $\{(0, y) : y \geq 1\}$ is $(0, 1)$. The scalar product of these two vectors is $u/\sqrt{u^2 + 4}$ and thus

$$\beta = \arccos \frac{u}{\sqrt{u^2 + 4}}.$$

Finally, the area is

$$\pi - 2(\alpha + \beta) = \pi - 2\left(\arccos \frac{2+u^2}{\sqrt{(u^2+1)(u^2+4)}} + \arccos \frac{u}{\sqrt{u^2+4}}\right).$$

Now, the analysis of this function. Since the fractions tend to 1 as $u \rightarrow \infty$, the arc-cosines tend to 0, and the area of the triangle tends to π . The first fraction tends to 1 as $u \rightarrow 0$, but the second fraction tends to 0. Thus, the second arc-cosine tends to $\pi/2$, and thereby the area will go to 0 as $u \rightarrow 0$. To complete the analysis, you would need to compute the derivative in u of the area function to see where the function is increasing or decreasing. For this function, this would be too lengthy in the exam setting and have little to do with Geometry, and so, I am skipping this computation here. Anyway, the area will be changing on the interval $(0, \infty)$ from 0 to π .

Problem (1). The Poincaré line $(x+4)^2 + y^2 = 16$ has $(-8, 0)$ and $(0, 0)$ as the direction indicators. The Poincaré lines asymptotically parallel to this line must have one of these points as a direction indicator and pass through $(0, 6)$. One of these lines, with direction indicator $(0, 0)$, is the vertical line $x = 0$. The other one should be a semi-circle $(x - \omega)^2 + y^2 = \rho^2$ such that

$$\begin{aligned} (8 + \omega)^2 &= \rho^2, \\ \omega^2 + 6^2 &= \rho^2. \end{aligned}$$

Subtract the second one from the first one:

$$64 + 16\omega - 36 = 0$$

or $\omega = -7/4$ and from the second equation $\rho = 25/4$. Finally, the equation of the second Poincaré line is

$$\left(x + \frac{7}{4}\right)^2 + y^2 = \left(\frac{25}{4}\right)^2.$$