SAMPLE MIDTERM EXAM - MATH 5378, SPRING 2013

THIS IS A CLOSED-BOOK, CLOSED-NOTES EXAM. YOU CAN USE IN YOUR SOLUTIONS ANY RESULT THAT WAS COVERED IN CLASS OR THE TEXT. YOU CAN ALSO USE ANY OF THE RESULTS FROM THE HOMEWORK.

Note: This exam is rather a set of sample problems. It is actually longer than an actual one-hour exam.

(1) Let $\alpha : I \longrightarrow \mathbb{R}^2$ be a regular parametrized plane curve and $N(t)$ and $\kappa(t)$ be the normal vector, positively oriented with respect to the tanget vector, and the directed curvature of α, respectively. Assume $\kappa(t) \neq 0$ for all $t \in I$. Recall that in this situation the curve

$$\mathcal{E}(t) = \alpha(t) + \frac{1}{\kappa(t)} N(t)$$

is called the *evolute* of α. Show that the tangent line of the evolute is the normal line to α at t.

(2) Show that the knowledge of the vector function $B(s)$ (the binormal vector) of a curve α, with nonzero torsion everywhere, determines the curvature $\kappa(s)$ and the absolute value of the torsion $\tau(s)$ of a unit-speed curve $\alpha(s)$. In other words, if $B(s)$ is given, but $\alpha(s)$ is not, show how to find $|\tau(s)|$ and $\kappa(s)$.

(3) One way to define a coordinate chart for the sphere S^2, given as $x^2 + y^2 + (z-1)^2 = 1$, is to consider the so-called *stereographic projection* $\pi : S^2 \setminus \{N\} \to \mathbb{R}^2$ which carries a point $p = (x, y, z)$ on the sphere minus the north pole $N = (0, 0, 2)$ onto the intersection of the xy-plane with the straight line which connects N to p. Let $(u, v) = \pi(x, y, z)$. (1) Show that $\pi^{-1} : \mathbb{R}^2 \to S^2$ is given by

$$\begin{aligned} x &= \frac{4u}{u^2 + v^2 + 4} \\ y &= \frac{4v}{u^2 + v^2 + 4} \\ z &= \frac{2(u^2 + v^2)}{u^2 + v^2 + 4} \end{aligned}$$

(2) Use this chart to give another coordinate chart, so as both charts cover the sphere. Write down formulas for the corresponding change of coordinates.

(4) Suppose we have a surface parametrized by coordinates u and v and whose first fundamental form is $E = 1$, $F = 0$, $G = u^2 + a^2$, where a is a constant. Find the area of the triangle on the surface bounded by the curves

$$u = \pm av, \quad v = 1.$$

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[Hint: Use the fact from Multivariable Calculus that the surface area element dA is $\|x_u \times x_v\| dudv$, i.e., use Theorem 7.16.]

(5) Let $\lambda_1, \ldots, \lambda_m$, $m > 2$, be the normal curvatures of a surface S at $p \in S$ along the directions making angles $0, 2\pi/m, \ldots, (m-1)2\pi/m$ with a principal direction. Prove that $\lambda_1 + \cdots + \lambda_m = mH$, where H is the mean curvature at p. [Hint: Use the fact that for $\theta = 2\pi/m$

$$1 + \cos^2\theta + \cdots + \cos^2(m-1)\theta = \frac{m}{2}.]$$

(6) Compute the first fundamental forms for the following surfaces:
   (a) $x(u,v) = ((a + b\cos v)\cos u, (a + b\cos v)\sin u, b\sin v)$ (the torus);
   (b) $x(u,v) = (av\cos u, av\sin u, bv)$ (the cone).

(7) Compute the second fundamental form and the Gaussian curvature of the pseudosphere $x(u,v) = (a\sin u\cos v, a\sin u\sin v, a(\ln\tan\frac{u}{2} + \cos u))$.