Math 5615H, Fall 2020 **Final Exam**

Posted: 12/17; minor clarification added to #1 at 4:00 pm 12/17; due Saturday, 12/19, 3:30 p.m.

Please scan handwritten pages and upload the resulting image file or the pdf file produced by LaTeX of whatever document preparation software you use to Canvas.

Rules: **Unlike working on your homework, no study groups or cooperation when doing the exam, no asking questions on internet forums, etc.! You may use any textbooks and internet sources, but just copying arguments you might occasionally find will not gain any credit and will be regarded as plagiarism. You have to present all solutions in your own words.**

Regarding justifying your solutions: You may use any statement stated in class or in our textbook, baby Rudin, or stated in the homework, unless it makes your solution ridiculous, such as just "Stated in class" and nothing more. You may also use one exam problem in your solution of another exam problem. You may use whatever theorems of algebra you wish to use.

You should also write on your paper the following *Honor Pledge*: "I pledge my honor that I have not violated the Honor Code during this examination" and sign your name under it.

Problem 1. Let A be the set of infinite sequences whose elements are the digits 0, 1, and 2, each sequence in A ending in all 2's, such as

$$\{0,0,0,2,0,1,2,1,0,2,2,2,\dots\}.$$

Determine whether A is countable or uncountable. Explain your answer.

Problem 2. Let $X, Y \subset \mathbb{R}$ be sets of **nonnegative** real numbers. Let $X+Y$ be the set of all possible sums of numbers from X and Y:

$$X+Y := \{x+y \,|\, x \in X, y \in Y\}.$$

Show that $\inf(X+Y) = \inf X + \inf Y$, where inf is the greatest lower bound of a set, also known as the *infimum*.

Problem 3. Show that the union of two compact sets in a metric space is compact.

Problem 4. Prove that every Cauchy sequence in a metric space is bounded.

Problem 5. Suppose a series $\sum_{n=1}^{\infty} a_n$ of real numbers is absolutely convergent. Show that the series $\sum_{n=1}^{\infty} a_n^2$ converges.

Problem 6. Let $\sum_{n=0}^{\infty} a_n$ be a convergent series. Prove that $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for each real x such that $|x| < 1$ and

$$\lim_{x\to 1-} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n.$$

Hint: Use summation by parts to deal with the second question.

Problem 7. Show that the image of a closed set under a continuous function between metric spaces is not always closed.

Problem 8. Let $f : (a,b) \to \mathbb{R}$ be differentiable. Prove that if $f'(x) \neq 0$ for all $x \in (a,b)$, then it must be that $f'(x) > 0$ for all $x \in (a,b)$ or $f'(x) < 0$ for all $x \in (a,b)$.