Math 5615H **Homework 4**

Posted: 10/3; #1 clarified: 10/8; due Friday, 10/9

The problem set is due at the beginning of the class on Friday. Please scan handwritten pages and upload the resulting image file or the pdf file produced by LaTeX of whatever document preparation software you use to Canvas.

Reading: **Class notes (some available on the Course Outlines page** `http://www-users.math.umn.edu/~voronov/5615-20/outline.html`**).**
(Baby) Rudin: Sections 2.34-2.42, 2.44-2.47, 2.7, 3.1-3.2.

Problem 1. Let X be a metric space. Suppose $A \subset X$ has the Bolzano-Weierstrass property, *i.e.*, every infinite subset of A has a limit point in A. Show that there exists a *countable base* for A, that is to say, a countable collection $\{U_i\}_{i\in\mathbb{N}}$ (or a finite one, $\{U_i\}_{i=1}^n$) of open sets of X, such that if V is an open set in X and $x \in A \cap V$, then there exists some $i \in \mathbb{N}$ such that $x \in U_i \subset V$. (A metric space that has a countable base is called *second-countable*.) Do not use the fact that the Bolzano-Weierstrass property implies compactness, which I mentioned in class. The matter is that you will be proving that B-W implies compactness on the take-home midterm, and this homework problem (or rather, your solution of it) may be used as a step to proving that theorem: B-W $\Rightarrow$ compactness, on the exam. *Hint*: To construct such a base, show for each $n \in \mathbb{N}$, there is a finite number of balls of radius $1/n$ covering A.

Problem 2. Show that $\mathbb{R}^n$ is separable in the sense of Problem 9 on HW 3.

Problem 3. Show that the definition of a disconnected set in the textbook is equivalent to our definition, which uses open sets for separation.

Problem 4. Show that any convex subset of $\mathbb{R}^n$ is connected.

Problem 5. Prove that for every $\varepsilon > 0$, there exists a finite collection of closed intervals $\{I_1, \dots, I_n\}$ such that $C \subset \bigcup_{j=1}^n I_j$ and $\sum_{j=1}^n \mu(I_j) < \varepsilon$, where $\mu([a, b]) := b-a$ is the length of an interval $[a, b]$. (By definition, this implies that the Cantor set has *Lebesgue measure zero*.) *Hint*: You may take it for granted that $\lim_{k\to\infty}(2/3)^k = 0$.

Problem 6. Show that the complement $D = [0, 1] \setminus C$ of the Cantor set C is dense in $[0, 1]$.

Problem 7. Find an example of a connected subset of $\mathbb{R}^2$ whose interior is disconnected. Explain why the interior is disconnected, but do not explain why the subset is connected, as it makes the problem unnecessarily hard.

Problem 8. Give an example to show that a nested sequence of open intervals can have empty intersection.