Math 5615H **Fall 2020** **Homework 6**

Posted: 10/24; Problem 7 removed on 10/28; due Friday, 10/30

The problem set is due at the beginning of the class on Friday. Please scan handwritten pages and upload the resulting image file or the pdf file produced by LaTeX of whatever document preparation software you use to Canvas.

Reading: **Class notes (available on the Course Outlines page** `http://www-users.math.umn.edu/~voronov/5615-20/outline.html`**).** **(Baby) Rudin**: Sections 3.43, 3.45-3.46, 3.25, 3.15-3.19, 3.33-3.37 (skipping the proof of 3.37).

Problem 1. Suppose that $a_1 \geq a_2 \geq a_3 \geq \dots$ and $\lim_{n\to\infty} a_n = 0$. Show that if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} na_n = 0$. (This is easy if you know that $\lim_{n\to\infty} na_n$ exists, but why should it exists?) *Hint*: Write what it means that 0 is not a limit of $\{na_n\}$.

Problem 2. Show that the series $\sum_{k=1}(-1)^{k+1}x^k/k$ converges for $0 < x < 1$. (The sum is $\log(1+x)$ for these values of x, where log denotes the natural logarithm function.) Show that the series converges when $x = 1$. (The sum when $x = 1$ can be shown to equal $\log 2$.) What about $x = -1$?

Problem 3. If $b_k \geq 0$ for all k, $\sum_{k=1}^{\infty} b_k$ converges with sum b, and a_k are complex numbers such that $|a_k| \leq b_k$ for all k, then $\sum_{k=1}^{\infty} a_k$ converges to a complex number a such that $|a| \leq b$. *Hint* : Denote the nth partial sums for the two series by $s_n = \sum_{k=1}^{n} a_k$ and $S_n = \sum_{k=1}^{n} b_k$, and use the triangle inequality to compare them. In particular, show that $|s_n| \leq S_n$ and that for $m > n$, $|s_m - s_n| \leq |S_m - S_n|$.

Problem 4 (A limit comparison test)**.** Prove: If $a_k > 0$ and $b_k > 0$for each k, and if

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L > 0,$$

then $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges. *Hint*: By the limit hypothesis, there exists N such that if $k \geq N$, then $L/2 < a_k/b_k < 3L/2$.

Problem 5. For a sequence $\{p_n\}$ of real numbers, in class, we will have defined $\limsup_{n\to\infty} p_n$ as

$$\lim_{n\to\infty} \sup_{m\geq n}\{p_m\}$$

and $\liminf_{n\to\infty} p_n$ as

$$\lim_{n\to\infty} \inf_{m\geq n}\{p_m\}.$$

Show that this definition is equivalent to the one in the text, which uses subsequential limits. You may assume the sequence $\{p_n\}$ is **bounded**, so as to avoid the unbounded above and below cases to be treated separately. *Hint*: Of course, you should prove only the statement for lim sup and turn the statement about lim inf into the first one by a one-symbol algebraic trick.

Problem 6. Show that for a sequence $\{p_n\}$ of real numbers,

$$\limsup_{n\to\infty} p_n < \infty,$$

iff $\{p_n\}$ is bounded above.